Directed packings with block size 5

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Abstract Let $v \geq 5$ and λ be positive integers and let $DD(v,k,\lambda)$ denote the packing number of a directed packing with block size 5 and index λ. The values of $DD(v, 5, \lambda)$ have been determined for $\lambda = 1,2$ with the possible exceptions of $(v, \lambda) = (15,1)$ $(19,1)$ $(27,1)$, [7, 8, 19]. In this paper we determine the values of $DD(v, 5, \lambda)$ for all $v \geq 5$ and $\lambda \geq 3$ except possibly $(v, \lambda) = (43,3)$.

1. Introduction

Let v, k and λ be positive integers. A transitively ordered k-tuple $(a_1, \dots, a_k)$ is defined to be the set $\{(a_i, a_j) \mid 1 \leq i < j \leq k\}$ consisting of $\frac{k(k-1)}{2}$ ordered pairs. A directed packing (covering), denoted by $DP(v, k, \lambda)$, $(DC(v, k, \lambda))$ is a pair (V,A) where V is a set of v elements and A is a collection of transitively ordered k-tuples (called blocks) of V such that every ordered pair of V occurs in at most (at least) λ blocks. Let $DD(v, k, \lambda)$ denote the maximum number of blocks in a DP (v, k, λ) and $DE(v, k, \lambda)$ denote the minimum number of blocks in a DC (v, k, λ). A $DP(v, k, \lambda)$ directed packing design with $|A| = DD(v, k, \lambda)$ will be called a maximum packing. Similarly, a $DC(v, k, \lambda)$ directed covering with $|A| = DE(v, k, \lambda)$ is called a minimum covering. If one ignores the order of the blocks, a $DP(v, k, \lambda)$ $(DC(v, k, \lambda))$ is a standard $(v, k, 2\lambda)$ packing (covering).

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So we have the following:

$DD(v, k, \lambda) \le \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} 2\lambda \right\rfloor \right\rfloor = DU(v, k, \lambda)$ and $DE(v, k, \lambda) \ge \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} 2\lambda \right\rceil \right\rceil = DL(v, k, \lambda)$.

where $\lfloor x \rfloor$ is the largest and $\lceil x \rceil$ is the smallest integer satisfying $\lfloor x \rfloor < x \le \lceil x \rceil$. When $DD(v, k, \lambda) = DU(v, k, \lambda)$ the directed packing is called optimal and denoted by $ODP(v, k, \lambda)$. Similarly, when $DE(v, k, \lambda) = DL(v, k, \lambda)$ the directed covering is called minimal and denoted by $MDC(v, k, \lambda)$. Further as a consequence of Hanani's result [15, P.362], this bound can be sharpened in certain cases.

<u>Theorem 1.1</u> If $2\lambda(v - 1) \equiv 0 \bmod(k - 1)$ and $2\lambda v \frac{(v-1)}{(k-1)} \equiv 1 (\bmod\ k)$ then $DD(v, k, \lambda) \le DU(v, k, \lambda) - 1$.

A directed packing with a hole of size h, $DP(v, k, \lambda)$ is a triple (V, H, A) where V is a v-set, H is a subset of V of cardinality h, and A is a collection of transitively ordered k-tuples, called blocks, of V such that
1) no 2-subset of H appears in any block;
2) every 2-subset of V appears in at most λ blocks.

Further, a $DP(v, k, \lambda)$ with a hole of size h is said to be maximum if it contains $DU(v, k, \lambda) - DU(h, k, \lambda)$ blocks.

A directed balanced incomplete block design, $DB[v, k, \lambda]$ is a $DP(v, k, \lambda)$, where every ordered pair of V is contained in exactly λ blocks. If a $DB[v, k, \lambda]$ exists then it is clear that $DD(v, k, \lambda) = 2\lambda \frac{v(v-1)}{k(k-1)} = DU(v,k,\lambda)$ and D.J. Street and W.H. Wilson [20] have shown the following:

<u>Theorem 1.2</u> Let λ and $v \ge 5$ be positive integers. Necessary and sufficient conditions for the existence of a $DB[v, 5, \lambda]$ are that $(v,\lambda) \ne (15, 1)$ and that $\lambda(v - 1) \equiv 0 (\bmod\ 2)$ and $\lambda v(v - 1) \equiv 0 (\bmod\ 10)$.

The following result was established in [7,8,9].

<u>Theorem 1.3</u> Let $v \ge 5$ be an integer. Then $DD(v, 5, \lambda) = DU(v,5,\lambda) - e$ for $\lambda = 1,2$ $(v, \lambda) \ne (15, 1)$ where $e = 1$ if $2\lambda(v - 1) \equiv 0 (\bmod\ 4)$ and $\frac{\lambda v(v-1)}{2} \equiv 1 (\bmod\ 5)$ and $e = 0$ otherwise with the possible exception of $(v,\lambda) = (19, 1)$ $(27, 1)$.

<u>Corollary 1.1</u> Let $v \equiv 0$ or $1 (\bmod\ 5)$, $v \ge 5$, be an integer. Then

DU(v, 5, λ) = DD(v, 5, λ) for all integers $\lambda \geq 1$ with the exception of (v, λ) = (15,1).

Proof For λ = 1 or 2 the result was established in Theorem 1.3. For λ = 2r, where r is a positive integer, notice that there exists a DB[v, 5, 2r]. For λ = 2r + 1, $v \neq 15$ it is easy to see that DU(v,5, 2r + 1) = DU(v,5,2r) + DU(v,5,1). For v=15 notice that a DB[15,5,3] exists by Theorem 1.2 and then DU(v,5,2r +1) = DU(v,5,2r-2) + DU(v,5,3).

In this paper we are interested in determining the values of DD(v, 5, λ) for $\lambda > 2$. Our goal is to prove the following.

Theorem 1.4 Let $v \geq 5$ and $\lambda > 2$ be integers. Then DD(v, 5, λ) = DU(v, 5, λ) − e where e = 1 if $2\lambda(v-1) \equiv 0$ and e = 0 otherwise, with the possible exception of (v,λ) = (43,3).

In what follows, we will use the following obvious fact. For brevity, we will not mention it subsequently.

Lemma 1.1 If there exists a DB[v,5,λ] and DU(v,5,λ') = DD(v, 5,λ') then DU(v,5,λ' + λ) = DD(v, 5, λ' + λ).

Finally, about the notation of <a b c d> ∪ $\{h_1, h_2\}$ we refer the reader to [9], and a block <a b − c d> ∪ $\{h_1, h_2\}$ means that h_i, i = 1, 2 is to be inserted in the middle.

2. The Structure of Packing and Covering Designs

Let (V, A) be a (v, k, λ) packing design, and for each 2-subset e = {x, y} of V define m(e) to be the number of blocks in A which contains e. The complement of (V, A) denoted by C(V, A) is defined to be the multigraph spanned by the edges not packed in (V, A). It is clear that the number of edges in C(V, A) is $\lambda\binom{v}{2} - |A|\binom{k}{2}$. The degree of a vertex x in C(V, A) is $\lambda(v - 1) - r_x(k - 1)$ where r_x is the number of blocks through x. In a similar way, one can define the excess graph, E(V, A), of a (v, k, λ) covering design to be the multigraph spanned by the edges covered more than λ times in (V, A). The number of edges in E(V, A) is $|A|\binom{k}{2} - \lambda\binom{v}{2}$ and the degree of each vertex x in C(V, A) is $r_x(k - 1) - \lambda(v - 1)$ where r_x is as above.

Lemma 2.1 [10] Let $v \equiv 2$ or 4 (mod 5), $v \geq 9$, be a positive integer. Then the complement graph of a (v, 5, 4) optimal packing design consists of two vertices joined by 4 edges.

Lemma 2.2 [16] Let $v \equiv 3 \pmod{10}$ $v \geq 23$, $v \neq 53, 63, 73, 83$. Then the complement graph of a (v, 5, 2) minimal covering design consists of two vertices joined by 4 edges.

