Near-Linear-Time Deterministic Plane Steiner Spanners and TSP Approximation for Well-Spaced Point Sets

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Abstract

We describe an algorithm that takes as input n points in the plane and a parameter ϵ, and produces as output an embedded planar graph having the given points as a subset of its vertices in which the graph distances are a $(1 + \epsilon)$-approximation to the geometric distances between the given points. For point sets in which the Delaunay triangulation has bounded sharpest angle, our algorithm's output has $O(n)$ vertices, its weight is $O(1)$ times the minimum spanning tree weight, and the algorithm's running time is bounded by $O(n\sqrt{\log \log n})$. We use this result in a similarly fast deterministic approximation scheme for the traveling salesperson problem.

1   Introduction

A *spanner* of a set of points in a geometric space [13] is a sparse graph having those points as its vertices, and with its edge lengths equal to the geometric distance between the endpoints, such that the graph distance between any two points accurately approximates their geometric distance. More precisely, the *dilation* of a spanner is the smallest number δ for which the graph distance of every pair of points is at most δ times their geometric distance. It has long been known that very good spanners exist: for every constant $\epsilon > 0$ and constant dimension d, it is possible to find a spanner for every set of n points in $O(n \log n)$ time such that the dilation of the spanner is at most $1 + \epsilon$, its weight is at most a constant times the weight of the minimum spanning tree, and its degree is constant [4].

A spanner is *plane* if no two of its edges (represented as planar line segments) intersect except at their shared endpoints [10]. Plane spanners with bounded dilation are known; for instance, the Delaunay triangulation is a spanner in this sense [9]. However, it is not possible for these spanners to have dilation arbitrarily close to one. For instance, for four points at the corners of a square, any plane graph must avoid one of the diago-

nals and have dilation at least $\sqrt{2}$. However, the addition of *Steiner points* allows smaller dilation for pairs of original points. For instance, the plane graph formed by overlaying all possible line segments between pairs of input points has dilation exactly one, although its $\Theta(n^4)$ combinatorial complexity is high. Less trivially, in the *pinwheel tiling*, a certain aperiodic tiling of the plane, any two vertices of the tiling at geometric distance D from each other have graph distance $D + o(D)$ [18]. We define a *plane Steiner δ-spanner* for a set of points to be a graph that contains the points as a subset of its vertices, is embedded with straight line edges and no crossings in the plane, and achieves dilation δ for pairs of points in the original point set. We do not require pairs of points that are not both original to be connected by short paths.

Arikati et al. [2] show how to construct a plane Steiner spanner in $O(n \log n)$ time, but do not bound the total weight of the graph. Of course, spanners may also be constructed by forming an arrangement of line segments [12] representing the edges of a nonplanar spanner graph; this planarization does not change the spanner's weight, but may add a large number of edges and vertices. A paper of Klein [15] on graph spanners provides an alternative basis for plane Steiner spanner construction. Generalizing a previous result of Althöfer et al. [1], Klein shows that any n-vertex planar graph with a specified subset of vertices may be thinned to provide a planar Steiner $(1 + \epsilon)$-spanner for the graph distances on the specified subset, with weight $O(1/\epsilon^4)$ times the weight of the minimum Steiner tree of the subset, in time $O((n \log n)/\epsilon)$. Klein combined this result with methods from another paper [16] to provide a polynomial time approximation scheme for the traveling salesperson problem in weighted planar graphs. Using Klein's method to reduce the weight of the geometric spanner formed by the arrangement of all line segments connecting pairs of a given point set would lead to a low weight plane $(1 + \epsilon)$ Steiner spanner for the point set, but again with a large number of vertices and edges. Ideally, we would prefer plane Steiner spanners that not only have low weight, but also have a linear number of edges and vertices.

Small and low-weight plane Steiner spanners in turn could be used with Klein's planar graph algorithms to derive a deterministic polynomial time approximation

scheme for the Euclidean TSP. The previous randomly shifted quadtree approximation scheme of Arora [3] and guillotine subdivision approximation scheme of Mitchell [17] have runtimes that are polynomial for fixed ϵ but with an exponent depending on ϵ; in contrast, Klein's method takes time linear in the spanner size for any fixed ϵ. However, combining Klein's method with the nonlinear-size Steiner spanners described above would not improve on a different deterministic TSP approximation scheme announced by Rao and Smith [19]. Their method is based on *banyans*, a generalized type of spanner that must accurately approximate all Steiner trees, and it takes $O(n \log n)$ time for any fixed ϵ and any fixed dimension, although its details do not appear to have been published yet.

