A criterion for some Hamiltonian graphs to be Hamilton-connected

A.S. Asratian
University of Twente
Department of Applied Mathematics
P.O. Box 217, 7500 AE Enschede
The Netherlands

Abstract. Let $\mathcal{M} = \{G / K_{n,n} \subseteq G \subseteq K_n \vee \overline{K_n}$ for some $n \geq 3\}$ where $\vee$ is the join operation. The author and N.K. Khachatrian proved that a connected graph G of order at least 3 is Hamiltonian if

$$d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)|$$

for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$ (where $N(x)$ is the neighborhood of x).

Here we prove that a graph G satisfying the above conditions is Hamilton-connected if and only if G is 3-connected and $G \notin \mathcal{M}$.

1 Introduction

We use Bondy and Murty [4] for terminology and notation not defined here and consider finite simple graphs only.

For each vertex u of a graph G we denote by $N(u)$ the set of all vertices of G adjacent to u.

Let P be a path of G. We denote by $\vec{P}$ the path P with a given orientation, and by $\overleftarrow{P}$ the path P with the reverse orientation. If $u, v \in V(P)$, then $u\vec{P}v$ denotes the consecutive vertices of P from u to v in the direction specified by $\vec{P}$. The same vertices, in reverse order, are given by $v\overleftarrow{P}u$. We use u^+ to denote the successor of u on $\vec{P}$ and u^- to denote its predecessor.

A path with x and y as end-vertices is called an $x-y$ path. An $x-y$ path is called a Hamilton path if it contains all the vertices of G. A graph G is Hamilton-connected if every two vertices of G are connected by a Hamilton path.

Let A and B be two disjoint sets of vertices of a graph G. We denote by $e(A, B)$ the number of edges in G with one end in A and the other in B.

A graph G of order $p \geq 3$ is called pancyclic if G contains a cycle of length l for each l satisfying $3 \leq l \leq p$.

Australasian Journal of Combinatorics 10(1994), pp.193-198

The author[1] and N.K. Khachatrian proved in [5] the following.

Theorem 1([5]). Let G be a connected graph of order at least 3 where

$$d(u) + d(v) \geq \mid N(u) \cup N(v) \cup N(w) \mid$$

for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$. Then G is Hamiltonian.

Clearly, Theorem 1 implies Ore's theorem [6]. A simpler proof of Theorem 1 was suggested in [2]. Other related results were obtained in [1] and [3].

Theorem 2([3]). Let a graph G satisfy the conditions of Theorem 1. Then either G is pancyclic or $\mid V(G) \mid = 2n$ and $G = K_{n,n}$ for some $n \geq 3$.

Theorem 3([1]). Let G be a connected graph of order at least 3 where

$$d(u) + d(v) \geq \mid N(u) \cup N(v) \cup N(w) \mid +1$$

for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$. Then G is Hamilton-connected.

Denote by L_0 the set of all graphs satisfying the conditions of Theorem 1. Let

$$\mathcal{M} = \{G / K_{n,n} \subseteq G \subseteq K_n \vee \overline{K}_n \text{ for some } n \geq 3\}$$

where $\vee$ is the join operation.

We prove here the following theorem.

Theorem 4. A graph G from the set L_0 is Hamilton-connected if and only if G is 3-connected and $G \notin \mathcal{M}$.

We use arguments similar to those in [2].

2 Results

Lemma 1([5]). If $G \in L_0$ then

$$\mid N(u) \cap N(v) \mid \geq \mid N(w) \setminus (N(u) \cup N(v)) \mid$$

for each triple of vertices u, v, w with $d(u, v) = 2$ and $w \in N(u) \cap N(v)$.

Corollary 1. If $G \in L_0$ then $\mid N(u) \cap N(v) \mid \geq 2$ for each pair of vertices u, v with $d(u, v) = 2$.

Proof. Let $w \in N(u) \cap N(v)$. Then $u, v \in N(w) \setminus (N(u) \cup N(v))$. Hence, by Lemma 1, $\mid N(u) \cap N(v) \mid \geq \mid N(w) \setminus (N(u) \cup N(v)) \mid \geq 2$.

Lemma 2. Let $G \in L_0$ and x, y be two distinct vertices of G. Furthermore, let P be an $x - y$ path and $v \in V(G) \setminus V(P), N(v) \cap V(P) \neq \emptyset$. If $vx \notin E(G)$ or $vy \notin E(G)$ then there exists an $x - y$ path longer than P.