3. Recursive Constructions

In order to describe our recursive constructions we need the notions of transversal designs and group divisible designs. For the definition of these combinatorial designs see [15]. We shall use the following notations: a $T[k,\lambda,m]$ stands for a transversal design with block size k, index λ and group size m. A (K,λ) – GDD of type $1^i, 2^r, 3^s \ldots$ stands for a group divisible design with block size from K, index λ, and there are i groups of order 1, r groups of order 2, s groups of order 3, etc. We remark that the notions of transversal designs and group divisible design can be easily extended to the directed case, and we write DT and DGDD with the appropriate parameters.

The following theorem is most useful to us. For a proof see [3] and references therein.

Theorem 3.1 There exists a T[6, 1, m] for all positive integers $m \neq 2, 3, 4, 6$ with the possible exceptions of $m \in \{10, 14, 18, 22\}$.

Let k, λ, m and v be positive integers. A modified group divisible design, $MGD[k, \lambda, m, mn]$ is a quadruple $(v, \beta, \gamma, \Delta)$ where V is a set of points with $|V| = mn$, $\gamma = \{G_1, \ldots, G_m\}$ is a partition of V into m sets, called groups, $\Delta = \{R_1, \ldots, R_n\}$ is a partition of V into n sets, called rows, and β is a family of k-subsets of V, called blocks, with the following properties.

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_j \in \gamma$.

2) $|B \cap R_j| \leq 1$ for all $B \in \beta$ and $R_j \in \Delta$.

3) $|R_j| = m$ for all $R_j \in \Delta$ and $|G_i| = n$ for all $G_j \in \gamma$.

4) Every 2-subset {x, y} of V such that x and y are neither in the same group nor same row is contained in exactly λ blocks.

5) $|G_i \cap R_j| = 1$ for all $G_j \in \gamma$ and $R_j \in \Delta$.

A resolvable modified group divisible design, $RMGD[k, \lambda, m, v]$ is a modified group divisible design the blocks of which can be partitioned into parallel classes. It is clear that RMGD[5, 1, 5, 5m] is the same as RT[5, 1, m] with one parallel class of blocks singled out, and since a RT[5, 1, m] is equivalent to T[6,1,m] we have the following.

Theorem 3.3 There exists a RMGD[5, 1, 5, 5m] for all $m \neq 2, 3, 4, 6$ with the possible exceptions of $m \in \{10, 14, 18, 22\}$.

The following is our main recursive construction [6].

Theorem 3.4 If there exists a RMGD[5, 1, 5, 5m] and a $(5, \lambda)$ – DGDD of type $4^m s^1$ and there exists a maximum DP$(20 + h, 5, \lambda)$ with a hole of size h then there exists a maximum DP$(20m + 4u + h + s, 5, \lambda)$ with a hole of size $4u + h + s$ where $0 \leq u \leq m - 1$.

In a similar way one can show

Theorem 3.5 If there exists a RMGD[5, 1, 5, 5m], a $(5, \lambda)$ – DGDD of type 2^m or 2^{m+1} and there exists a maximum DP$(10 + h, 5, \lambda)$ with a hole of size h then there exists a maximum DP$(10m + 2u + h + 2e, 5, \lambda)$ with a hole of size $2u + h + 2e$ where $e = 0$ if the DGDD is of type 2^m and $e = 1$ if the DGDD is of type 2^{m+1} and $0 \leq u \leq m - 1$.

The application of theorem 3.4 requires a $(5, \lambda)$ – DGDD of type $4^m s^1$. Notice that we may choose $s = 0$ if $m \equiv 1 \pmod 5$, $s = 4$ if $m \equiv 0$ or 4 (mod 5) and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$. Further, we may apply the following [14]

Theorem 3.6 There exists a $(5, 1)$ – DGDD of type $4^m 8^1$ for all $m \equiv 0$ or 2(mod 5), $m \geq 7$ with the possible exception of $m = 10$.

The following two theorems are the directed versions of theorem 2.11 and theorem 2.18 of [6].

Theorem 3.7 If there exist a RMGD[5, 1, 5, 5m]; a $(5, \lambda)$ – DGDD of type 4^m; a maximum DP$(20 + h, 5, \lambda)$ with a hole of size h and an ODP$(20 + h, 5, \lambda)$ then there exists and an ODP$(20m + h, 5, \lambda)$.

Theorem 3.8 If there exists a $(k, 1)$ – DGDD of type 5^m; a $(5, \lambda)$ – GDD of type 4^k and a maximum DP$(20 + h, 5, \lambda)$ with a hole of size h and an ODP$(20 +h, 5, \lambda)$ then there exists an ODP$(20m + h, 5, \lambda)$.

Again the following theorem is the directed version of theorem 2.4 of [6]

Theorem 3.9 If there exists a $(6, 1)$ – DGDD of type 5^m and a maximum DP$(20 + h, 5, \lambda)$ with a hole of size h then there exists a maximum DP$(20(m - 1) + 4u + h, 5, \lambda)$ with a hole of size $4u + h$.

Lemma 3.1 There exists a (6, 1) – DGDD of type 5^7.

Proof Let $X = Z_{35}$ with groups {i, i + 7, i + 14, i + 21, i + 28}, $i \in Z_7$. Then the blocks are <1 5 17 0 2 11> (mod 35) <22 10 13 18 2 0> (mod 35).

4. Directed Packing With Index 3

We first mention that the result for v ≡ 0 or 1 (mod 5) was established in corollary 1.1. For all other values of v we proceed as follows.

Lemma 4.1 Let v ≡ 2(mod 10), v ≥ 12, be an integer. Then DU(v, 5, 3) = DD(v, 5, 3).

Proof DU(v, 5, 3) = DU(v, 5, 2) + DU(v, 5, 1).

Lemma 4.2 Let v ≡ 4(mod 10) be a positive integer. If there exists a maximum DP(v, 5, 1) with a hole of size 4 and a MDC(v, 5, 2) then DU(v, 5, 3) = DD(v, 5, 3).

Proof For all v ≡ 4(mod 10) v ≥ 14 the construction is as follows:

1) Take a maximum DP(v, 5, 1) with a hole of size 4, say, {a, b, c, d}.

2) Take a MDC(v, 5, 2). This design has a triple say {a, b, c} the ordered pairs of which appear in three blocks [5].

Lemma 4.3 (i) There exists a maximum DP(26, 5, 3) with a hole of size 6.
(ii) There exists a maximum DP(v, 5, 1) with a hole of size 4 for v = 34, 54, 74, 94.

Proof (i) for a maximum DP(26, 5, 3) with a hole of size 6 take the blocks of maximum DP(26, 5, 1) with a hole of size 6 [19] together with the blocks of maximum DP(26, 5, 2) with a hole of size 6 which can be constructed by taking a DT[5, 2, 5]. Add a point to the groups and on the first four groups construct a DB(6, 5, 2) [20] and then take this point with the last group to be the hole.

(ii) For v = 34 let $X = Z_2 \times Z_{15} \cup \{\infty_i\}_{i=1}^{4}$. Then the blocks are

< (0,1) (1,11) (1,2) (1,7) (0,0) > mod(-, 15)
< (1 ,1) (0, 0) (1, 2) (1, 0) (1, 4) > mod (-, 15)
< (1, 12) (0, 2) ∞_1 (0, 0) (1, 9) > mod (-, 15)
< (1, 14) (0, 0) ∞_2 (0, 5) (1, 8) > mod (-, 15)
< (0, 7) (1, 5) ∞_3 (0, 0) (1, 12) > mod (-, 15)

< (1, 6) (0, 3) (0, 0) ∞_4 (1, 14) > < (1, 7) (0, 4) (0, 1) ∞_4 (1, 0) >
< (1, 8) (0, 5) (0, 2) ∞_4 (1, 1) > < (1, 9) (0, 6) ∞_4 (0, 3) (1, 2) >
< (1, 10) (0, 7) ∞_4 (0, 4) (1, 3) > < (1, 11) (0, 8) ∞_4 (0, 5) (1, 4)>
< (1, 12) (0, 9) ∞_4 (0, 6) (1, 5) > < (1, 13) (0, 10) ∞_4 (0, 7) (1, 6) >
< (1, 14) (0, 11) ∞_4 (0, 8) (1, 7) > <(1, 0) ∞_4 (0, 12) (0, 9) (1, 8) >
< (1, 1) ∞_4 (0, 13) (0, 10) (1, 9) > < (1, 2) ∞_4 (0, 14) (0, 11) (1, 10) >
< (1, 3) (0, 12) ∞_4 (0, 0) (1, 11) > < (1, 4) (0, 13) ∞_4 (0, 1) (1, 12) >
< (1, 5) (0, 14) ∞_4 (0, 2) (1, 13) >
< (1, 4 + j) (0, 4 + j) (0, 11 + j) (0, 6 + j) (0, j) > $j \in Z_9$,
< (1, 4 + t) (0, 4 + t) (0, 11 + t) (0, t) (0, 6 + t) > t = 9, 10, ... , 14
< (0, k) (0, 3 + k) (0, 6 + k) (0, 9 + k) (0, 12 + k) > k = 0, 1, 2.

