Interpolation theorems for the (r, s)-domination number of spanning trees

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Abstract

If G is a graph without isolated vertices, and if r and s are positive integers, then the (r, s)-domination number $\gamma_{r,s}(G)$ of G is the cardinality of a smallest vertex set D such that every vertex not in D is within distance r from some vertex in D, while every vertex in D is within distance s from another vertex in D. This generalizes the total domination number $\gamma_t(G) = \gamma_{1,1}(G)$.

Let $\mathcal{T}(G)$ denote the set of all spanning trees of a connected graph G. We prove that $\gamma_{r,s}(\mathcal{T}(G))$ is a set of consecutive integers for every connected graph G of order at least two when $s \geq 2r + 1$. This is not true if $1 \leq s \leq 2r - 1$, and for $s = 2r$ the problem is open. We prove that $\gamma_{r,2r}(\mathcal{T}(G))$ is a set of consecutive integers for $r = 1$ and we conjecture this also holds for $r \geq 2$. We also prove that $\gamma_{r,s}(\mathcal{T}(G))$ is a set of consecutive integers for every 2-connected graph G and for any two positive integers r and s.

Let G be a simple undirected graph with vertices $V(G)$ and edges $E(G)$. The *neighbourhood* of a vertex v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighbourhood* is $N_G[v] = N_G(v) \cup \{v\}$. For a connected graph G, let $d_G(v, u)$ denote the distance between vertices v and u in G. If S is a set of vertices of G and v is a vertex of G, then $d_G(v, S)$ denotes the distance between v and S, the shortest distance between v and a vertex of S.

Let r and s be two positive integers. A vertex set D of a graph G is an *$(r, -)$-set* of G if $d_G(v, D) \leq r$ for every $v \in V(G) - D$. Similarly, a subset D of $V(G)$ is a $(-,s)$-*set* of G if $d_G(u, D-\{u\}) \leq s$ for every $u \in D$. A subset D of $V(G)$ is an (r,s)-*dominating set* of G if D is both an $(r,-)$-set and a $(-,s)$-set of G. The cardinality of a minimum (r,s)-dominating set in G is called the (r,s)-*domination number* of G and is denoted by $\gamma_{r,s}(G)$. Note that this parameter is only defined for graphs without isolated vertices and if G is a graph without isolated vertices, then $\gamma_{r,s}(G) \geq 2$. The (r,s)-domination number introduced by Mo and Williams [11] is related to other graphical parameters. In particular, the $(1,1)$-domination number $\gamma_{1,1}(G)$ of a graph G is the *total domination number* $\gamma_t(G)$ of G defined by Cockayne, Dawes and Hedetniemi [1]. The (r,r)-domination number was studied in [8] as the *total* $P_{\leq r+1}$-*domination number.* $(r,-)$-sets are also known as *distance* r-*dominating sets* or r-*coverings* (in [10]) and the minimum cardinality of a distance r-dominating set of a graph G is called the *distance* r-*domination number* of G and is denoted by $\gamma_k(G)$.

An invariant π defined for all spanning trees of a connected graph G is said to *interpolate* over G if the set $\pi(\mathcal{T}(G)) = \{\pi(T) : T \in \mathcal{T}(G)\}$ consists of consecutive integers, i.e. $\pi(\mathcal{T}(G))$ is an integer interval. We shall call π an *interpolating function* if π interpolates over each connected graph. The interpolating character of different graphical parameters was investigated in a number of papers. In particular, the interpolation of domination related parameters was studied in [2, 4, 5, 6, 7, 12, 13], to quote a few. In this paper we study the interpolating character of the (r,s)-domination number. The following four lemmas will be useful in our proofs.

