# Radius-Forcing Sets in Graphs

Peter Dankelmann, Vivienne Smithdorf and Henda C. Swart

University of Natal, Durban 4001
South Africa

**Abstract**

Let $G$ be a connected graph of order $p$ and let $\emptyset \neq S \subseteq V(G)$. Then $S$ is a *rad(G)-forcing set* (or a *radius-forcing set of* $G$) if, for each $v \in V(G)$, there exists $v' \in S$ with $d_G(v, v') \geq \text{rad}(G)$. The cardinality of a smallest radius-forcing set of $G$ is called the *radius-forcing number of* $G$ and is denoted by $\text{rf}(G)$. A graph $G$ is called a *randomly k-forcing graph* for a positive integer $k$ if every $k$-subset of $V(G)$ is a radius-forcing set of $G$. We investigate the value of $\text{rf}(G)$ for various graphs $G$, and obtain some general bounds, and we characterize graphs for which rf achieves the values of $1, 2, p-1$, and $p$, respectively. We establish the NP-completeness of the calculation of rf for arbitrary graphs, and conclude with an investigation of $k$-randomly forcing graphs.

## 1. Introductory definitions and examples

Let $G$ be a connected graph of order $p$ and vertex set $V(G)$. Suppose that the vertices of $G$ represent $p$ facilities in which essential data or materials are storeable (for example, warehouses, rooms, computers in an information network). Two vertices in $G$ are joined by an edge if the corresponding facilities are linked or adjacent or are somehow "close" to each other. Suppose that it has been determined that, for some $k \in \mathbb{N}$, if a disaster or failure of some kind occurs at a facility (represented by a vertex $v$, say), then all facilities represented by vertices at distance at most $k-1$ from $v$ will be jeopardized. The problem at hand now is to select the smallest possible subset of $V(G)$ so that, if our essential material is stored in the facilities corresponding to this subset, then our system, in the most economical way, has the property that our material, or information, is retrievable from somewhere in the system even in the case when an arbitrary facility fails. One option, of course, is to design $G$ to have radius at least $k$ and to store all essential data in *each* facility, but this is an expensive option. However, if $\text{rad}(G) \geq k$ and if $S$ is a smallest subset of $V(G)$ with the property that, for each $w \in V(G)$, there exists $w' \in S$ such that $d_G(w, w') \geq k$, then selecting the $|S|$ facilities represented by $S$ as the set of facilities at which to store our essential data will produce a choice that may be considerably cheaper, but which still provides the required security. In this paper, we will consider the case where $\text{rad}(G) = k$.

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Let $G$ be a (connected) graph, $a \in V(G)$ and $A, S \subseteq V(G)$. We define the generalized *$S$-eccentricity of $a$ in $G$*, $e_G(a, S)$, by

$$e_G(a, S) = \max\{d_G(a, s); \quad s \in S\}.$$

If $w$ is a vertex in $S$ for which $e_G(a, S) = d_G(a, w)$, we will call $w$ an *$S$-eccentric vertex of $a$*. We define $\mathrm{rad}(A, S, G)$, the *radius of $A$ with respect to $S$ in $G$*, by

$$\mathrm{rad}(A, S, G) = \min\{e_G(a, S); \quad a \in A\}.$$

If $S$ is such that $\mathrm{rad}(A, S, G) = \mathrm{rad}(G)$, then $S$ is called an *$A$-rad($G$)-forcing set*; the size of a smallest $A$-rad($G$)-forcing set is denoted by $\mathrm{rf}(A, G)$ and called the *$A$-rad($G$)-forcing number.* If $\mathrm{rad}(V(G), S, G) = \mathrm{rad}(G)$, then (briefly) $S$ is a *rad($G$)-forcing set*, or simply a *radius-forcing set* if no ambiguity is possible; the size of a smallest rad($G$)-forcing set, denoted by $\mathrm{rf}(G)$, is called the *radius-forcing number of $G$*. Also, we abbreviate $\mathrm{rad}(V(G), S, G)$ by $\mathrm{rad}(S, G)$. (Notice that $\mathrm{rf}(G)$ can be seen as the smallest number of vertices in a subset $S$ of $V(G)$ such that each vertex of $G$ is at distance at least $\mathrm{rad}(G)$ from some vertex in $S$.)

In [4], Fajtlowicz introduced the class of graphs called *$r$-ciliates* and the following notion of $r$-criticality.

**Definition 1.** For $a, b \in \mathbf{N}$ with $b \geq 3$, let $C_{b,a}$ be a graph obtained from $b$ disjoint copies of $P_{a+1}$ by linking together one end-vertex of each in a cycle $C_b$. For $r, a \in \mathbf{N}$ with $r \geq a$, the graphs $C_{2a,r-a}$ are called *$r$-ciliates.* A graph is *$r$-critical* if it has radius $r$ and every proper induced connected subgraph has radius strictly smaller than $r$.

Finally, for a connected graph $G$ of radius $r$, we define the graph $G^*$ to be the graph given by $V(G^*) = V(G)$ and $uv \in E(G^*)$ if and only if $d_G(u, v) \geq r$. Notice that this graph $G^*$ provides a link between total domination and radius-forcing number since, by the definition of rf, $\gamma_t(G^*) = \mathrm{rf}(G)$. Furthermore, it is not difficult to see that $G^* = \overline{G^{\mathrm{rad}(G)-1}}$. (This graph is a generalization, in a sense, of the antipodal graph $A(G)$ of a graph $G$ defined by R. R. Singleton [6], where $A(G) \subset G^*$ and $uv \in E(A(G))$ if and only if $d_G(u, v) = \mathrm{diam}(G)$.)