These past results raise several questions. Are banyans necessary for fast TSP approximation, or is it possible to make do with more vanilla forms of spanners? How quickly may low-weight plane Steiner spanners be constructed, and how quickly may the TSP be approximated? And how few vertices are necessary in a plane Steiner spanner?

In this work we provide some partial answers, for planar point sets that are well-spaced in the sense that their Delaunay triangulation avoids sharp angles. We show that, when both ϵ and the sharpest angle in the Delaunay triangulation are bounded by fixed constants, then there exist plane Steiner $(1+\epsilon)$-spanners with $O(n)$ vertices whose weight is $O(1)$ times the minimum spanning tree weight (with a near-linear dependence on ϵ, improving the quartic dependence in Klein's construction). The dependence on ϵ and the sharpest angle is given in Theorem 8. Our spanners may be constructed in linear time given the Delaunay triangulation, or (by combining a fast Delaunay triangulation algorithm of Buchin and Mulzer [11] with fast integer sorting algorithms [14]) in time $O(n\sqrt{\log \log n})$ for points with integer coordinates. Combining these spanners with the methods from Klein [16] leads to near-linear-time TSP approximation for the same class of point sets.

2 Delaunay triangulations without sharp angles

The *Delaunay triangulation* DT of a set S of points (called *sites*) is a triangulation in which the circumcircle of each triangle does not contain any sites in its interior. For points in general position (no four cocircular) the Delaunay triangulation is uniquely defined and its sharpest angle α is at least as large as the sharpest angle in any other triangulation. As we show in this section, Delaunay triangulations that do not have any triangles with sharp angles have two key properties: their total weight $w(DT)$ is small relative to the weight $w(MST)$ of the minimum spanning tree, and every point in the plane is covered by only a few circumcircles.

Lemma 1

$$w(DT) = f_w(\alpha)w(MST) \ where \ f_w(\alpha) = \frac{1 + \cos \alpha}{1 - \cos \alpha}.$$

Proof. The proof follows closely that of Lemma 3.1 of Klein [15]. Let T be the MST. (Recall $T \subseteq DT$.) Let H_0 be the non-self-crossing Euler tour of T. We consider the edges of $DT \setminus T$ in a leaf-to-root order with respect to the dual tree T^*: $e_1, e_2, \ldots, e_k$. e_1 makes a triangle with edges a_1 and b_1 of H_0. Recursively define H_i as the tour resulting from removing a_i and b_i from H_{i-1} and adding e_i: $w(H_i) = w(H_{i-1}) + w(e_i) - w(a_i) - w(b_i)$.

Since the α is the smallest angle of triangle $e_i a_i b_i$ and e_i is longest when $w(a_i) = w(b_i)$, we get $w(e_i) \leq (w(a_i) + w(b_i)) \cos \alpha$.

Combining, we get:
$w(H_i) \leq w(H_{i-1}) + (1 - 1/\cos \alpha)\, w(e_i)$.
Summing:
$w(H_k) \leq w(H_0) + (1 - 1/\cos \alpha) \sum_i w(e_i)$.
By rearranging and using the facts $w(H_k) \geq 0$ and $w(H_0) = 2\,w(MST)$, the lemma follows. $\square$

Lemma 2 *The number of Delaunay circumdisks whose interiors contain a given point in the plane is at most*

$$f_e(\alpha) = 2\pi/\alpha. \tag{1}$$

Proof. The lemma trivially holds for points that are sites. Let x be a non-site point in the plane. Then the Delaunay triangles whose circumcircles contain x are exactly the ones that get removed from the Delaunay triangulation if we add x to S and re-triangulate. Therefore, the number of Delaunay circumcircles that contain x is the same as the degree of x in the Delaunay triangulation, DT_x of $S \cup \{x\}$.