For v = 54 apply Theorem 3.5 with m = 5 , h = 2, e = 0, u = 1 and $\lambda = 3$ and see [20] for a (5, 3) – DGDD of type 2^5 and 2^6 and for a maximum DP(12, 5, 3) with a hole of size 2 take a maximum DP(12, 5, 2) and a maximum DP(12, 5, 1) with a hole of size 2 [8, 19].

For v = 74 take a T[5, 1, 7] and inflate the design by a factor of 2, that is, replace each quintuple by the blocks of a (5, 1) – DGDD of type 2^5 [20]. To the groups add 4 new points and construct a maximum DP(18, 5, 1) with a hole of size 4 [19].

For v = 94 take a ({5, 6}, 1) – GDD of type $9^5 1^1$ and inflate the design by a factor of 2, that is, replace each block by the blocks of a (5, 1) – DGDD of type 2^5 and 2^6. To the groups add two points and on the first five groups construct a DP(20, 5, 1) with a hole of size 2 [19] and take these two points with the last group to be the hole of order 4.

Corollary DU(v, 5, 3) = DD(v, 5, 3) for v = 34, 54, 74, 94.

Proof By the previous lemma there exists a DP(v, 5, 1) with a hole of size 4. Furthermore there exists a DC(v, 5, 2) for the stated values of v in lemma [5] such that there is a triple the ordered pairs of which appear in three blocks. Hence DU(v, 5, 3) = DD(v, 5, 3) by lemma 4.2.

Lemma 4.4 Let $v \equiv 14 (\mathrm{mod}\ 20)$ be a positive integer then DU(v, 5, 3) = DD(v, 5, 3).

Proof For v = 14 the construction is as follows:
1) Take the following blocks of a DP(14, 5, 1) on $Z_{12} \cup \{a, b\}$ where the first three blocks are taken under the action of the permutation α and the last two under the action of the permutation β where α = (0 1 2 3) (4 5 6 7) (8 9 10 11) and β = (8 10) (4 6) (9 11) (5 7).
<7 1 11 a 0> <0 b 9 1 4> <0 6 11 8 2>, <a 8 4 11 5> <9 5 8 6 b>.

Close observation of this design shows that (0, 5), (5, 0), (5, 7), (7, 5), (1, 6), (6, 1), (2, 7), (7, 2), (7, 6) and (6, 5) appear in zero blocks. Further, this design has the following two blocks <9 5 8 6 b> <a 10 6 9 7>. Replace these blocks by the blocks <9 5 8 0 b> <a 10 2 9 7>.

2) Take a DC(14, 5, 2) [5]. Close observation of this design shows that we may permute the points so that the ordered pairs of {5, 6, 7} appear in three blocks and it has the two blocks <9 8 1 0 b> <a 10 2 1 9> which we replace by <9 8 1 6 b> <a 10 6 1 9>. It is easy to see that the above two steps yield the blocks of a DP(14, 5, 3).

For v = 34 , 54, 74, 94 the result follows from the corollary.
For v = 134 apply Theorem 3.9 with m = 7, h = 6 and u = 2 and $\lambda = 3$.
For all other values of v simple calculations show that v can be written in the form v = 20m + 4u + h + s where m, u, h and s are chosen so that
1) There exists a RMGD[5, 1, 5, 5m].
2) There exists a (5, 3) – DGDD of type $4^m\ s^1$.
3) $0 \le u \le m - 1$, $s \equiv 0 \pmod 4$ and h = 6.
4) 4u + h + s = 14, 34, 54, 74, 94.

Now apply Theorem 3.4 with $\lambda = 3$ to get the result.

<u>Lemma</u> 4.5 Let $v \equiv 4 \pmod{20}$, $v \ge 24$, be an integer. Then DU(v, 5, 3) = DD(v, 5, 3).

<u>Proof</u> For all such v, the construction is as follows:
1) Take a (v – 1, 5, 1) optimal packing design in decreasing order [4] [17].

The complement graph of this design contains v – 1 edges and v – 1 vertices each having degree 2. So we may assume that (e,a), (d, b) and (d, c) are arcs of the complement graph.

2) Take a B[v + 1, 5, 1] in increasing order and place the point v + 1 at the beginning of each block. Assume we have the block <v + 1 a b c v>. In this block change v + 1 to d and in all other blocks change v + 1 to v. Further, assume in this design we have the block <x e z w v> where {x, z, w} are arbitrary numbers, different from a and d, arranged in increasing order. In this block change v to a.

3) Take a DP(v, 5, 2) with a hole of size two say {d, v} [5].

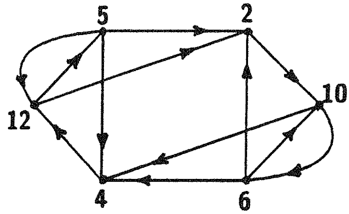
Further, permute the points of this design so that we have the block <w z d x a>. In this block change a to v.
It is clear that the above three steps yield the blocks of an ODP(v, 5, 3) for all $v \equiv 4 \pmod{20}$ $v \ge 24$.

Lemma 4.6 Let $v \equiv 7$ or $9 \pmod{10}$, $v \geq 7$ be an integer. Then $DU(v, 5, 3) = DD(v, 5, 3)$.

Proof For $v \equiv 7$ or $9 \pmod{20}$ $v \geq 29$ it has been shown that maximum $DP(v, 5, 1)$ with a hole of size 7 or 9 exists [7]. By taking 3 copies of a maximum $DP(v, 5, 1)$ with a hole of size 7 or 9 we get a maximum $DP(v, 5, 3)$ with a hole of size 7 or 9. Hence, to complete the proof of this lemma we need to show that $DU(v, 5, 3) = DD(v, 5, 3)$ for $v = 7, 9, 17, 19, 27$. For $v = 7, 9, 19, 27$ see next table.