**Examples 1.**

1. *The trivial graph is the only graph having radius-forcing number equal to 1.*

2. *Any graph having radius 1 has radius-forcing number equal to 2.*

3. *If $G \cong K_{m,n}, 2 \leq m \leq n$, with partite sets $V_1$ and $V_2$, then, for $S \subseteq V(G)$ such that $|S \cap V_i| \geq 2$ for $i \in \{1, 2\}$ we have* $\mathrm{rad}(S, G) = 2 = \mathrm{rad}(G)$, *whereas* $\mathrm{rad}(S, G) \leq 1$ *if $|S \cap V_i| \leq 1$ for some $i \in \{1, 2\}$. So,* $\mathrm{rf}(G) = 4$.

4. *If $G$ is a graph with* $\mathrm{rad}(G) = 1$, *then $G^* \cong K_{p(G)}$. If $G$ is a graph of radius 2, then $G^* = \overline{G}$. For $n \in \mathbf{N}$, $C_{2n}^* = nK_2$ $(n \geq 2)$ and $C_{2n+1}^* = C_{2n+1}$.*

Some preliminary results are listed in the following.

**Proposition 2.** *Let $G$ be a connected graph and $\emptyset \neq S \subseteq V(G)$. Then*

*(1)* $\mathrm{rad}(S, G) \leq \mathrm{rad}(G)$.

*(2)* $\mathrm{rad}(S, G) \leq \mathrm{rad}(\langle S \rangle_G)$.

*(3)* $\mathrm{rad}(S, G) \leq \mathrm{rad}(S, H)$ *for a connected subgraph $H$ of $G$ with $S \subseteq V(H)$.*

*(4) For any $T$, $S \subseteq T \subseteq V(G)$, $T$ is a radius-forcing set of $G$ if $S$ is a radius-forcing set of $G$.*

That there is a fundamental difference between $\mathrm{rad}(S, G)$ and $\mathrm{rad}(\langle S \rangle_G)$ can be easily shown by simple constructions of graphs $G$ where $\mathrm{rad}(S, G)$ and $\mathrm{rad}(\langle S \rangle_G)$ (and hence their difference) can be prescribed for selected sets $S \subseteq V(G)$.

## 2. The radius-forcing number of a graph

As we shall see in Section 3, the computation of $\mathrm{rf}(G)$ is an NP-complete problem. Hence, one cannot expect a simple characterization of graphs with given radius-forcing number. Graphs with radius-forcing number 2, however, can easily be characterized.

**Theorem 3.** *For any connected, nontrivial graph $G$,* $\mathrm{rf}(G) = 2$ *if and only if* $\mathrm{diam}(G) \geq 2\,\mathrm{rad}(G) - 1$.

*Proof.* Let $G$ be a non-trivial, connected graph. Suppose first that $\mathrm{diam}(G) \geq 2\,\mathrm{rad}(G) - 1$. Let $s_1$, $s_2 \in V(G)$ with $d_G(s_1, s_2) = \mathrm{diam}(G)$. Then, for any $w \in V(G)$,

$$d_G(s_1, w) + d_G(s_2, w) \geq d_G(s_1, s_2) = \mathrm{diam}(G) \geq 2\,\mathrm{rad}(G) - 1,$$

so that at least one of $d_G(s_1, w)$, $d_G(s_2, w)$ is at least $\mathrm{rad}(G)$, and thus $\{s_1, s_2\}$ is a radius-forcing set of $G$, and $\mathrm{rf}(G) \leq 2$. Since $G$ is connected and non-trivial, the desired result follows.

For the converse, let $S = \{s_1, s_2\}$ be a minimum $\mathrm{rad}(G)$-forcing set. Of course, for all $w \in V(G)$, $e_G(w, S) = \max\{d_G(w, s_1), d_G(w, s_2)\} \geq \mathrm{rad}(G)$. Let $P : (s_1 =) x_0, x_1, \ldots, x_m (= s_2)$ be a shortest $s_1 - s_2$ path. Then, for all $i \in \{0, 1, \ldots, m\}$, $\max\{d_G(x_i, x_0), d_G(x_i, x_m)\} \geq \mathrm{rad}(G)$; i.e., $\max\{i, m - i\} \geq \mathrm{rad}(G)$ for all $i \in \{0, \ldots, m\}$. So $\left\lceil \frac{m}{2} \right\rceil = \max\left\{\left\lceil \frac{m}{2} \right\rceil, m - \left\lceil \frac{m}{2} \right\rceil\right\} \geq \mathrm{rad}(G)$, whence we obtain $\mathrm{diam}(G) \geq m \geq 2\,\mathrm{rad}(G) - 1$. □

**Corollary 4.** *For every non-trivial tree $T$,* $\mathrm{rf}(T) = 2$.

**Proposition 5.** *Every non-trivial interval graph has radius-forcing number 2.*

*Proof.* Let $G$ be an interval graph and let $[a(v), b(v)]$ be the interval corresponding to the vertex $v$. Let $v'$, $v''$ be such that $b(v') = \min\{b(w) | w \in V(G)\}$ and $a(v'') = \max\{a(w) | w \in V(G)\}$. Then, for every vertex $v \in V(G)$, either $v'$ or $v''$ is an eccentric vertex of $v$. Hence, $\{v', v''\}$ is a radius-forcing set of $G$. □