Let d be the degree of x in DT_x. Then, one of the triangles, xqr, in DT_x incident to x has an angle at x of at most $2\pi/d$. Edge qr must be a side of a triangle qrs in DT replacing triangle xqr, because after the removal of x, line segment qr is still a chord of the empty circle that circumscribed xqr; however, triangle qrs has a circumcircle that extends at least as far from qr on the side of the triangle as the circumcircle of xqr, and therefore angle qsr is at least as sharp as angle qxr. So it must be that $2\pi/d > \alpha$ or $d < 2\pi/\alpha$, proving the lemma. $\square$

3 Portals for chords

As we now show, it is possible to space a set of *portals* along an edge of a Delaunay triangulation in such a way that any chord of a Delaunay circumcircle must pass close to one of the portals, relative to the chord length.

Lemma 3 *Let AD be a chord of a circle O, let B and C be points interior to segment AD, and let EF be another*
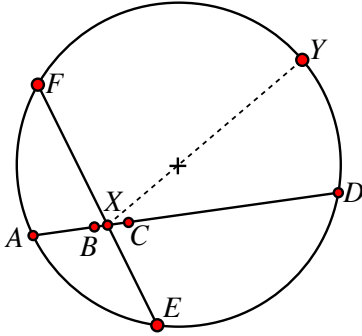
Figure 1: Figure for Lemma 3

chord of O, crossing AD between B and C. Then the distance from chord EF to the nearer of the two points B and C is at most $|EF| \cdot |BC|/2\min(|AB|,|CD|)$.

Proof. The points of the lemma are illustrated in Figure 1. We assume without loss of generality that F is on the side of AD that contains the center of O, as drawn in the figure; let Y be the point of O farthest from X, lying on the line through X and the center of O. Note that the distance from line EF to the closer of B and C is at most $\min(|BX|,|CX|) \le |BC|/2$, so it remains to prove that $|EF| \ge \min(|AB|,|CD|)$. But if F lies on the arc between A and Y, then $|EF| \ge |FX| \ge |AB|$, and if F lies on the arc between Y and D then $|EF| \ge |FX| \ge |CD|$. In either case the result follows. □

Lemma 4 *Let s be a line segment in the plane, and let $\epsilon > 0$. Then there exists a set $P_{s,\epsilon}$ of $O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$ points on s with the property that, for every circle O for which s is a chord, and for every chord t of O that crosses s, t passes within distance $\epsilon|t|$ of a point in $P_{s,\epsilon}$.*

Proof. Our set $P_{s,\epsilon}$ includes both endpoints of s and its midpoint. In the subset of s from one endpoint p_0 to the midpoint, we add a sequence of points p_i, where p_1 is at distance $O(\epsilon^2 s)$ from p_0 with a constant of proportionality to be determined later and where for each $i > 1$, p_i is at distance $\epsilon\, d(p_0, p_{i-1})$ from p_i. Because the distance from p_0 increases by a $(1+\epsilon)$ factor at each step, the set formed in this way contains $O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$ points.

If chord t crosses s between some two points p_i and p_{i+1} for $i \ge 1$, or between the last of these points and the midpoint of s, then Lemma 3 ensures that the nearer of these two points is within distance $\epsilon|t|$ of the chord.

Otherwise, t crosses s between p_0 and p_1. Let r be the radius of O, necessarily at least $|s|/2$, and suppose that t passes within distance δr of p_0. Because of the choice of p_1, $\delta = O(\epsilon^2)$. By the Pythagorean theorem, $|t| \ge r\sqrt{2\delta - \delta^2} = \Omega(|r|\sqrt{\delta})$, and combining this information with the definition of δ shows that t is within distance $O(\sqrt{\delta}|t|) = O(\epsilon|t|)$ of p_0. By choosing the constant of proportionality in the placement of p_1 appropriately we can ensure that this distance is at most $\epsilon|t|$. □

We call the points in $P_{s,\epsilon}$ *portals*.

4 Spanning the portals within each triangle

Within each triangle of the Delaunay triangulation, we will use a plane Steiner spanner that connects the portals that lie on the triangle edges. For this special case, we use a construction that generalizes to an arbitrary set P of points on the boundary of an arbitrary planar convex set K. Given a range of angles $\theta \pm \delta$, we say that a path is $(\theta \pm \delta)$-*angle-bounded* if it is piecewise linear and each linear segment remains within this range of angles, and we say that a point p on the boundary of K is $(\theta \pm \delta)$-*extreme* if there does not exist a $(\theta \pm \delta)$-angle-bounded path from p to a point interior to K.