Proof. Without loss of generality we suppose $vy \notin E(G)$. Let $\vec{P}$ be the path P with orientation from x to y and let $w_1, \cdots, w_n$ denote the vertices of $W = N(v) \cap V(P)$ occurring on P in the order of their indices.

[1]In [5] the last name of the present author was transcribed as Hasratian.

Case 1. $n = 1$. Then $d(v, w_1^+) = 2$ and, by Corollary 1, there is a vertex $z \in (N(v) \cap N(w_1^+)) \setminus V(P)$. The $x - y$ path $x\vec{P}w_1vzw_1^+\vec{P}y$ is longer than P.

Case 2. $n \geq 2$. Clearly, if v is adjacent to two consecutive vertices of P or $w_i^+w_j^+ \in E(G)$ for some pair $i, j, 1 \leq i < j \leq n$, then there is a longer $x - y$ path.

Now suppose:

a) v is not adjacent to two consecutive vertices of P,

b) $w_i^+w_j^+ \notin E(G)$ for $1 \leq i < j \leq n$, that is: the set $W^+ = \{w_1^+, \cdots, w_n^+\}$ is independent.

Since $d(v, w_i^+) = 2$ for each $i = 1, \cdots, n$ then, by Lemma 1, we have

$$(1) \qquad \sum_{i=1}^{n} | N(v) \cap N(w_i^+) | \geq \sum_{i=1}^{n} | N(w_i) \setminus (N(v) \cup N(w_i^+)) | .$$

If $N(v) \cap N(w_i^+) \subseteq V(P)$ for each $i = 1, \cdots, n$ then

$$(2) \qquad \sum_{i=1}^{n} | N(v) \cap N(w_i^+) | \leq e(W, W^+)$$

and

$$(3) \qquad \sum_{i=1}^{n} | N(w_i) \setminus (N(v) \cup N(w_i^+)) | \geq e(W, W^+) + n$$

because $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $i = 1, \cdots, n$.

But (2) and (3) contradict (1). Hence $(N(v) \cap N(w_i^+)) \setminus V(P) \neq \emptyset$ for some i. Let

$$z \in (N(v) \cap N(w_i^+)) \setminus V(P).$$

Then the $x - y$ path $x\vec{P}w_ivzw_i^+\vec{P}y$ is longer than P. □

Proof of Theorem 4.

Clearly, if a graph G is Hamilton-connected then G is 3-connected and $G \notin \mathcal{M}$. Now suppose that G is a 3-connected graph from the set L_0. Let x and y be two distinct vertices of G.

Consider a longest $x - y$ path $\vec{P}$ with orientation from x to y. Suppose P is not a Hamilton path. Since G is 3-connected, there exists a vertex v outside P such that

$$(N(v) \cap V(P)) \setminus \{x, y\} \neq \emptyset.$$

Let $w_1, \cdots, w_n$ denote the vertices of $W = N(v) \cap V(P)$ occurring on P in the order of their indices. Since P is a longest $x - y$ path then, by Lemma 2, $w_1 = x$ and $w_n = y$, that is $n \geq 3$. Moreover, $w_i^+ \neq w_{i+1}$ for each $i = 1, \cdots, n-1$. Set $W_1 = \{w_1, \cdots, w_{n-1}\}$ and $W_2 = \{w_2, \cdots, w_n\}$. Using similar arguments as in the proof of Lemma 2, we can show the following:

(4) the sets $W_1^+ = \{w_1^+, \cdots, w_{n-1}^+\}$ and $W_2^- = \{w_2^-, \cdots, w_n^-\}$ are independent.

(5) $N(v) \cap N(w_i^+) \subseteq V(P)$ for each $i = 1, \cdots, n-1$.

(6) $N(v) \cap N(w_j^-) \subseteq V(P)$ for each $j = 2, \cdots, n$.

(7) $\sum_{i=1}^{n-1} \mid N(v) \cap N(w_i^+) \mid \geq \sum_{i=1}^{n-1} \mid N(w_i) \setminus (N(v) \cup N(w_i^+)) \mid$.

(8) $\sum_{j=2}^{n} \mid N(v) \cap N(w_j^-) \mid \geq \sum_{j=2}^{n} \mid N(w_j) \setminus (N(v) \cup N(w_j^-)) \mid$.