Using Theorem 3, we can quickly calculate $\mathrm{rf}(P)$ for the Petersen graph $P$: $\mathrm{rad}(P) = 2 = \mathrm{diam}(P) = 2\,\mathrm{rad}(P) - 2$ shows that $\mathrm{rf}(G) \geq 3$. If $I$ is a maximum independent set of one of the 5-cycles $C$ of $P$, then $V(C) - I$ is a radius-forcing set of $P$, whence $\mathrm{rf}(P) = 3$.

A characterization of graphs having radius-forcing number 3 appears to be difficult. It is true, however, that a graph with radius-forcing number 3 can have arbitrarily large radius $r$ and maximum possible diameter $2r - 2$ (see Theorem 3); in fact, the diameter of a graph $H$ with $\mathrm{rf}(H) = 3$ and radius $r$ can be $2r-2$ or arbitrarily smaller rhan $2r - 2$, as Proposition 6 shows. On the other hand, having diameter $2r - 2$ and radius $r$ is not a sufficient condition for a graph to have radius-forcing number 3, as Proposition 8 shows. Furthermore, that having radius-forcing number 3 does not force a graph to have small girth is a consequence of Proposition 7, which shows that arbitrarily large girths (of odd parity) are possible.

**Proposition 6.** *Given any $a \in \mathbb{N}$, $a \geq 2$, there exists a graph $G$ with $\mathrm{rf}(G) = 3$ and $\mathrm{diam}(G) = 2\,\mathrm{rad}(G) - a$.*

*Proof.* Given $a \in \mathbb{N}$ with $a \geq 2$, let $b \in \mathbb{N}$ with $b \geq \frac{a}{2}$. Construct a graph $G$ from the cycle $C_{3a} : v_0, v_1, \ldots, v_{3a-1}, v_0$ and four additional vertices $x$, $u$, $v$, and $w$ by joining the vertices $u$, $v$ and $w$ to $v_0$, $v_a$, and $v_{2a}$, respectively, with paths $P_{u,0}$, $P_{v,a}$, and $P_{w,2a}$, respectively, of length $b$, and by joining $x$ to the vertices $v_0$, $v_a$, $v_{2a}$ by paths $P_{x,0}$, $P_{x,a}$, and $P_{x,2a}$, respectively, of length $a$, so that $P_{u,0}$, $P_{v,a}$, $P_{w,2a}$, $P_{x,0}$, $P_{x,a}$, and $P_{x,2a}$ are mutually internally disjoint. Then, $\mathrm{rad}(G) = a + b$ and $\mathrm{diam}(G) = a + 2b = 2\,\mathrm{rad}(G) - a \leq 2\,\mathrm{rad}(G) - 2$, whence $\mathrm{rf}(G) \geq 3$. However, $\mathrm{rad}(\{u, v, w\}, G) = \mathrm{rad}(G)$; so, $\mathrm{rf}(G) = 3$. □

**Proposition 7.** *For any $r \in \mathbb{N}$, $r \geq 3$, let $G$ be obtained by $r - 1$ subdivisions of each spoke of the wheel with $2n$ outer vertices (so there are $r + 1$ vertices (in total) on each spoke), where $n = 2r - 3$ or $n = 2r - 4$. Then, $\mathrm{rad}(G) = r$, $g(G) = 2r + 1$ and $\mathrm{rf}(G) = 3$.*

*Proof.* Let $r, n \in \mathbb{N}$ and $G$ be as defined above. Denote the centre of the wheel by $u$ and its outer vertices by $v_1, v_2, \ldots, v_{2n}$. Let $w_i$ be the neighbour of $v_i$ on the subdivided spoke of the wheel. It is easy to verify that $\{u, v_1, w_n\}$ is a radius-forcing set whence $\mathrm{rf}(G) \leq 3$. Since $\mathrm{diam}(G) = n \leq 2r - 3 = 2\ \mathrm{rad}(G) - 3$, Theorem 3 implies $\mathrm{rf}(G) \geq 3$. □

**Proposition 8.** *There exists an infinite class of graphs $G$ with $\mathrm{diam}(G) = 2\ \mathrm{rad}(G) - 2$ and $\mathrm{rf}(G) > 3$.*