Lemma 5 *Every $(\theta \pm \delta)$-angle-bounded path has length $1 + O(\delta^2)$ times the distance between its endpoints.*

Proof. The most extreme case is a path that follows two sides of an isosceles triangle having the endpoints of the path as base, for which the length is the length of the base multiplied by $1/\cos\delta = 1 + O(\delta^2)$. □

Lemma 6 *Let P be a set of n points on the boundary of a convex set K with perimeter ℓ, let θ be an angle, and let $\delta > 0$ be a positive number. Then in time $O(n\log n)$ we can construct a set S of $O(n)$ line segments within K, with total length $O((\ell \log n)/\delta)$, with the property that for every point p in P there exists a $(\theta \pm \delta)$-angle-bounded path in S from p to a $(\theta \pm \delta)$-extreme point of K.*

Proof. We consider the points of P in an order we will later define; for each such point p that is not itself $(\theta \pm \delta)$-extreme, we extend two line segments with angles $\theta - \delta$ and $\theta + \delta$ until reaching either an extreme point of K or one of the previously constructed line segments. Thus, a $(\theta \pm \delta)$-angle-bounded path from p may be found by following either of these two line segments, and continuing to follow each line segment hit in turn by the previous line segment on the path, until reaching an extreme point.

The non-extreme points of P, because K is convex, form a contiguous sequence along the boundary of K. We extend segments from the two endpoints of this sequence, then from its median, and then finally we continue recursively in the two subsequences to the left and right of the median, as shown in Figure 2.

The segments from the first two points of P contribute a total length of ℓ to S. For each subsequent point p, the length of each added segment is at most proportional to $1/\delta$ times the length of the part of the boundary of K that extends from p to the most recently previously considered point in the same direction. Because of the ordering of the points, each point along the boundary is
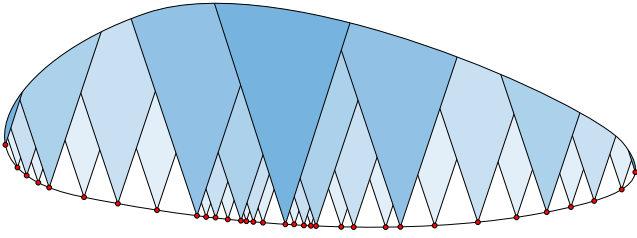
Figure 2: Figure for Lemma 6

charged in this way for $O(\log n)$ segments, so adding this quantity over all points, the total length of the segments is $O((\ell \log n)/\delta)$ as claimed. We may construct S in $O(n \log n)$ time by using binary search to determine the endpoint on K of each segment. □

Lemma 7 *Let P be a set of n points on the boundary of a convex set K with perimeter ℓ, and let $\epsilon > 0$ be a positive number. Then in time $O(n^2/\epsilon)$ we can construct a plane Steiner $(1 + \epsilon)$-spanner for P, with all spanner edges in K, with $O(n^2/\epsilon)$ edges and vertices, and with total length $O((\ell \log n)/\epsilon)$.*

Proof. We choose $\delta = O(\sqrt{\epsilon})$ (with a constant of proportionality determined later), partition the circle into $O(1/\delta)$ arcs of angle 2δ, let θ_i be the angle at the center of the ith arc, and apply Lemma 6 to each of the arcs $\theta_i \pm \delta$. We overlay the resulting system of $O(n/\delta)$ line segments; when two line segments from different arcs both have the same angle and starting point, we choose the longer of the two to use in the overlay. The resulting arrangement of line segments has $O(n^2/\epsilon)$ edges and vertices and total length $O((\ell \log n)/\epsilon)$ as required, and can be constructed in time $O(n^2/\epsilon)$ using standard line segment arrangement construction algorithms [12].