v	Point Set	Base Blocks
7	$Z_4 \cup H_3$	$\langle 0\ 1\ 2\ h_1\ h_2 \rangle + i,\ i \in Z_2$ $\langle h_2\ h_1\ 2\ 3\ 0 \rangle + i,\ i \in Z_2$
		$\langle h_3\ h_1\ 2\ 1\ 0 \rangle + i,\ i \in Z_2$ $\langle 0\ 3\ 2\ h_1\ h_3 \rangle + i,\ i \in Z_2$
		$\langle h_2\ 0\ 2\ 1\ h_3 \rangle + i,\ i \in Z_2$ $\langle h_3\ 2\ 0\ 3\ h_2 \rangle + i,\ i \in Z_2$
9	$Z_2 \times Z_3 \cup H_3$	$\langle (0,0)\ (0,1)\ h_1\ (1,1)\ h_2 \rangle$ $\langle h_1\ (1,0)\ (0,1)\ h_3\ (0,0) \rangle$
		$\langle h_2\ (0,0)\ (1,2)\ (0,1)\ h_3 \rangle$ $\langle (1,1)\ (0,0)\ (0,2)\ (0,1)\ (1,1) \rangle$
		$\langle (0,0)\ (1,1)\ (1,0)\ h_2\ h_1 \rangle$ $\langle (1,1)\ h_3\ h_1\ (1,2)\ (0,0) \rangle$
		$\langle h_3\ h_2\ (1,0)\ (0,0)\ (1,2) \rangle$
19	$Z_2 \times Z_8 \cup H_3$	On $\{i\} \times Z_8 \cup H_3$ construct a DB[11, 5, 2], i = 0, 1 [20].
		Further take the following blocks
		$\langle (0,0)\ (0,4)\ (1,1)\ (0,1)\ (1,0) \rangle$ $\langle (1,0)\ (0,1)\ (0,3)\ (1,1)\ (0,0) \rangle$
		$\langle (0,2)\ (0,0)\ (1,1)\ (1,0)\ (1,3) \rangle$ $\langle (1,3)\ (1,5)\ (1,0)\ (0,2)\ (0,0) \rangle$
		$\langle (1,6)\ (0,0)\ (1,2)\ (0,4)\ h_1 \rangle$ $\langle h_1\ (1,4)\ (1,2)\ (0,0)\ (0,1) \rangle$
		$\langle (0,1)\ (1,4)\ (0,0)\ (1,5)\ h_2 \rangle$ $\langle h_2\ (1,7)\ (0,0)\ (0,2)\ (1,4) \rangle$
		$\langle (0,0)\ (0,3)\ (1,6)\ (1,2)\ h_3 \rangle$ $\langle h_3\ (1,7)\ (0,0)\ (1,5)\ (0,3) \rangle$
27	$Z_2 \times Z_{12} \cup H_3$	$\langle (0,2)\ (1,0)\ (0,0)\ (1,0)\ (0,7) \rangle$ $\langle (1,3)\ (0,0)\ (0,2)\ (0,5)\ (1,8) \rangle$
		$\langle (0,1)\ (1,1)\ (0,5)\ (0,0)\ (1,0) \rangle$ $\langle (1,8)\ (0,5)\ (1,2)\ (1,6)\ (0,0) \rangle$
		$\langle (1,0)\ (0,0)\ (0,1)\ (1,2)\ (1,4) \rangle$ $\langle (0,0)\ (1,5)\ (0,1)\ (1,10)\ h_1 \rangle$
		$\langle (0,3)\ (0,0)\ (1,3)\ (0,2)\ (1,5) \rangle$ $\langle (0,2)\ (0,0)\ h_2\ (1,7)\ (1,10) \rangle$
		$\langle (1,1)\ (1,0)\ (0,3)\ (1,2)\ (0,0) \rangle$ $\langle (1,6)\ (1,9)\ h_2\ (0,3)\ (0,0) \rangle$
		$\langle (1,7)\ (0,0)\ (0,2)\ (0,10)\ (1,4) \rangle$ $\langle (1,7)\ h_3\ (0,0)\ (1,5)\ (1,1) \rangle$
		$\langle (0,4)\ (1,8)\ (1,4)\ (1,11)\ (0,0) \rangle$ $\langle (1,4)\ (0,0)\ h_3\ (1,11)\ (0,6) \rangle$
		$\langle (1,4)\ (1,3)\ -\ (1,0)\ (1,1) \rangle \cup \{h_2, h_3\}$
		$\langle (1,0)\ h_1\ (1,1)\ (1,7)\ (1,5) \rangle$
		$\langle h_1\ (0,0)\ (0,3)\ (0,6)\ (0,9) \rangle$ orbit length 3, twice.
		$\langle h_1\ h_2\ (0,8)\ (0,4)\ (0,0) \rangle$ orbit length two
		$\langle h_2\ h_1\ (0,10)\ (0,6)\ (0,2) \rangle$ orbit length two
		$\langle (0,0)\ (0,4)\ (0,8)\ h_1\ h_3 \rangle$ orbit length two
		$\langle (0,2)\ (0,6)\ (0,10)\ h_3\ h_1 \rangle$ orbit length two
		$\langle (0,0)\ (0,4)\ (0,8)\ h_2\ h_3 \rangle$ orbit length two
		$\langle (0,2)\ (0,6)\ (0,10)\ h_3\ h_2 \rangle$ orbit length two.

For v = 17 the construction is as follows:
1) Take a DP(17, 5, 1) (not optimal) [7]. Close observation of this design shows that its complement directed graph is the following:



Further, this design has the following two blocks <1 2 6 13 12> <7 10 3 2 5> which we replace by <1 2 6 10 13> <7 10 3 5 4>.
2) Take a DC(17, 5, 2) [5]. Close observation of this design shows that we may permute the points so the ordered pairs of the triple {2, 5, 12} appear in three blocks and it has the two blocks <6 4 1 10 13> <3 7 10 6 4>, which we replace by <6 4 1 13 12> <3 7 10 6 2>.
It is easy to check that the above construction yields the blocks of a DP(17, 5, 3).

<u>Lemma 4.7</u> Let $v \equiv 8 \pmod{10}$, $v \geq 8$, be a positive integer. Then DU(v, 5, 3) = DD(v, 5, 3).

<u>Proof</u> For v = 58 take a ({5, 6}, 1) - GDD of type $5^5 4^1$ and inflate this design by a factor of 2 with index 3, that is, replace each block of size 5 and 6 by the blocks of a (5, 3) - DGDD of type 2^5 and type 2^6. On the groups construct an optimal DP(v, 5, 3) for v = 10, 8.
For v = 98 apply Theorem 3.5 with m = 9, u = h = 2 and e = 1.
For v = 8, 18, 28, . . ., 88, $v \neq 58$, see next table.
For v = 128 apply Theorem 3.9 with m = 7, h = 0, $\lambda = 3$ and u = 2.
For v = 138 apply Theorem 3.9 with m = 7, h = 6, u = 2 and $\lambda = 3$.
For all other values of v simple calculations show that v can be written in the form v = 20m + 4u + h + s where m, u, h and s are chosen so that
1) There exists a RMGD[5, 1, 5, 5m]
2) There exists a (5, 3) - DGDD of type $4^m s^1$
3) $4u + h + s \equiv 8 \pmod{10}$, $8 \leq 4u + h + s \leq 98$
4) h = 6 if $v \equiv 18 \pmod{20}$, h = 0 if $v \equiv 8 \pmod{20}$, $0 \leq u \leq m - 1$, and $s \equiv 0 \pmod{4}$.
Now apply Theorem 3.4 to get the result.

v	Point Set	Base Blocks
8	Z_8	<0 1 2 3 5> <0 6 5 4 3>
18	Z_{18}	<0 1 2 3 5> <0 3 7 11 16> <0 5 15 12 11> <0 14 13 8 6> <0 16 10 7 8>
28	Z_{28}	<0 1 3 7 12> <0 5 13 23 21> <0 10 9 24 21> <0 27 19 17 13> <0 1 2 4 7> <0 4 10 19 26> <0 20 17 13 8> <0 25 17 12 11>
38	Z_{38}	<0 1 5 14 21> <0 2 37 30 17> <0 27 12 22 8> <0 1 3 7 12> <0 6 14 32 31> <0 29 27 13 10> <0 33 23 15 11> <0 1 3 6 10> <0 8 20 19 34> <0 32 24 22 17> <0 35 29 18 16>
48	$Z_{40} \cup H_8$	On $Z_{40} \cup H_8$ construct a DB[45, 5, 1] such that H_5 is a block which we delete. Furthermore, take the following blocks <13 33 h_6 0 20> + i, $i \in Z_{13}$ <26 6 h_6 33 13> + i, $i \in Z_7$ <0 1 3 7 15 <0 5 16 34 26> <0 13 - 30 9> $\cup \{h_1, h_2\}$ <0 25 h_1 23 22> <0 31 h_2 24 19> <0 1 h_3 3 7> <0 5 h_4 13 22> <0 10 h_5 21 35> <0 12 h_6 28 27> <0 30 h_7 26 24> <0 31 h_8 23 20>
68	$Z_{60} \cup H_8$	On $Z_{60} \cup H_7$ take a DP(67, 5, 1) with a hole of size 7, say, H_7 [7]. <10 40 h_8 0 30> + i, $i \in Z_{10}$ <20 50 h_8 40 10> + i, $i \in Z_{20}$ <0 12 24 36 48> + i, $i \in Z_{12}$; <48 36 24 12 0> + i, $i \in Z_{12}$ twice <0 1 3 7 12> <0 8 18 31 45> <0 15 59 39 32> <0 21 49 41 19> <0 43 34 29 16> <0 56 50 25 22> <0 1 h_1 3 7> <0 5 h_2 13 22> <0 10 h_3 21 35> <0 15 h_4 31 49> <0 19 h_5 46 45> <0 28 h_6 23 20> <0 51 h_7 44 38> <0 56 h_8 39 37>
78	$Z_{70} \cup H_8$	On $Z_{70} \cup H_3$ construct a DP(73, 5, 1) with a hole of size, 3, say H_3 [7]. <17 52 h_8 0 35> + i, $i \in Z_{17}$ <34 69 h_8 52 17> + i, $i \in Z_{18}$ <0 14 28 42 56> + i, $i \in Z_{14}$ <56 42 28 14 0> + i, $i \in Z_{14}$ <0 1 3 7 12> <0 8 18 31 47> <0 15 32 52 51> <0 37 33 27 67> <0 21 54 45 43> <0 62 48 41 36> <0 7 46 50 26> <0 53 h_1 38 25> <0 5 - 15 26> $\cup \{h_4, h_5\}$ <0 12 - 25 41> $\cup \{h_6, h_7\}$