*Proof.* Let $r \in \mathbb{N}$ with $r \geq 3$ and let $G$ be a graph obtained from the disjoint union of a $2r$-cycle, $C : u_1, u_2, \ldots, u_{2r}, u_1$, and a path of order $2r - 3$, $P : v_1, v_2, \ldots, v_{2r-3}$, by identifying the vertices $u_3$ and $v_{r-1}$. We note that $e_G(u_i) = r$ for $i \in \{1, 2, 3, \ldots, 5\}$ and that $e_G(w) > r$ for $w \in V(G) - \{u_1, u_2, \ldots, u_5\}$; so $\mathrm{rad}(G) = r$ and $\mathrm{diam}(G) = r + (r - 2) = 2r - 2$. Furthermore, each of the vertices $u_2, u_3, u_4$ has a unique eccentric vertex in $G$, namely $u_{2+r}, u_{3+r}$ and $u_{4+r}$, respectively. Hence, if $S$ is a minimum radius-forcing set of $G$, then, as $e_G(u_i, S) \geq r$ for $i \in \{2, 3, 4\}$, it follows that $u_{2+r}, u_{3+r}, u_{4+r} \in S$; furthermore, as $e_G(u_{i+r}, S) \geq r \geq 3$ for $i \in \{2,3,4\}$, $S - \{u_{2+r}, u_{3+r}, u_{4+r}\}$ contains at least one vertex and so $\mathrm{rf}(G) = |S| \geq 4$. (More specifically, $\mathrm{rf}(G) = 4$ follows from the observation that $\{u_{r+2}, u_{r+3}, u_{r+4}, v_1\}$ is a radius-forcing set of $G$.) $\square$

Before moving on from considering graphs which have radius-forcing number three, we ask the question whether any graph $H$ having $\mathrm{rf}(H) = 3$ satisfies $\mathrm{diam}(G) = 2\,\mathrm{rad}(G) - 2$ (by Proposition 8, this condition is, of course, not sufficient).

In [4], Fajtlowicz proved that a graph is $r$-critical if and only if it is an $r$-ciliate.

**Proposition 9.** *Let $G$ be a radius-critical graph that is neither a path nor a cycle. Then, $G \cong C_{2a,r-a}$ for some $a, r \in \mathbb{N}$, $2 \leq a < r$ and $\mathrm{rf}(G) = 2a$.*

*Proof.* Let $a, r \in \mathbb{N}$ with $2 \leq a < r$ and let $G \cong C_{2a,r-a}$. It is easy to verify that, in every radius-forcing set of $G$, each vertex can be replaced by the closest end-vertex. Hence, there is a minimum radius-forcing set containing only end-vertices. On the other hand, no proper subset of the end-vertices is a radius-forcing set. $\square$

Having considered graphs of minimum possible radius-forcing number, we now turn to the graphs having maximum possible radius-forcing number.

A graph $G$ is a *unique eccentric vertex graph* if every vertex $v \in V(G)$ has exactly one $V(G)$-eccentric vertex $w \in V(G)$, i.e. if for every $v \in V(G)$ there exists exactly one vertex $w \in V(G)$ with $d_G(v, w) = e_G(v, V(G))$.

**Theorem 10.** *A graph $G$ satisfies $\mathrm{rf}(G) = p(G)$ if and only if $G$ is a self-centred unique eccentric vertex graph.*

*Proof.* Let $G$ be a graph with $\mathrm{rf}(G) = p = p(G)$. Since any connected graph $F$ of order at least 3 satisfies $\gamma_t(F) \leq p(F) - 1$, it follows that $G^* \cong nK_1 \cup mK_2$ for some non-negative integers $m, n$. But $G^*$ has no isolated vertex. So, $G^* \cong \frac{p}{2}K_2$ and, for every vertex $v \in V(G)$, there is only one vertex $G$ at distance at least $\mathrm{rad}(G)$ from $v$. Hence, every vertex has a unique eccentric vertex and $G$ is self-centred.

Conversely, suppose $G$ is a self-centred, unique eccentric vertex graph. Then, for any vertex $v$ of $G$, there is exactly one vertex $v^*$ that is at distance at least $\mathrm{rad}(G)$ from $v$. So, in $G^*$, every vertex has degree 1. So, $G^* = mK_2$ for some $m \in \mathbb{N}$ and $\mathrm{rf}(G) = \gamma_t(G^*) = p(G)$, as required. $\square$

Now, considering the statement of Theorem 3 that a graph $G$ has $\mathrm{rf}(G) = 2$ if and only if $\mathrm{diam}(G) \geq 2\,\mathrm{rad}(G) - 1$, and the statement of Theorem 10, one may be inclined to believe that, relative to its order, a graph's radius-forcing number is large if the diameter is "close" to the radius. However, for $k, n \in \mathbb{N}$ $(n \geq 2)$, the graph $F$ which is the lexicographic product $C_{2n}[K_k]$ of $C_{2n}$ and $K_k$ is such that $\mathrm{rf}(F) = \mathrm{rf}(C_{2n}) = 2n$ and $p(F) = 2kn$, i.e., $\frac{\mathrm{rf}(F)}{p(F)} = \frac{1}{k}$, while $\mathrm{diam}(F) = n = \mathrm{rad}(F)$.

That the simple operation of subdivision of an edge can have the effect of almost halving the radius-forcing number of a graph is illustrated by Proposition 5. That the contraction of an edge can produce a graph with a radius-forcing number that is an arbitrarily large factor smaller than the the radius-forcing number of the original graph is seen as follows: If $n \geq 2$ is an integer, $G \cong K_{2n}$, $F$ is a perfect matching of $G$, $H = G - F$, and $V(H) = A \cup B$ such that $\langle A \rangle_H \cong \langle B \rangle_H \cong K_n$, then the contraction of any edge $e$ of $\langle A \rangle_H$ or $\langle B \rangle_H$ yields a graph $G'$ having $\text{rf}(G') = 2$, while $H$, being a self-centred, unique eccentric vertex graph, satisfies $\text{rf}(H) = 2n$.