To see that this is a spanner, we must show that every pair (p, q) of points in P may be connected by a short path. Let θ be the angle formed by the segment from p to q, choose i such that $\theta + \delta \leq \theta_i \leq \theta + 3\delta$, and use Lemma 6 to find a $(\theta_i \pm \delta)$-angle-bounded path pp' in the spanner from p to a $(\theta_i \pm \delta)$-extreme point p'. Because of the angle bound, p' must be clockwise of q. Similarly, we may choose θ_j within $O(\delta)$ of $\pi + \theta$, and find a $(\theta_j \pm \delta)$-angle-bounded path qq' to a $(\theta_j \pm \delta)$-extreme point q' that is counterclockwise of p. These two paths (depicted in Figure 3) must cross at at least one point x, and the combination of the path from p to x and from x to q lies within the spanner and is $(\theta \pm O(\delta))$-angle-bounded. By Lemma 5, this path has length at most $1 + O(\delta^2)$ times the distance between its endpoints, and by choosing the constant of proportionality in the definition of δ appropriately we can cause this factor to be at most $1 + \epsilon$. □
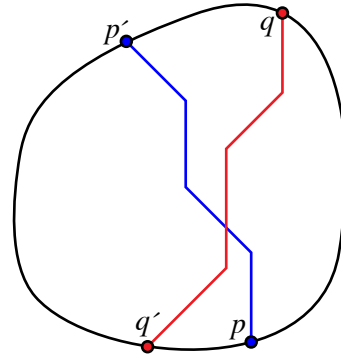

Figure 3: Figure for Lemma 7

5 Spanner construction

We now have all the pieces for our overall spanner construction.

Theorem 8 *Let P be a planar point set whose Delaunay triangulation is given and has sharpest angle α, and let $\epsilon > 0$ be given. Then in time $O(n \log^2(1/(\alpha\epsilon))/(\alpha^2\epsilon^3))$ we can construct a plane Steiner $(1 + \epsilon)$-spanner for P with $O(n \log^2(1/(\alpha\epsilon))/(\alpha^2\epsilon^3))$ vertices and edges, and with total length $O(w(MST) \log(1/(\alpha\epsilon))/(\alpha^2\epsilon))$.*

Proof. We apply Lemma 3 to place portals along the edges of the triangulation, such that each chord s of a Delaunay circle passes within distance $O(\alpha\epsilon|s|)$ of a portal on each Delaunay edge that it crosses. We then apply Lemma 7 within each Delaunay triangle to construct a $1 + O(\epsilon)$-spanner for the portals on the boundary of that triangle.

The construction time is bounded by the time to construct the spanners within each triangle. Since there are $O(\log(1/(\alpha\epsilon))/(\alpha\epsilon))$ portals on each triangle, the time to construct the spanner for a single triangle is $O(\log^2(1/(\alpha\epsilon))/(\alpha^2\epsilon^3))$ and the total time over the whole graph is $O(n \log^2(1/(\alpha\epsilon))/(\alpha^2\epsilon^3))$. This bound also applies to the number of vertices and edges in the constructed spanner. By Lemma 1, the total perimeter of the Delaunay triangles is $O(w(MST)/\alpha^2)$, and combining this bound with the length bound of Lemma 7 gives total length $O(w(MST) \log(1/(\alpha\epsilon))/(\alpha^2\epsilon))$ for the spanner edges.

To show that this is a spanner, we must find a short path between any two of the input points p and q. By Lemma 3, the line segment pq passes within distance $O(\alpha\epsilon|s|)$ of a portal on every Delaunay edge that it crosses, where s is the chord of one of the Delaunay circles for the crossed edge. By Lemma 2, the total length of all of these chords is $O(|pq|/\alpha)$, so we may replace pq by a polygonal path that contains a portal on each crossed Delaunay edge, expanding the total length by a

factor of at most $1 + O((\alpha\epsilon)/\alpha) = 1 + O(\epsilon)$. Then, by Lemma 7 we may replace each portal-to-portal segment in this path by a path within the spanner for the portals in a single Delaunay triangle, again expanding the total length by a factor of at most $1 + O(\epsilon)$. By choosing constants of proportionality appropriately, we may make the total length expansion be at most $1 + \epsilon$. $\quad\square$

6 Approximating the TSP

An algorithm of Klein [16] provides a linear time approximation scheme for the traveling salesperson problem in a planar graph. Its first step is to find a low-weight spanner of the graph. A subsequent paper, also by Klein [15] describes an algorithm that, given a planar graph G and a subset S of the nodes, finds a subgraph of G whose weight is $O(\epsilon^{-4})$ times that of the minimum-weight tree spanning S and that is a $(1+\epsilon)$-spanner for the shortest-path metric on S [15]. This subset spanner construction can be substituted for the first step of Klein's approximation scheme, resulting in an algorithm for approximating the TSP on the subset S. However, in this more general result, the spanner construction takes time $O(n \log n)$, so the total time for the approximation scheme is $O(n \log n)$ for any fixed $\epsilon > 0$.