<0 1 h_2 3 9> <0 14 h_3 31 54> <0 20 h_4 48 44>
<0 22 h_5 60 57> <0 32 h_6 62 27> <0 52 h_7 51 45>
<0 58 h_8 49 47>

88 $Z_{80} \cup H_8$ On $Z_{80} \cup H_7$ construct a DP(87, 5, 1) with a hole of size 7, say H_7, [7].
<21 61 $h_{ç}$ 0 40> + i, i $\in Z_{21}$ <42 2 $h_{ç}$ 61 21> + i, i $\in Z_{19}$
<0 16 32 48 64> + i, i $\in Z_{16}$ <64 48 32 16 0> + i, i $\in Z_{16}$
twice
<0 1 5 7 31> <0 10 21 44 62> <0 19 57 56 54>
<0 22 71 65 61> <0 66 58 53 46> <0 3 12 20 45>
<0 14 69 58 41> <0 15 61 51 28> <0 56 54 50 29>
<0 75 62 45 42> <0 1 h_1 3 7> <0 5 h_2 13 22>
<0 10 h_3 21 33> <0 14 h_4 29 49> <0 16 h_5 43 34>
<0 26 h_6 65 51> <0 36 h_7 28 68> <0 73 h_8 31 30>.

Lemma 4.8 Let $v \equiv 3 \pmod{10}$ be a positive integer, $v \geq 13$. Then DU(v, 5, 3) = DD (v, 5, 3) with the possible exception of v=43.

Proof For $v \neq$ 13, 43, 53, 63, 73, 83 the construction is as follows:
1) Take a (v, 5, 2) minimal covering in increasing order. This design has a pair, say, (a, b) that appears in 6 blocks, [16]. We may permute the points such that (a, b) and (b, a) appear in 3 blocks.

2) Take a (v, 5, 2) optimal packing in decreasing order and assume that the ordered pairs of {a, b, c} appear in zero blocks, [8].
3) Take an optimal DP(v, 5, 1) and assume that the ordered pairs of {a, b, c} appear in zero blocks, [7].

For v = 53, 73 take 6 copies of a PBD(v, {5, 13*}, 1) : 3 copies in increasing order and 3 copies in decreasing order, where * means there is exactly one block of order 13 [14]. On the block of size 13 construct an optimal DP(13, 5, 3).

For v=63, 83, see next table.

For v=13 let X = {1, 2, . . ., 13} then the blocks are
<1 2 3 11 5> <1 2 9 13 6> <1 2 8 11 13> <1 3 9 12 4> <1 5 7 6 10>
<1 9 3 7 13> <1 12 7 6 5> <2 3 6 4 13> <2 3 10 9 5> <3 4 7 12 8>
<3 8 10 7 2> <4 3 5 8 11> <4 6 10 7 13> <4 5 2 8 9> <4 7 9 10 11>
<5 6 2 12 4> <5 6 8 7 13> <6 11 9 7 4> <7 5 3 10 11> <7 13 9 5 2>
<8 3 9 6 12> <8 4 5 13 11> <8 4 6 9 12> <8 2 10 7 12> <9 1 10 4 5>
<9 8 7 3 1> <9 11 8 6 5> <10 6 1 8 11> <10 6 11 4 3> <10 13 9 6 3>
<11 2 7 8 6> <11 4 10 12 13> <11 5 9 12 13> <11 7 12 9 2> <11 10 9 2 1>
<11 12 3 13 6> <12 5 7 4 1> <12 6 10 5 3> <12 10 8 2 1> <12 11 3 2 1>
<13 5 2 10 12> <13 7 4 3 2> <13 1 8 10 4> <13 8 5 3 1> <13 12 7 1 11>
<13 12 9 10 8>

v	point set	Base Blocks
63	$Z_{50} \cup H_{13}$	<13 38 h_{13} 0 25> + i, v ∈ Z_{13}; <26 1 h_{13} 38 13> + i ∈ Z_{12} <0 1 3 9 14> <0 4 20 30 48> <0 15 47 39 38> <0 36 - 21 17> ∪ {h_1,h_2} <0 43 - 27 22> ∪ {h_3,h_4} <0 9 - 19 30} ∪ {h_5,h_6} <0 13 - 27 44> ∪ {h_7,h_8} <0 41 - 28 23> ∪ {h_9,h_{10}} <0 5 - 13 22} ∪ {h_{11},h_{12}} <0 5 h_2 12 20> <0 33 h_3 25 23> <0 1 h_1 3 7> <0 38 h_4 27 24} <0 16 h_6 35 34> <0 1 h_7 3 7> <0 43 h_5 26 22> <0 10 h_8 21 35> <0 15 h_{10} 45 39> <0 32 h_{11} 20 19> <0 12 h_9 28 46> <0 36 h_{12} 29 26> <0 41 h_{12} 33 31>
83	$Z_{70} \cup H_{13}$	On $Z_{70} \cup \{h_1\}$ construct a DB[71, 5, 1]. Further, take the following base blocks: <0 14 28 42 56> + i, i ∈ Z_{14} <56 42 28 14 0> + i, i ∈ Z_{14} <0 2 6 32 44> <54 30 18 0 8> <0 38 h_1 4 20> <33 25 h_2 3 0> <0 5 h_3 11 31> <3 1 h_4 0 8> <4 0 h_5 13 23> <55 19 h_6 6 0> <10 0 h_7 25 43> <0 27 h_8 50 11> <53 29 h_9 0 12> <0 1 h_{10} 3 10> <29 9 h_{11} 4 0> <0 6 h_2 21 39> <11 0 h_{13} 28 47> <48 27 - 13 0> ∪ {h_2, h_3} <9 2 - 0 1> ∪ {h_4, h_5} <26 15 - 0 3> ∪ {h_6, h_7} <43 0 - 5 30> ∪ {h_8, h_9} <7 0 - 46 29> ∪ {h_{10}, h_{11}} <33 0 - 14 49> ∪ {h_{12}, h_{13}}

5. Directed Packing With Index 4, 6, 8

Lemma 5.1 Let v ≥ 5 be a positive integer. Then DU(v, 5, 4) = DD (v, 5, 4).

Proof For v ≡ 0 or 1 (mod5) the result is contained in Corollary 1.1.

For v ≡ 2 or 4 (mod 5), v ≠ 7, DU(v, 5, 4) = 2 DU(v, 5, 2) [8].

For v ≡ 3 (mod 5), v ≠ 8, the construction is as follows:

1) Take a (v – 1, 5, 4) optimal packing in increasing order [10].

2) Take a (v + 1, 5, 4) optimal packing in decreasing order. This design has a pair say (v + 1, v) that appears in zero blocks. Place the point v + 1 at the end of each block and change it to v.