**Proposition 11.** *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} \text{rf}(C_{2n+1}) &= n+1 \\ \text{rf}(C_{2n}) &= 2n, \quad n \geq 2. \end{aligned}$$

*Proof.* Let $n \in \mathbb{N}$. For $n \geq 2$, that $\text{rf}(C_{2n}) = 2n$ follows immediately from Theorem 10. Since $C_{2n+1}$ is self-centred and $C^*_{2n+1} \cong C_{2n+2}$ and $\gamma_t(C_{2n+1}) = n+1$, $\text{rf}(C_{2n+2}) = n+1$. □

**Lemma 12.** *If $G$ is a connected graph with $p(G) \geq 4$, then $\gamma_t(G) \leq p(G) - 2$.*

**Lemma 13.** *Let $G$ be a connected graph of radius $r \geq 2$ and order $p$ with $\text{rf}(G) = p - 1$. Then $p$ is odd and $V(G) = \{u, v, w\} \cup \{x_{1i}, x_{2i};\quad i = 1, 2, \ldots, \frac{p-3}{2}\}$, where*

*(i) $d_G(u, v) = d_G(u, w) = r$, $d_G(v, w) \leq r$,*

*(ii) $d_G(y, x_{ji}) < r$ for $y \in \{u, v, w\}, j \in \{1, 2\}, i \in \{1, 2, \ldots, \frac{p-3}{2}\}$,*

*(iii) $d_G(x_{1i}, x_{2i}) = r$, $i = 1, 2, \ldots, \frac{p-3}{2}$.*

*Proof.* Let $G$ be a graph of order $p$ having $\text{rf}(G) = p - 1$. Then, no component of $G$ has order more than three (by Lemma 12. Furthermore, at most one component of $G^*$ has order three since any connected graph of order three has total domination number two. So, $G^* = \frac{p}{2}K_2$ or $G^* \cong \frac{p-3}{2}K_2 \cup P_3$ or $G^* \cong \frac{p-3}{2}K_2 \cup K_3$. However, $\gamma_t(\frac{p}{2}K_2) = p \neq \text{rf}(G)$, and the desired result follows. □

We can now describe all (connected) graphs $G$ having $\text{rf}(G) = p(G) - 1$.

**Theorem 14.** *Let $G$ be a connected graph of order $p$ with $\text{rf}(G) = p(G) - 1$.*

*(1) If $\text{rad}(G) = 1$, then $G \cong K_3$ or $G \cong K_{1,2}$.*

*(2) If $\text{rad}(G) = 2$, then, for $H$ the complete $\frac{p-1}{2}$-partite graph $K(3, 2, 2, \ldots, 2)$, we have $G \cong H$ or $G \cong H + e$ where $e \in E(\bar{H})$ joins two vertices in the partite set of cardinality 3.*

*(3) If $\text{rad}(G) \geq 3$, then $V(G) = \{u, v, w\} \cup \{x_{1i}, x_{2i};\ i = 1, 2, \ldots, \frac{p-3}{2}\}$ where*

*(i) $d_G(u, v) = d_G(u, w) = r$, $d_G(v, w) \leq r$,*

*(ii) $d_G(y, x_{ji}) < r$ for $y \in \{u, v, w\}$, $j \in \{1, 2\}$, $i \in \{1, 2, \ldots, \frac{p-3}{2}\}$,*

(iii) $d_G(x_{1i}, x_{2i}) = r$, $i \in \{1, 2, \ldots, \frac{p-3}{2}\}$.

*Proof.* Since $\mathrm{rf}(H) = 2$ for any graph $H$ having radius 1, (1) follows immediately. Statement (3) holds by Lemma 13. Let $G$ be a connected graph of radius two. By Lemma 13, $V(G) = \{u, v, w\} \cup \{x_{1i}, x_{2i};\ 1 \le i \le \frac{p-3}{2}\}$ where $d_G(u, v) = d_G(u, w) = 2$, $d_G(v, w) \in \{1, 2\}$, each of $u, v$ and $w$ is adjacent to every vertex of $V(G) - \{u, v, w\}$ and, for each $i$, $1 \le i \le \frac{p-3}{2}$, $x_{1i}$ (respectively, $x_{2i}$) is adjacent to each vertex of $V(G) - \{x_{2i}\}$ (respectively, $V(G) - x_{1i}\}$). Clearly, (2) holds. □

We conclude this section with four bounds on rf. Based on the observation that $\mathrm{rf}(G) = \gamma_t(G^*) \le \frac{2}{3}p(G^*)$ (see [2]) for any connected graph of order at least three, it follows that $\mathrm{rf}(G) \le \frac{2}{3}p(G)$ whenever $G$ is a connected graph of order at least three, having no vertex with a unique eccentric vertex. Three lower bounds are given next.

**Proposition 15.** *For a connected graph $G$ of order $p$, finite radius $r \ge 2$, minimum degree $\delta$, and connectivity $\kappa$,*

1. $\mathrm{rf}(G) \ge \left\lceil \frac{p}{p-1-(r-1)\kappa} \right\rceil$

2. $\mathrm{rf}(G) \ge \begin{cases} \left\lceil \frac{p}{p-[\delta-2]\lfloor\frac{r-1}{3}\rfloor-r} \right\rceil & \text{if } r \ge 4, \\ \left\lceil \frac{p}{p-\delta-2} \right\rceil & \text{if } r = 3, \\ \left\lceil \frac{p}{p-\delta-1} \right\rceil & \text{if } r = 2. \end{cases}$

3. $\mathrm{rf}(G) \ge \left\lceil \frac{p}{t} \right\rceil$ *where* $t = \max\{|\{y \in V(G); d_G(y, v) \ge r\}|; v \in V(G)\}$.