Lemma 1 [13]. *An integer-valued graph function π is an interpolating function if and only if π interpolates over every unicyclic graph.*

Lemma 2 [11]. *Let G be a connected graph of order at least two, and let r and s be positive integers. Then $\gamma_{r,s}(G) = \gamma_{r,s}(T)$ for some spanning tree T of G.*

Lemma 3 [13]. *For any positive integer r, the distance r-domination number γ_r is an interpolating function.*

Lemma 4. *If G is a connected graph of order at least two and if r and s are positive integers such that $s \geq 2r+1$, then $\gamma_{r,s}(G) = \max\{2, \gamma_r(G)\}$.*

Proof. Let D be a minimum distance r-dominating set of G. If $|D| = 1$, then for any $x \in V(G) - D$, $D \cup \{x\}$ is a minimum (r,s)-dominating set of G and $\gamma_{r,s}(G) = 2 = \max\{2, \gamma_r(G)\}$. If $|D| \geq 2$, then D is an (r,s)-dominating set in G; for if not, then there is a vertex x in D such that $d_G(x, D-\{x\}) > s \geq 2r+1$ and any shortest path joining x to a vertex of $D-\{x\}$ contains a vertex y for which $d_G(y, D) > r$, which contradicts the fact that D is a distance r-dominating set in G. In addition, since D is a minimum distance r-dominating set of G, D is a minimum (r,s)-dominating set of G and therefore $\gamma_{r,s}(G) = \gamma_r(G) = \max\{2, \gamma_r(G)\}$. □

Theorem 1. *The (r,s)-domination number $\gamma_{r,s}$ is an interpolating function if $s \geq 2r+1$.*

Proof. Since γ_r is an interpolating function (by Lemma 3), $\max\{2, \gamma_r\}$ is an interpolating function. Now, by Lemma 4, $\gamma_{r,s}$ is an interpolating function. □