For v = 7 let X = Z_7. Then take the following blocks

<5 0 1 2 4> <6 2 4 1 0> <0 2 5 1 6> <5 1 4 6 0>
<0 1 3 4 5> <6 3 4 1 0> <0 1 3 6 5> <6 5 4 1 2>
<2 3 0 4 6> <4 5 3 2 0> <6 5 4 3 2> <0 2 5 3 6>
<1 2 3 4 5> <4 6 3 2 1> <1 2 3 5 6> <4 6 5 3 0>

For v = 8 let X = $Z_6 \cup \{a, b\}$. Then take the following blocks

<0 1 a 2 3> (mod 6) <3 2 b 1 a> (mod 6) <0 1 - 4 3> ∪{a, b}
<a b 0 2 4> + i, i ∈ Z_2, <4 2 0 b a > + i, i ∈ Z_2

Lemma 5.2 Let $v \geq 5$ be an integer. Then
1) DU(v, 5, 6) = DD (v, 5, 6) – 1 if $v \equiv 2$ or 4 (mod 5)
2) DU(v, 5, 6) = DD (v, 5, 6) if $v \equiv 0$, 1 or 3 (mod 5).

Proof For $v \equiv 0$ or 1 (mod 5) the result is contained in Corollary 1.1.
For $v \equiv 2$ or 4 (mod 5), $v \neq 7$, DU(v, 5, 6) = DU(v, 5, 4) + DU(v, 5, 2) holds.
For v = 7 let $X = Z_6 \cup \{\infty\}$. Then the blocks are
<0 1 3 4 2> (mod 6) <2 1 0 5 ∞> (mod 6)
<5 0 ∞ 1 2> (mod 6) <∞ 4 3 1 0> (mod 6)
For $v \equiv 3$ (mod 5), we first treat the values under 100. For v = 8 let X = $Z_5 \cup H_3$. Then the blocks are <0 4 1 h_1 h_2> (mod 5), <h_2 h_1 0 4 1> (mod 5), <0 3 4 h_1 h_3> (mod 5), <h_3 h_1 0 2 3> (mod 5), <h_2 h_3 0 3 2> (mod 5), <0 2 3 h_3 h_2> <1 3 4 h_3 h_2> <2 0 4 h_3 h_2> <0 3 1 h_3 h_2> <1 4 2 h_3 h_2> <0 1 2 3 4> and <4 3 2 1 0> twice.
For $v \geq 13$ the construction is as follows.
1) Take a MDC(v – 1, 5, 2) [5]. This design has a triple say, {a, b, c} the ordered pairs of which appear in 3 blocks.
2) Take a DP(v + 1, 5, 2) with a hole of size 2 say {v, v+ 1} [8]. Replace the point v + 1 by v.
3) Take a ODP(v, 5, 2) [8]. Close observation of this design shows that every ordered pair of this design occurs in exactly two blocks except the pairs of a triple say {a, b, c}, the ordered pairs of which appear in zero blocks, except v = 68. We now construct a maximum DP(68, 5, 2) with a hole of size 3 by taking a T[12, 1, 11]. Let B be a block of size 12. Remove the last 6 groups and all but 5 points of the 6th group but we leave the points of B. Let the remaining six groups be $G_1, \ldots, G_6$ where $|G_6| = 5$. Let $B \cap G_i = \infty_i, i = 1, \ldots, 5$. Adjoin a new point ∞_6 to the groups and on each of the five groups $G_1, \ldots, G_5$ construct a maximum DP(12, 5, 2) with a hole of size two where the hole is $\{\infty_i, \infty_6\}$, i = 1, . . ., 5 and on G_6 construct a DB[6, 5, 2]. On the blocks of size 5 and 6 of the truncated transversal design we construct a DB[v, 5, 2], v = 5, 6. Finally, adjoin the point ∞_6 to the block B and construct a DP(13, 5, 2) with a hole of size 3 [8].

It is clear that the above three steps yield the blocks of a DP(v, 5, 6) $v \equiv 3$ (mod 5) $13 \leq v \leq 98$.
For v = 138 apply theorem 3.9 with m = 7, u = 4, h = 2 and $\lambda = 6$ and notice that a maximum DP(22, 5, 6) with a hole of size 2 is obtained by taking three copies of a maximum DP(22, 5, 2) with a hole of size 2 [8].
For $v \geq 108$ write v = 20m + 4u + h + s and then the proof is similar to that of lemma 4.7.

Lemma 5.3 Let $v \geq 5$ be a positive integer. Then DU(v, 5, 8) = DD(v, 5, 8).

Proof For v ≡ 0 or 1 (mod 5) there exists a DB[v, 5, 8].
For v ≡ 3 (mod 5) DU(v, 5, 8) = 2 DU(v, 5, 4) holds.
For v ≡ 2 or 4 (mod 5), v ≠ 7, the construction is as follows:
1) Take a MDC(v, 5, 2) [5]. This design has a triple, say, {a, b, c} the ordered pairs of which appear in three blocks.
2) Take three copies of a maximum DP(v, 5, 2) with a hole of size 2 [8]. Assume the hole is {a, b}, {a, c} and {b, c} respectively.
It is clear that the above two steps yield the blocks of an ODP(v, 5, 8) for all v ≡ 2 or 4 (mod 5) v ≠ 7. For v = 7 let $X = Z_2 \times Z_3 \cup \{\infty\}$. Then the blocks are

<(1,0) (0,0) (1,2) (0,1) ∞> mod (-, 3)
<(0,0) (0,1) (1,0) (1,1) ∞> mod (-, 3)
<∞ (1,2) (1,1) (0,1) (0,0)> mod (-, 3)
<(0,0) (0,1) ∞ (0,2) (1,2)> mod (-, 3)
<(1,1) (0,2) (0,1) (1,0) (0,0)> mod (-, 3)
<(1,2) (0,0) (0,1) (1,0) (1,1)> mod (-, 3)
<(1,0) (0,1) (0,0) (0,2) (1,1)> mod (-, 3)
<(0,1) (1,1) (0,0) (1,2) ∞> mod (-, 3)
<∞ (0,1) (1,0) (1,2) (0,0)> mod (-, 3)
<∞ (1,0) (0,0) (0,1) (1,1)> mod (-, 3)
<(1,0) (1,1) ∞ (1,2) (0,0)> mod (-, 3)

We now turn our attention to deal with directed packing with index λ=5, 7, 9. Notice that if v is odd then, by Theorem 1.2, there exists a DB[v, 5, 5]. Hence, the result for λ=7, 9 is obtained by applying Lemma 1.1. When v ≡ 0 or 1 (mod 5), the result has been established in corollary 1.1. Therefore, we need only consider the cases v ≡ 2, 4, or 8 (mod 10), which is done in the next three sections.

6. Directed Packing with Index 5

Lemma 6.1 Let v ≡ 4 (mod 20) v ≥ 24 be an integer. Then DU(v, 5, 5) = DD(v, 5, 5).

Proof For all v ≡ 4 (mod 20), v ≥ 24 the construction is as follows:
1) Take a maximum DP(v, 5, 2) with a hole of size 2, say, {v – 1, v} [8]. Assume in this design we have a block containing {v v – 3 a b c} where {v – 3 a b c} are on the right side of v in any order. In this block change v to v – 1.
2) Take an ODP(v – 2, 5, 1) [8].
3) Take 2 copies of a B[v + 1, 5, 1] the first in increasing order and the second in decreasing order. Assume in the first copy we have the block < e f g v v + 1>. In this block change v + 1 to v – 1 and in all other blocks change v + 1 to v. Furthermore, assume in the second copy we have the block <v + 1 v v – 1 v – 2 v – 3>. Delete this block and in all other blocks change v + 1 to v.

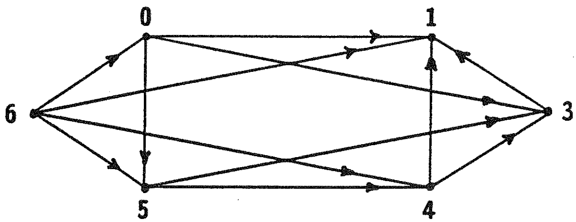
4) Again take 2 copies of B[v + 1, 5, 1] the first in increasing order and the second in decreasing order. Assume in the first copy we have the block <e f g v – 1 v + 1> and in the second we have the block <v + 1 v – 1 c b a>. In these two blocks change v + 1 to v and in all other blocks of the 2 copies change v + 1 to v – 1.