*Proof.* Let $G$, $p$, $r$, $\delta$ and $\kappa$ be as described above. Let $v \in V(G)$ and $A_i = \{y \in V(G); d_G(y, v) = i\}$ for $i$, $1 \le i \le e_G(v)$. Clearly, $N_{G^*}(v) = \bigcup_{i=r}^{e_G(v)} A_i$ so that $\deg_{G^*} v = p - 1 - \sum_{i=1}^{r-1} |A_i|$. Observing that $|A_i| \ge \kappa$ for $1 \le i \le r-1$, we have $\Delta(G^*) \le p - 1 - (r-1)\kappa$ and so

$$\mathrm{rf}(G) = \gamma_t(G^*) \ge \left\lceil \frac{p}{\Delta(G^*)} \right\rceil \ge \left\lceil \frac{p}{p-1-(r-1)\kappa} \right\rceil.$$

Moreover, observing that, for any $j$, $2 \le j \le r-2$, $|A_{j-1} \cup A_j \cup A_{j+1}| \ge \delta + 1$, we have, for $r \ge 4$, that $\Delta(G^*) \le p - 1 - \{\lfloor\frac{r-1}{3}\rfloor(\delta+1) + r - 1 - 3\lfloor\frac{r-1}{3}\rfloor\} = p - (\delta-2)\lfloor\frac{r-1}{3}\rfloor - r$, whence

$$\mathrm{rf}(G) \ge \left\lceil \frac{p}{p-(\delta-2)\lfloor\frac{r-1}{3}\rfloor - r} \right\rceil.$$

For $r = 3$, $\Delta(G^*) \le p - 1 - (\delta+1) = p - \delta - 2$ so that $\mathrm{rf}(G) \ge \left\lceil \frac{p}{p-\delta-2} \right\rceil$; and for $r = 2$, $\Delta(G^*) \le p - 1 - \delta$ so that $\mathrm{rf}(G) \ge \left\lceil \frac{p}{p-\delta-1} \right\rceil$. Result 3 follows from the fact that $\Delta(G^*) = \max\{|A_v|; v \in V(G)\}$, where $A_v = \{y \in V(G); d_G(y, v) \ge r\}$ and $\mathrm{rf}(G) \ge \left\lceil \frac{p(G)}{\Delta(G^*)} \right\rceil$. □

Consideration of the even cycles shows that the first bound in Proposition 15 is sharp. To show that the next three are also sharp, let $k$ and $\delta$ be positive integers with $\delta \geq 2$, and consider the graph $G$ obtained from a path $P : v_1, v_2, \ldots, v_{6k}$ by the replacement of each of the vertices $v_{2+3i}$ $(0 \leq i \leq 2k-1)$ by a graph $G_{2+3i} \cong K_{\delta-1}$, the deletion of the edges $v_{1+3i}v_{2+3i}$ and $v_{2+3i}v_{3+3i}$, the addition of the edges $av_{1+3i}$, $av_{2+3i}$ for all $a \in V(G_{2+3i})$, the addition of two new vertices $u$ and $v$, where $u$ is joined to $v_1$ and to every vertex of $V(G_2)$, and where $v$ is joined to $v_{6k}$ and to every vertex of $V(G_{6k-1})$. Then, $\mathrm{rad}(G) = 3k$, $\mathrm{diam}(G) = 6k-1$ (whence $\mathrm{rf}(G) = 2$), $\delta(G) = \delta$, and $p(G) = 2k\delta+2k+2$. If $k \geq 2$ (so that $\mathrm{rad}(G) > 3$), then Proposition 15 gives $\mathrm{rf}(G) \geq \left\lceil \frac{p}{p-[\delta-2]\lfloor\frac{r-1}{3}\rfloor-r} \right\rceil = \left\lceil \frac{2\delta+2+\frac{2}{k}}{\delta+1+\frac{\delta}{k}} \right\rceil$, where $\left\lceil \frac{2\delta+2+\frac{2}{k}}{\delta+1+\frac{\delta}{k}} \right\rceil \to 2$ as $k \to \infty$. If $k = 1$ (so that $\mathrm{rad}(G) = 3$), Proposition 15 gives $\mathrm{rf}(G) \geq \left\lceil \frac{p}{p-\delta-2} \right\rceil = 2$. Finally, if $H$ is the graph obtained from $G$ by the deletion of the set $\{v_4\} \cup V(G_5) \cup \{v_6\} \cup \ldots \cup V(G_{6k-4}) \cup \{v_{6k-3}\}$ of vertices and the identification of the vertices $v_3$ and $v_{6k-2}$, then $\mathrm{rad}(H) = 2$, $\mathrm{diam}(G) = 4$, $\mathrm{rf}(H) = 2$, $\delta(H) = \delta$ and $p(H) = 2\delta+3$, and Proposition 15 gives $\mathrm{rf}(G) \geq \left\lceil \frac{p}{p-\delta-1} \right\rceil = \left\lceil \frac{2\delta+3}{p-\delta-1} \right\rceil = 2$.