We now turn our attention to interpolation properties of the (r, s)-domination number $\gamma_{r,s}$ with $1 \le s \le 2r$. For a positive integer r, let G_r be the graph given in Fig. 1. Since G_r is a unicyclic graph, every spanning tree of G_r is an edge-deleted subgraph $G_r - vu$, where vu is an edge of the unique cycle of G_r. One can verify that if $1 \le s \le r$, then $\gamma_{r,s}(G_r - v_i v_{i+1}) = \gamma_{r,s}(G_r - u_i u_{i+1}) = 4$ for each $i = r, \ldots, 2r$, while $\gamma_{r,s}(G_r - v_r u_r) = \gamma_{r,s}(G_r - v_{2r+1} u_{2r+1}) = 6$. Consequently, $\gamma_{r,s}(\mathcal{T}(G_{r,s})) = \{4, 6\}$ and this implies that the (r, s)-domination number $\gamma_{r,s}$ with $1 \le s \le r$ and, in particular, the total domination number $\gamma_t = \gamma_{1,1}$ are not interpolating functions. The next example proves that the (r, s)-domination number $\gamma_{r,s}$ is not an interpolating function if $r + 1 \le s \le 2r - 1$. Let r, s and l be positive integers such that $3 \le r + 1 \le s \le 2r - 1$ and $l \ge \lceil (r + 1)/3 \rceil$, and let $H_{r,s}$ be the unicyclic graph of girth $2l(2r - s + 2)$ given in Fig. 2. Let vu be an edge belonging to the unique cycle of $H_{r,s}$. It is evident that every (r, s)-dominating set of the tree $H_{r,s} - vu$ contains at least one vertex of the path $v_s^{(i)} - v_{s-r}^{(i)}$ and at least one vertex of the path $u_s^{(i)} - u_{s-r}^{(i)}$ for every $i \in \{1, \ldots, 2l\}$, so that $\gamma_{r,s}(H_{r,s} - vu) \ge 4l$. If $vu \notin \{v_0^{(1)} u_0^{(1)}, \ldots, v_0^{(2l)} u_0^{(2l)}\}$, then, since $\{v_{s-r}^{(1)}, \ldots, v_{s-r}^{(2l)}, u_{s-r}^{(1)}, \ldots, u_{s-r}^{(2l)}\}$ is an (r, s)-dominating set of $H_{r,s} - vu$, we also have $\gamma_{r,s}(H_{r,s} - vu) = 4l$. We now show that $\gamma_{r,s}(T_i) = 4l + 2$ if $T_i = H_{r,s} - v_0^{(i)} u_0^{(i)}$ for $i \in \{1, \ldots, 2l\}$. Since trees $T_1, T_2, \ldots, T_{2l}$ are mutually isomorphic, it suffices to show that $\gamma_{r,s}(T) = 4l + 2$ where $T = T_1$. Let D be a minimum (r, s)-dominating set of T, and let $v^{(i)}$ ($u^{(i)}$, resp.) denote that vertex of D for which $d_T(v_s^{(i)}, D) = d_T(v_s^{(i)}, v^{(i)})$ ($d_T(u_s^{(i)}, D) = d_T(u_s^{(i)}, u^{(i)})$, resp.), $i = 1, \ldots, 2l$. Certainly, $v^{(i)}$ ($u^{(i)}$, resp.) belongs to the path $v_s^{(i)} - v_{s-r}^{(i)}$ ($u_s^{(i)} - u_{s-r}^{(i)}$, resp.) for every $i \in \{1, \ldots, 2l\}$. Let v (u, resp.) be a vertex in D for which $d_T(v^{(1)}, D - \{v^{(1)}\}) = d_T(v^{(1)}, v)$ ($d_T(u^{(1)}, D - \{u^{(1)}\}) = d_T(u^{(1)}, u)$, resp.). Since $d_T(v^{(1)}, v) \le s$ and $d_T(u^{(1)}, u) \le s$ while $d_T(v^{(1)}, \{v^{(2)}, \ldots, v^{(2l)}, u^{(1)}, \ldots, u^{(2l)}\}) \ge d_T(v_{s-r}^{(1)}, u_{s-r}^{(2)}) > s$ and $d_T(u^{(1)}, \{v^{(1)}, \ldots, v^{(2l)}, u^{(2)}, \ldots, u^{(2l)}\}) \ge d_T(u_{s-r}^{(1)}, v_{s-r}^{(2l)}) > s$, neither v nor u belongs to $\{v^{(1)}, \ldots, v^{(2l)}, u^{(1)}, \ldots, u^{(2l)}\}$. In addition, vertices v and u are distinct, for otherwise $d_T(v^{(1)}, u^{(1)}) \le d_T(v^{(1)}, v) + d_T(u, u^{(1)}) \le 2s$ which is impossible as $d_T(v^{(1)}, u^{(1)}) \ge d_T(v_{s-r}^{(1)}, u_{s-r}^{(1)}) = 2(s - r) + 2l(2r - s + 2) - 1 \ge 2s + 1$. We conclude that $\gamma_{r,s}(T) \ge |\{v^{(1)}, \ldots, v^{(2l)}, u^{(1)}, \ldots, u^{(2l)}\} \cup \{v, u\}| = 4l + 2$. Since $\{v_{s-r}^{(1)}, \ldots, v_{s-r}^{(2l)}, u_{s-r}^{(1)}, \ldots, u_{s-r}^{(2l)}\} \cup \{v_0^{(1)}, u_0^{(1)}\}$ is an (r, s)-dominating set of T, we also have that $\gamma_{r,s}(T) \le 4l + 2$, whence $\gamma_{r,s}(T) = 4l + 2$. It follows that $\gamma_{r,s}(\mathcal{T}(H_{r,s})) = \{4l, 4l + 2\}$. Consequently, the (r, s)-domination number $\gamma_{r,s}$ with $r + 1 \le s \le 2r - 1$ (and therefore with $1 \le s \le 2r - 1$) is not an interpolating function.

Since $\gamma_{r,s}$ is an interpolating function when $s \ge 2r + 1$, a question to be considered here is whether $\gamma_{r,2r}$ is an interpolating function. We suspect that $\gamma_{r,2r}$ is an interpolating function for every positive integer r, but we are able to prove it only for $r = 1$. We also prove that for any positive integers r and s, $\gamma_{r,s}$ interpolates over every 2-connected graph. First we analyze how the (r, s)-domination number varies as we delete an edge from a graph.

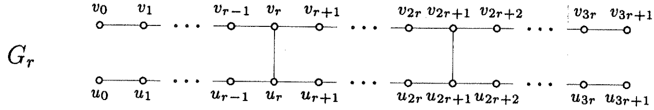

Fig. 1. A graph $G = G_r$ for which $\gamma_{r,s}(\mathcal{T}(G)) = \{4, 6\}$ where $1 \leq s \leq r$.