Lemma 6.2 Let v ≡ 8 (mod 20) be a positive integer. Then DU(v, 5, 5) = DD(v, 5, 5).

Proof For v = 8 the construction is as follows:
1) Take the following blocks of a ODP(8, 5, 2) on X = Z_7 ∪ {a}:

<1 0 4 5 2>...<2 5 1 3 6>...<3 2 4 0 6>...<1 0 a 4 2>
<2 1 a 3 5>...<6 3 a 2 4>...<4 3 a 5 0>...<5 4 a 1 6>
<6 5 a 2 0>...<0 6 a 3 1>

Close observation of this design shows that the complement graph of this design consists of 2 isolated vertices and the following directed graph on the remaining 6 vertices.



2) Take an optimal DP(8, 5, 3) and assume that (6, 3) and (5, 1) appear at most twice.
3) Add the block <6 5 4 3 1>.
For v = 28 the construction is as follows:
1) Take a MDC(27, 5, 2) [5]. This design has a triple, say, {a, b, c} the ordered pairs of which appear in 3 blocks.
2) Take a maximum DP(29, 5, 2) with a hole of size 2 say {28, 29}. Change the point 29 to 28.
3) Take an ODP(28, 5, 1) with a hole of size 4, say, {a, b, c, d} [19].
For v = 48, 68, 88 the construction is the same as for v = 8 but in the first step we take a maximum DP(v, 5, 2) with a hole of size 8 for v = 48, 68, 88. So to complete the construction we need to show that there exists a DP(v, 5, 2) with a hole of size 8 for the stated values of v.
For v = 48 let X = $Z_{40} \cup H_8$ and take the following blocks under the action of the permutation α = (0, 1, 2, . . ., 39).

<0 2 5 11 23> <23 11 5 2 0> <0 4 10 24 36> <0 1 h_1 8 39>
<0 11 h_2 34 27> <0 13 h_3 3 28> <0 1 h_4 5 14> <0 2 h_5 20 27>
<0 15 h_6 10 36> <0 22 h_7 30 19> <0 24 h_8 17 16>
For v = 68 see [19].

For v = 88 take a ({5, 6}, 1)-GDD of type $8^5 4^1$ and inflate the design by a factor of 2, that is, replace each block of size 5 and 6 by the blocks of (5, 1)-DGDD of type 2^5 and 2^6 respectively [20]. On the first five groups construct a DB[16, 5, 2] and take the last group to be the hole. For v = 128 apply Theorem 3.9 with m = 7, h = 0, u = 2 and λ = 5. For all other values of v write v = 20m + 4u + h + s where m, u, h and s are chosen as in Lemma 4.4 with the difference that h = 0 and 4u + h + s = 8, 28, 48, 68, 88. Now apply Theorem 3.4 to get the result.

Lemma 6.3 (i) Let v ≡ 2 (mod 10) v ≥ 12 be a positive integer. Then DU(v, 5, 5) = DD(v, 5, 5). (ii) There exists a DP(26, 5, 5) with a hole of size 6.

Proof For the first part of the lemma notice that DU(v, 5, 5) = DU(v, 5, 4) + DU(v, 5, 1).
For the second part of the lemma take the blocks of a DP(26, 5, 1) with a hole of size 6 [19] and two copies of a DP(26, 5, 2) with a hole of size 6, lemma 4.3

Lemma 6.4 Let v ≡ 14 (mod 20) be a positive integer. Then DU(v, 5, 5) = DD(v, 5, 5).

Proof For v = 14 proceed as follows:
1) Take the following blocks of a DP(14, 5, 1) on X = $Z_3 \times Z_4 \cup \{a, b\}$
<(0,0 a (1,3) (2,3) (0,1)> mod (-, 4) <(1,0) (0,1) (2,1) b (0,0)> mod (-, 4)
<(0,2) (2,3) (2,0) (0,0) (1,2)> mod (-, 4)
<(2,i) (1,i) (1, i + 1) (2, i + 3) a>, i = 0, 2
<B (2, j + 1) (1, j + 1) (1, j + 2) (2, j)> j = 0, 2
Close observation of this design shows that the pairs ((0,0), (1,1)), ((1,1), (0,0)), ((1,1), (1,3)) and ((1,3), (1,1)) appears in zero blocks. We may relabel the points such that the pairs (12, 13) (13, 12) (13, 14) and (14, 13) appear in zero blocks.
2) Take a MDC(14, 5, 2) [5]. This design has a triple say {12, 13, 14} the ordered pairs of which appear in 3 blocks.
3) Take a DP(14, 5, 2) with a hole of size 2, say, {12, 14}.
For v = 34, 54, 74, 94 the construction is as follows:
1) Take an ODP(v, 5, 2) [8].
2) Take a MDC(v, 5, 2) [5]. This design has a triple, say, {a, b, c} the ordered pairs of which appear in 3 blocks.
3) Take a DP(v, 5, 1) with a hole of size 4, Lemma 4.3
for v = 134 apply Theorem 3.9 with m = 7, h = 6, u = 2 and λ = 5.
For all other values write v = 20m + 4u + h + s where m, u, h and s are chosen as in lemma 4.4. Now apply theorem 3.4 to get the result.

Lemma 6.5 Let v ≡ 18 (mod 20) be a positive integer. Then there exists a DP(v, 5, 1) with a hole of size 4.

Proof For v = 18, 38 see [19].
For v = 58, 78 see next table.
For v = 98 take a ({5, 6}, 1) – GDD of type $9^5 2^1$ and inflate it by a factor of 2. Replace each block of size 5 and 6 by the blocks of a (5, 1) – DGDD of type 2^5 and 2^6 respectively [20]. Add two points to the groups and on the first five groups construct a maximum DP(20, 5, 1) with a hole of size 2 [19]. Then take these two points with the last group to be the hole.
For v = 138 apply Theorem 3.9 with m = 7, h = 2, u = 4, λ = 1 and notice that a DP(22, 5, 1) with a hole of size 2 can be constructed on $Z_{20} \cup H_2$ by taking the following blocks:
<16 12 8 4 0> + i, i ∈ Z_4, <0 5 - 14 13> ∪ $\{h_1, h_2\}$ (mod 20),
<0 1 3 7 18> (mod 20).
For all other values of v write v = 20m + 4u + h + s where m, u, h and s are chosen as in Lemma 4.4 with the difference that 4u + h + 2 = 18, 38, 58, 78, 98 and h = 2 or 6. Now apply Theorem 3.4 with λ = 5 to get the result and for a DP(26, 5, 1) with a hole of size 6 see [19].

v	Point Set	Base Blocks
58	$Z_{54} \cup H_4$	<0 1 3 7 12> <0 8 18 31 45> <0 15 32 53 51> <0 30 24 16 50> <0 41 - 29 22> ∪ $\{h_1, h_2\}$ <0 43 - 33 28> ∪ $\{h_3, h_4\}$
78	$Z_{74} \cup H_4$	<0 1 3 7 12> <0 8 18 31 45> <0 15 32 48 67> <0 20 41 66 63> <0 54 53 36 30> <0 60 58 28 24> <0 26 - 65 55> ∪ $\{h_1, h_2\}$ <0 59 - 47 34> ∪ $\{h_3, h_4\}$

Lemma 6.6 Let v ≡ 18 (mod 20) be a positive integer. Then DU(v, 5, 5) = DD(v, 5, 5).

Proof For all v ≡ 18 (mod 20) the construction is as follows:
1) Take a MDC(v – 1, 5, 2) and assume that the directed pairs of the triple {a, b, c} appear in 3 blocks [5].
2) Take a maximum DP(v + 1, 5, 2) with a hole of size 2, say, {v, v + 1} [8] and change the point v + 1 to v.
3) Take a maximum DP(v, 5, 1) with a hole of size 4, say, {a, b, c, d}.

7. Directed Packing with Index 7

Lemma 7.1 (i) There exists a maximum DP(v, 5, 1) with a hole of size 4 for v = 24,34, 44, . . ., 94.
(ii) There exists a maximum DP(26, 5, 7) with a hole of size 6.
(iii) There exists a maximum DP(24, 5, 7) with a hole of size 4.