## 3. NP-Completeness considerations

It would be very interesting to characterize the class of graphs $G^*$ (for a connected graph $G$) since, if this class is "large enough," the decision problem RF (see below) associated with $\mathrm{rf}(G)$ would perhaps likely be NP-complete (since determining $\mathrm{rf}(G)$ is essentially determining $\gamma_t(G^*)$ and the total domination problem is NP-complete). Unfortunately, the problem of characterizing the graphs $G^*$ seems to be very difficult, since it is related to the problem of characterizing powers of graphs (see Section 1). Fortunately, that the problem of total domination for bipartite graphs is NP-complete is sufficient to show the NP-completeness of RF.

**Definition 2.** We define the Radius-Forcing Number Problem RF as follows:

INSTANCE: A connected graph $G$, integer $M \geq 1$.

QUESTION: Is $\mathrm{rf}(G) \leq M$?

**Theorem 16.** *RF is NP-complete.*

*Proof.* That RF is in NP follows from the fact that it can be efficiently verified whether a given set of vertices of a connected graph is a radius-forcing set of the graph.

The problem of computing the total domination number for bipartite graphs is NP-complete ([5]). We shall show that RF is NP-complete by showing that BTD is reducible in polynomial time to RF, where BTD shall refer to the problem "Given a non-complete bipartite graph $B$ (without isolated vertices) and a positive integer $M$, is $\gamma_t(B) \leq M$?"

Let $B$ be any non-complete bipartite graph without isolated vertices with partite sets $V_1$ and $V_2$, and let $M$ be a positive integer. Let $G = \overline{B}$ (we can construct $G$

in polynomial time). Notice that, since $B$ is non-complete, $G$ is connected and has radius 2. Hence, by definition of the graph $G^*$, $B = G^*$ and thus $\gamma_t(B) = \mathrm{rf}(G)$. $\square$

## 4. Randomly k-forcing graphs

We refer the reader to the motivation provided in Section 1 where we discussed the selection of a smallest set of facilities at which to store material to ensure the survival of that material in the event of a disaster occurring at any one of the facilities. Imagine now the situation where the time and cost of finding such a set is sufficiently high to warrant re-evaluation by management of this method of ensuring security (after all, RF is NP-complete). In other words, suppose that there are other factors more important than the size of our security-ensuring collection of facilities. The question is, does there exist a number $k$ such that *every* subset of $V(G)$ of size $k$ is a radius-forcing set (where $G$ is, again, the graph that models our system of facilities). If such a number $k$ exists, and is not too much bigger than $\mathrm{rf}(G)$, then those other factors can be allowed to determine where our material is stored. Clearly, picking the smallest such $k$ is the most cost-effective. A formal definition is as follows.

**Definition 3.** We call a connected graph $G$ a *randomly $k$-forcing graph* ($k \in \mathrm{N}$) if $\mathrm{rad}(S, G) = \mathrm{rad}(G)$ for every $S \subseteq V(G)$ with $|S| = k$ (i.e., every $k$-set of $V(G)$ is a radius-forcing set of $G$).

Notice that every connected graph $G$ is a randomly $p(G)$-forcing graph, which justifies the following definition.

**Definition 4.** For a connected graph $G$, let $\mathrm{RF}(G)$, the *randomly forcing number of $G$*, denote the smallest $k$ for which $G$ is a randomly $k$-forcing graph.

*Observation.*

1. For all connected graphs $G$, $\mathrm{rf}(G) \leq \mathrm{RF}(G) \leq p(G)$.
2. For all connected graphs $G$ and $\ell \in \mathrm{N}$, $\mathrm{RF}(G) \leq \ell \leq p(G)$, $G$ is randomly $\ell$-forcing.
3. For a connected graph $G$,

$$\mathrm{RF}(G) = 1 + \max\{\ell \in \mathrm{N}; \exists T \subseteq V(G), |T| = \ell, \mathrm{rad}(T, G) < \mathrm{rad}(G)\}.$$

**Proposition 17.** *For any connected subgraph $H$ of $G$ satisfying* $\mathrm{rad}(H) < \mathrm{rad}(G)$, $\mathrm{RF}(G) > p(H)$.

*Proof.* For $G$ and $H$ satisfying the hypothesis of the proposition and $S \subseteq V(H)$,

$$\mathrm{rad}(S, G) \leq \mathrm{rad}(S, H) \leq \mathrm{rad}(H) < \mathrm{rad}(G).$$

So, $\mathrm{RF}(G) > \max\{\,|S|\,; S \subseteq V(H)\} = p(H)$. $\square$

**Corollary 18.** *For a connected graph $G$ of order $p$, radius $r \in \mathbf{N}$ and maximum degree $\Delta$,*

$$\mathrm{RF}(G) \geq p - \Delta(\Delta - 1)^{r-1} + 1.$$

*Proof.* Let $G$ be a connected graph of order $p$, finite radius $r$ and maximum degree $\Delta$. Construct a breadth first search tree $T$ rooted at any central vertex $c$ of $G$, and let $L$ be the leaves of $T$ that are the eccentric vertices of $c$. Then, $\mathrm{rad}(G-L) = r-1$. So, by Proposition 17,

$$\mathrm{RF}(G) > p - |L| \geq p - \Delta(\Delta - 1)^{r-1}.$$

□

The bound given by the above proposition is best possible since it is attained by any $\Delta$-ary tree.