Furthermore, from (4) and (5) we have

(9) $\sum_{i=1}^{n-1} \mid N(v) \cap N(w_i^+) \mid \leq e(W_1^+, W_1) + n - 1$.

Now let us prove that $w_i^+ = w_{i+1}^-$ for each $i = 1, \cdots, n-1$. First, note that

(10) if $w_i^+ \neq w_{i+1}^-$ then $w_{i+1}^- w_{i+1}^+ \in E(G), 1 \leq i \leq n-2$.

Assuming $w_{i_0}^+ \neq w_{1+i_0}^-$ and $w_{1+i_0}^- w_{1+i_0}^+ \notin E(G)$ for some $i_0, 1 \leq i_0 \leq n-2$, we obtain

(11) $\sum_{i=1}^{n-1} \mid N(w_i) \setminus (N(v) \cup N(w_i^+)) \mid \geq e(W_1^+, W_1) + n$

because $v, w_{1+i_0}^- \notin W_1^+, w_{1+i_0}^- \in N(w_{1+i_0}) \setminus (N(v) \cup N(w_{1+i_0}^+))$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $i = 1, \cdots, n-1$.

But (9) and (11) contradict (7). So, (10) is proved.

Case 1. $w_1^+ \neq w_2^-$.

Then, by (10), $w_2^- w_2^+ \in E(G)$. Since, by (4), $w_2^- w_3^- \notin E(G)$ then $w_2^+ \neq w_3^-$. Hence, by (10), $w_3^- w_3^+ \in E(G)$.

Repetition of this argument shows that $w_i^+ \neq w_{i+1}^-$ and $w_i^- w_i^+ \in E(G)$ for each $i = 2, \cdots, n-1$.

Consider the set $D_1 = N(v) \cap N(w_1^+)$. Since $d(v, w_1^+) = 2$ then, by Corollary 1, $\mid D_1 \mid \geq 2$. If $w_i \in D_1$ for some $i, 2 \leq i \leq n-1$, then the $x-y$ path $w_1 v w_i w_1^+ \vec{P} w_i^- w_i^+ \vec{P} w_n$ is longer than P. Hence $w_i \notin D_1$ for each $i = 2, \cdots, n-1$. Since, by (5), $D_1 \subseteq V(P)$ then $D_1 = \{w_1, w_n\}$.

By similar reasoning we have for the set $D_2 = N(v) \cap N(w_2^+) : D_2 \subseteq V(P), \mid D_2 \mid \geq 2$ and if $n \geq 4$, then $w_i \notin D_2$ for each $i = 3, \cdots, n-1$.

Subcase 1.1. $w_1 \in D_2$. It means $v, w_1^+, w_2^+ \in N(w_1) \setminus (N(v) \cup N(w_1^+))$. Since $d(v, w_1^+) = 2$ then, using Lemma 1, we have

$$2 = \mid N(v) \cap N(w_1^+) \mid \geq \mid N(w_1) \setminus (N(v) \cup N(w_1^+)) \mid \geq 3,$$

a contradiction.

Subcase 1.2. $w_1 \notin D_2$. Then $D_2 = \{w_2, w_n\}$ and $v, w_1^+, w_2^+ \in N(w_n) \setminus (N(v) \cup N(w_2^+))$.

Using Lemma 1 we have

$$2 = \mid N(v) \cap N(w_2^+) \mid \geq \mid N(w_n) \setminus (N(v) \cup N(w_2^+)) \mid \geq 3,$$

a contradiction.

Case 2. $w_i^+ = w_{i+1}^-$ for each $i, 1 \leq i \leq t-1 < n-1$, but $w_t^+ \neq w_{t+1}^-$.

Then, by (4) and (6), we have

(12) $\sum_{j=2}^{n} | N(v) \cap N(w_j^-) | \leq e(W_2^-, W_2) + n - 1.$

If $w_t^- w_t^+ \notin E(G)$ then

(13) $\sum_{j=2}^{n} | N(w_j) \setminus (N(v) \cup N(w_j^-)) | \geq e(W_2^-, W_2) + n$

because $w_t^+ \in N(w_t) \setminus (N(v) \cup N(w_t^-))$ and

$v \in N(w_j) \setminus (N(v) \cup N(w_j^-))$ for each $j = 2, \cdots, n$. But (12) and (13) contradict (8).