Proof For v = 34, 54, 74, 94 see Lemma 4.3.
For v = 24, 44, 64, 84 the construction is as follows:

1) Take a (v – 1, 5, 1) optimal packing in increasing order. The complement graph of this design is the circuit graph C_n [17]. So we may assume that the pairs {d, v – 3}, {v – 3, v – 2}, {v – 2, v – 1} and {e, v – 1} appear in zero blocks [4, 17]. Further, assume we have the block <a b c v – 3 v – 1>. Replace this block by <d a b c v – 3>.
(2) Take a B[v + 1, 5, 1] in decreasing order and delete the block <v + 1 v v - 1 v – 2 v – 3>. Assume we have the block <e d c b a> which we replace by <e c b a v – 1>. In all other blocks place the point v + 1 at the end of each block and then replace it by v. Now it is readily checked that the above construction yields a DP(v, 5, 1) with a hole of size 4 for all v ≡ 4(mod 20) v ≥ 24 where the hole is {v -3, v-2, v-1, v}.
(ii) For a maximum DP(26, 5, 7) with a hole of size 6 take a maximum DP(26, 5, 1) with a hole of size 6 [19] and 3 copies of a maximum DP(26, 5, 2) with a hole of size 6, Lemma 4.3.
(iii) For a (24, 5, 7) with a hole of size 4 take a maximum DP(24, 5, 1) with a hole of size 4 [19], three copies of a maximum (23, 5, 2) packing design with a hole of size 3 in increasing order [8] and six copies of a B[25, 5, 1] in decreasing order. Delete the six blocks <25 24 23 22 21>. Place the point 25 at the end of each block and then replace it by 24.

<u>Lemma 7.2</u> Let v ≡ 4 (mod 10), v ≥ 14 be an integer. Then DU(v, 5, 7) = DD(v, 5, 7).

<u>Proof</u> For v = 14 let X = Z_{14} then take the following blocks under the action of the group Z_{14}
<6 3 2 1 0> twice, <0 1 2 3 6> <0 1 4 8 10> <10 8 4 1 0> <0 2 5 7 10> <10 7 5 2 0> <0 1 3 7 9> <1 4 9 8 0>.
For v = 24, 34, . . ., 94 the construction is as follows:
1) Take 2 copies of a MDC(v, 5, 2). This design has a triple the directed pairs of which appear in 3 blocks [5]. Assume the triple in the firs copy is {a, b, c} and in the second is {a, c, d}.
2) Take a maximum DP(v, 5, 2) with a hole of size 2, say {a, c} [8].
3) Take a maximum DP(v, 5, 1) with a hole of size 4, say {a, b, c, d}.
For v = 134 apply Theorem 3.9 with m = 7, u = 2, h = 6 and λ = 7.
For v = 144 apply Theorem 3.8 with m = 7, k = 6, h = 4 and λ = 7.
For v = 104, 224, apply Theorem 3.7 with h = 4, λ = 7 and m = 5,11 respectively . For v = 184 apply Theorem 3.8 with k = 5, λ = 7, h = 4 and m = 9. For all other values of v write v = 20m + 4u + h + s where m, u, h and s are chosen as in Lemma 4.4 where 4u + h + s = 14, 24, ... , 94 and h = 0 if v ≡ 4(mod 20) and h = 6 if v ≡ 14(mod 20). Now apply Theorem 3.4 with λ = 7 to get the result.

<u>Lemma</u> 7.3 Let v ≡ 8 (mod 10) be a positive integer. Then DU(v, 5, 7) = DD(v, 5, 7).

<u>Proof</u> DU(v, 5, 7) = DU(v, 5, 4) + DU(v, 5, 3).

Lemma 7.4 Let v ≡ 12(mod 20) be a positive integer. Then DU(v, 5, 7) = DD(v, 5, 7)

Proof For all v ≡ 12(mod 20) v ≥ 12 the construction is as follows:
1) Take a MDC(v, 5, 2) and assume that the directed pairs of {a, b, c} appear in 3 blocks [5].
2) Take two copies of a maximum DP(v, 5, 2) with a hole of size 2. Assume the holes are {a, b} and {a, c} respectively [8].
3) Take a maximum DP(v, 5, 1) with a hole of size 2, say, {b, c}.
To complete the proof of this lemma we need to show that a maximum DP(v, 5, 1) with a hole of size 2 exists for all v ≡ 12 (mod 20).
For v = 12 see [19].
For v = 52 take a T[5, 1, 5] and inflate the design by a factor of 2 and replace each of its quintuples by the blocks of (5, 1) - DGDD of type 2^5.
Add two points to the groups and on each group construct a maximum DP(12, 5, 1) with a hole of size 2.
For v = 92 take a DT[5, 1, 18]. Add two points to the groups and on each group construct a maximum DP(20, 5, 1) with a hole of size two [19].
For v = 32, 72 see next table.
For v = 132 apply Theorem 3.9 with m = 7, h = 0, u = 3 and λ = 7.
For all other values of v the proof is the same as in Lemma 4.7 with the difference that 4u + h + s = 12, 32, 52, 72, 92.

v	Point Set	Base Blocks
32	$Z_{30} \cup H_2$	<24 18 12 6 0> + i, i $\in Z_6$ <0 9 - 26 19> $\cup \{h_1, h_2\}$ <0 1 3 7 15> <0 16 21 13 11>
72	$Z_{70} \cup H_2$	<56 42 28 14 0> + i, i $\in Z_{14}$, <0 55 - 39 38> $\cup \{h_1, h_2\}$ <0 1 3 7 12> <0 8 18 31 45> <0 15 32 48 67> <0 20 50 46 41> <0 24 68 60 49> <0 64 57 34 22>

Lemma 7.5 Let v ≡ 2(mod 20) v ≥ 22 be an integer. Then DU(v, 5, 7) = DD(v, 5, 7).

Proof The proof of this lemma is the same as the previous one. So we need to show that a maximum DP(v, 5, 1) with a hole of size 2 exists for all v ≡ 2 (mod 20).
For v = 22 see lemma 6.5.
For v =42, 62 see [19 P 138].
For v = 82 let x = $Z_{80} \cup H_2$ and take the following blocks under the action of the group Z_{80} : <64 48 32 16 0> +i, i $\in Z_{16}$.
<0 1 3 7 12> <0 8 18 31 45> <0 15 32 51 67>
<0 20 41 66 63 > <0 26 79 59 55> <0 38 28 78 72>
<0 65 54 47 42 > <0 61 - 39 30> $\cup \{h_1, h_2\}$
For all other values of v, v ≠ 142, 182 the proof is the same as in Lemma

4.4 with the difference that $4u + h + s = 22, 42, 62, 82$.
For $v = 142$ apply theorem 3.8 with $k = 6$, $m = 7$, $h = 2$ and $\lambda = 7$.
For $v = 182$ apply theorem 3.8 with $k = 5$, $m = 9$, $h = 2$ and $\lambda = 7$.

8. Directed Packing With Index 9

Lemma 8.1 Let $v \geq 5$ be a positive integer. Then $DU(v, 5, 9) = DD(v, 5, 9)$.

Proof If $v \equiv 2$ or $4 \pmod{10}$ then $DU(v, 5, 9) = DU(v, 5, 7) + DU(v, 5, 2)$.
If $v \equiv 8 \pmod{10}$ then $DU(v, 5, 9) = DU(v, 5, 6) + DU(v, 5, 3)$.

Conclusion We have determined the values of $DD(v,5,\lambda)$ for all $v \geq 5$ and $3 \leq \lambda \leq 9$ in section 4-8. These results together with Theorem 1.3, making use of Lemma 1.1 and Theorem 1.2, give the following:

Main Theorem Let $v \geq 5$ be an integer. Then $DD(v,5,\lambda) = DU(v,5,\lambda) - e$ where $e=1$ if $2\lambda(v-1) \equiv 0$ and $\frac{\lambda v(v-1)}{2} \equiv 1 \pmod 5$ and $e=0$ otherwise with the exception of $(v, \lambda)=(15,1)$ and the possible exception of $(v, \lambda)= (19,1)$ $(27,1)$ $(43,3)$.

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