In [3], Erdős, Saks and Sós proved that every connected graph of radius $r$ contains a path $P_{2r-1}$ as an induced subgraph, whence the following.

**Corollary 19.** *If $G$ is a connected graph, then $\mathrm{RF}(G) \geq 2\,\mathrm{rad}(G)$.*

**Examples 20.**

1. *For $n \in \mathbf{N}$, $\mathrm{rad}(P_{2n-1}) < \mathrm{rad}(C_{2n+1}) = n$, so that $\mathrm{RF}(C_{2n+1}) \geq 2n$; obviously, $\mathrm{RF}(C_{2n+1}) = 2n$. Since $\mathrm{rf}(C_{2n}) = 2n$, $\mathrm{RF}(C_{2n}) = 2n$ follows trivially.*

2. *Any graph $G$ of radius 1 has $\mathrm{RF}(G) = 2$.*

3. *If $v$ is an end-vertex of an $r$-ciliate $C_{2a,r-a}$ $(2 \leq a < r)$, then $\mathrm{rad}(C_{2a,r-a} - v) < \mathrm{rad}(C_{2a,r-a})$, so that $\mathrm{RF}(C_{2a,r-a}) > p(C_{2a,r-a} - v)$ and it follows that $\mathrm{RF}(C_{2a,r-a}) = p(C_{2a,r-a})$.*

4. *For $n \in \mathbf{N}$, $\mathrm{rad}(P_{2n-1}) = n - 1 < n = \mathrm{rad}(P_{2n})$, so that $\mathrm{RF}(P_{2n}) > 2n - 1$, and $\mathrm{RF}(P_{2n}) = p(P_{2n})$ follows. Furthermore, $\mathrm{rad}(P_{2n}) = n = \mathrm{rad}(P_{2n+1})$, while $\mathrm{rad}(P_{2n-1}) = n - 1 < \mathrm{rad}(P_{2n+1})$, whence $\mathrm{RF}(P_{2n+1}) \geq 2n$. However, it is easy to see that any $2n$-set of $V(G)$ is a radius-forcing set of $P_{2n+1}$. So, $\mathrm{RF}(P_{2n+1}) = p(P_{2n+1}) - 1$.*

Obviously, a graph $G$ being randomly $\mathrm{RF}(G)$-forcing does not imply $\mathrm{rf}(G) = \mathrm{RF}(G)$, which leads naturally to the problem of determining which graphs $G$ *do* satisfy $\mathrm{rf}(G) = \mathrm{RF}(G)$.

**Proposition 21.** *A connected graph $G$ is a randomly $a$-forcing graph of order $p$ with $a = \mathrm{rf}(G)$ if and only if $a = p$ or $a = 2 < p$ and $\mathrm{rad}(G) = 1$.*

*Proof.* Let $G$ be a connected graph of order $p$. If $\mathrm{rf}(G) = p$, then obviously $G$ is randomly $\mathrm{rf}(G)$-forcing. Otherwise, if $\mathrm{rad}(G) = 1$, then $\mathrm{rf}(G) = 2$ and every pair of distinct vertices of $G$ form a radius-forcing set, so that $G$ is randomly $\mathrm{rf}(G)$-forcing. Conversely, suppose that $G$ is randomly $a$-forcing graph with $a = \mathrm{rf}(G)$. Suppose $a < p$. Then, every $a$-set of $V(G^*)$ is a minimum total dominating set of $G^*$. Let

$D$ be a minimum total dominating set of $G$; let $J = \langle D \rangle_{G^*}$. Suppose $J$ contains a path of length greater than one; let $P : x_1, x_2, \ldots, x_k$ $(k \geq 3)$ be a longest path in $J$. Then, $N_J(x_1) \subseteq V(P)$ and $x_1$ has no private neighbour in $V(P)$, so that $x_1$ must have a private neigbour $y$ (say) in $V(G^*) - D$. Then, $D' = (D - \{x_1\}) \cup \{y\}$ is not a total dominating set (since $y$ has no neighbour in $D'$); however, this contradicts the fact that $|D'| = a$. Hence, $J$ contains precisely paths of length one. So, $\langle A \rangle_{G^*} \cong \frac{a}{2} K_2$ for every $a$-set $A$ in $V(G^*)$.

**Case 1:** Suppose $a \geq 3$ (and hence $p \geq 4$) and $G^*$ is connected. Then, if $u, v, w$ is a path of length 2 in $G^*$, the set $\{u, v, w\}$ can be extended to an $a$-set $A'$ of $G^*$, where $\delta(\langle A' \rangle_{G^*}) \geq 2$, which is impossible.

**Case 2:** Suppose $a \geq 3$ and $G^*$ is disconnected. Then, by an argument similar to that used in Case 1, it follows that every component of $G^*$ is a copy of $K_2$. However, since $a < p$, there exists an $a$-set $A''$ in $V(G^*)$ that contains a single vertex of some component of $G^*$, so that $\delta(\langle A'' \rangle_{G^*}) = 0$, which is impossible.

**Case 3:** Suppose $a = 2$ (and hence $p \geq 3$). Then, every two vertices of $G^*$ are joined by an edge, so that $G^*$ is complete. Therefore, $\text{rad}(G) = 1$. □

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