For points in the plane, we may substitute our own faster low-weight spanner construction for the first step of the approximation scheme. The remaining steps of the approximation use only the facts that the points we are seeking to connect into a tour are vertices in a planar graph, and that the whole graph has total weight proportional to the minimum spanning tree of the given points. Thus, we obtain the following result:

Theorem 9 *For any fixed α and ϵ, the optimal traveling salesman tour of sets of n points in the plane with sharpest Delaunay triangulation angle at most α may be approximated to within a $1 + \epsilon$ factor in time $O(n)$ plus the time needed to construct the Delaunay triangulation.*

It would also be possible to design a TSP approximation scheme more directly using a framework used by Borradaile, Klein and Mathieu [8] to solve the Steiner tree problem; details on how this framework applies to TSP were given by Borradaile, Demaine and Tazari [7] in generalizing the planar framework to bounded-genus graphs. Their algorithm, as interpreted for point sets in the Euclidean plane, would partition the triangles of the Delaunay triangulation into layers according to their depth from the infinite face in the dual graph so that the boundary between layers is an ϵ fraction of the optimal solution. This can be achieved with depth $f_w(\alpha)/\epsilon = O\left(\frac{1}{\alpha^2\epsilon}\right)$; each layer has tree-width polynomial in this depth. The problem is then solved using dynamic programming, where the dynamic programs are additionally indexed by the portals. The base cases

are made to correspond to the triangles in which the intersection with any tour can be enumerated. The size of the dynamic program is therefore bounded singly-exponentially in $1/\epsilon$ and $1/\alpha$. This task is slightly easier in the geometric setting than in the planar graph setting as computing shortest paths is trivial.

7 Looking ahead

The most obvious question posed by this work is: how do we remove the dependence on α? The dependence on α appears in two places: in the number of circumcircles that enclose a point and in the weight of the Delaunay triangulation. We believe that it should be possible to remove these dependencies on α by treating groups of skinny triangles as a single region. In fact, using this idea, we are able to remove each dependency separately, but not together.

An alternative approach to removing this dependence would be to augment the input to remove all sharp angles from its Delaunay triangulation, but this may sometimes need a number of added points that cannot be bounded by a function of n [5]. A construction based on quadtrees shows that every point set may be augmented with $O(n)$ points so that the Delaunay triangulation has no obtuse angles [5]; the resulting triangulation may also be modified to have the bounded circumcircle enclosure property, despite having some sharp angles, and may be constructed as efficiently as sorting [6]. Applying our spanner construction method to the augmented input would allow us to completely eliminate the dependence on α in the time and output complexity of our spanners, but at the expense of losing control over their total weight. Once a spanner is constructed in this way, Klein's subset spanner [15] can be used to reduce its weight, allowing it to be used in an algorithm to approximate the TSP for arbitrary planar point sets in time $O(n \log n)$ for any fixed $\epsilon > 0$, but this does not improve on the time bound of Rao and Smith [19].

Unlike in the methods of Arora [3], Mitchell [17], Rao and Smith [19] and Borradaile, Klein and Mathieu [8], the approximation error in our method is charged locally as opposed to globally. In the quad-tree based approximation schemes, the error incurred is charged to the dissection lines that form the quad tree. In the planar approximation-scheme framework for Steiner tree, the error incurred is charged to an $O(MST)$-weight subgraph called the mortar graph which acts much like the quad-tree decomposition. Our charging scheme is much more similar to that used by Klein for the subset tour problem in planar graphs [15]. However, in applying the planar approximation-scheme frameworks of either Klein or Borradaile, Klein and Mathieu, some error is incurred in partitioning the graph into pieces of bounded treewidth. This error is proportional to the

graph that is partitioned, which in our case is either the spanner (for Klein's scheme) or the triangulations (for Borradaile et. al.'s scheme). This error is indirectly related to OPT by way of the $O(\text{MST})$ weight of the spanner and triangulation. By current techniques, this source of error does not seem avoidable.

Finally, our spanner construction more closely ties Euclidean and planar distance metrics together. By unifying the approximation schemes in these two related metrics, it may be possible to generalize these methods to other two dimensional metrics.

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