Hence $w_t^- w_t^+ \in E(G)$. But then the $x - y$ path $w_1 \vec{P} w_{t-1} v w_t w_t^- w_t^+ \vec{P} w_n$ is longer than P, a contradiction.

So $w_i^+ = w_{i+1}^-$ for each $i = 1, \cdots, n-1$. Clearly, the path $P_i = w_1 \vec{P} w_i v w_{i+1} \vec{P} w_n$ is a longest $x - y$ path for each $i = 1, \cdots, n-1$. Repeating the arguments above with P_i and w_i^+ instead of P and v we obtain $w_i^+ w_j \in E(G)$ for each pair $i, j, 1 \leq i \leq n-1, 1 \leq j \leq n$. Hence, by (5), $| N(v) \cap N(w_i^+) | = n$ for each $i = 1, \cdots, n-1$. Since

$$v, w_1^+, \cdots, w_{n-1}^+ \in N(x) \setminus (N(v) \cup N(w_1^+))$$

then, using Lemma 1, we obtain

$$n = | N(v) \cap N(w_1^+) | \geq | N(x) \setminus (N(v) \cup N(w_1^+)) | \geq n.$$

Therefore

(14) $$N(x) \setminus (N(v) \cup N(w_1^+)) = \{v, w_1^+, \cdots, w_{n-1}^+\}.$$

Let us prove that the set $V_0 = V(G) \setminus (V(P) \cup \{v\})$ is empty. Suppose $V_0 \neq \emptyset$. Since G is connected then there exists a vertex $z \in V_0$ with $N(z) \cap V(P) \neq \emptyset$. Then, by Lemma 2, z is adjacent to x. By (14), $z \notin N(x) \setminus (N(v) \cup N(w_1^+))$. Furthermore, $zw_1^+ \notin E(G)$ because P is a longest $x - y$ path. Hence $zv \in E(G)$. But then the $x - y$ path $xzvw_2 \vec{P} w_n$ is longer than P, a contradiction.Therefore $V_0 = \emptyset, V(G) = V(P) \cup \{v\}$ and $G \in \mathcal{M}$. □

A graph G of order at least 3 is called an Ore graph if $d(u) + d(v) \geq | V(G) |$ for each pair of nonadjacent vertices u, v of G.

Corollary 2. An Ore graph G is Hamilton-connected if and only if G is 3-connected and $G \notin \mathcal{M}$.

Finally note the following. If G is a graph satisfying the conditons of Theorem 3 then $G \notin \mathcal{M}$. Moreover, $| N(u) \cap N(v) | \geq 3$ for each pair of vertices u, v of G with $d(u, v) = 2$. (It is possible to prove this using the same argument as in the proof of Corollary 1). We deduce that G is 3-connected. Therefore Theorem 3 is a corollary of Theorem 4. From Theorem 3 we have the following.

Corollary 3. ([7]). A graph G of order at least 3 is Hamilton-connected if $d(u) + d(v) \geq |V(G)| + 1$ for each pair of nonadjacent vertices u, v of G.

Acknowledgements

I thank H.J. Veldman and H.J. Broersma for their helpful comments. Also I thank all members of the discrete mathematics group of the Department of Applied Mathematics for their kind hospitality during my visit.

References

[1] A.S. Asratian, O.A. Ambartsumian and G.V. Sarkisian, Some local conditions for the hamiltonicity and pancyclicity of a graph (Russian), Doclady Academ. Nawk Armenian SSR, v. 91, No. 1, 1990, 19-22.

[2] A.S. Asratian, H.J. Broersma, J. van den Heuvel and H.J. Veldman, On graphs satisfying a local Ore-type condition. Preprint of Twente University,1993.

[3] A.S. Asratian and G.V. Sarkisian, On cyclic properties of some hamiltonian graphs (Russian), Discretnaja Mathematica, v. 3, No. 4, 1991, 91-104.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[5] A.S. Hasratian and N.K. Khachatrian, Some localization theorems on hamiltonian circuits, J. of Combin. Theory B49, No. 2, 1990, 287-294.

[6] O. Ore, Note on hamiltonian circuits, Amer. Math. Monthly, 67, 1969, 55.

[7] O. Ore, Hamilton-connected graphs, J. Math. Pures Appl. (9) 42 (1963) 21-27.

(Received 22/2/94)