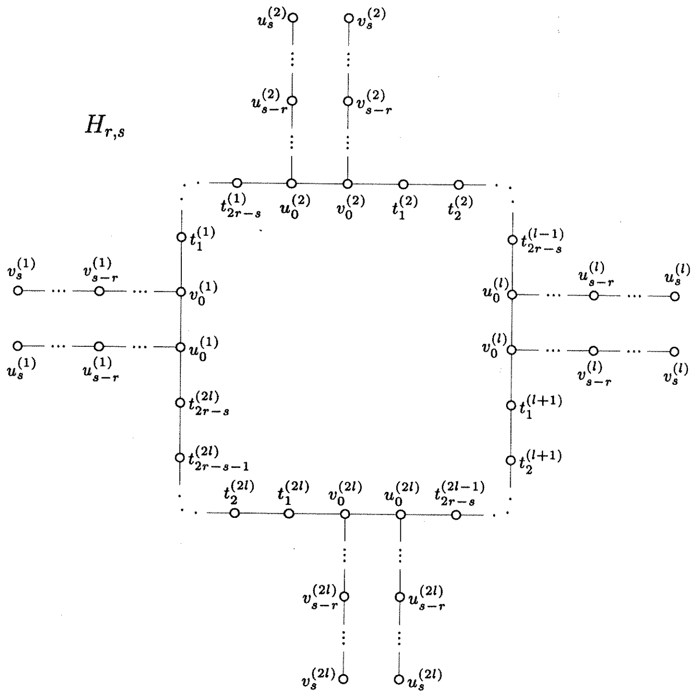

Fig. 2. A graph $G = H_{r,s}$ for which $\gamma_{r,s}(\mathcal{T}(G)) = \{4l, 4l+2\}$ where $3 \leq r+1 \leq s \leq 2r-1$ and $l \geq \lceil (r+1)/3 \rceil$.

Lemma 5. *Let r and s be positive integers, and let vu be an edge of a graph G. If vu is not an end-edge of G, then*

$$\gamma_{r,s}(G) \leq \gamma_{r,s}(G - vu) \leq \gamma_{r,s}(G) + 2.$$

Proof. Since any (r, s)-dominating set of $G - vu$ is (r, s)-dominating in G, the inequality $\gamma_{r,s}(G) \leq \gamma_{r,s}(G - vu)$ is obvious.

By definition $\gamma_{r,s}(G) \geq 2$, so the inequality $\gamma_{r,s}(G - vu) \leq \gamma_{r,s}(G) + 2$ is obvious if $|V(G)| \leq 4$. Thus assume that $|V(G)| \geq 5$ and let D be a minimum (r, s)-dominating set of G. We consider four possible cases.

Case 1. D is both an $(r,-)$- and $(-,s)$-set of $G-vu$. Then D is (r,s)-dominating in $G-vu$ and therefore $\gamma_{r,s}(G-vu) \le |D| \le \gamma_{r,s}(G)+2$.

Case 2. D is a $(-,s)$-set but it is not an $(r,-)$-set of $G-vu$. In this case the set $V' = \{x \in V(G) - D : d_{G-vu}(x,D) > r\}$ is nonempty and for every $x \in V'$, since $d_G(x,D) \le r$, any path of length at most r joining x to a vertex of D in G contains the edge vu. This implies that $d_G(v,D) \neq d_G(u,D)$, say $d_G(v,D) < d_G(u,D)$. Now, for any $u' \in N_G(u) - \{v\}$, the set $D \cup \{u,u'\}$ is (r,s)-dominating in $G-vu$ and so $\gamma_{r,s}(G-vu) \le |D \cup \{u,u'\}| \le \gamma_{r,s}(G)+2$.

Case 3. D is an $(r,-)$-set but it is not a $(-,s)$-set of $G-vu$. Now the set $D' = \{x \in D : d_{G-vu}(x, D-\{x\}) > s\}$ is nonempty and for every $x \in D'$, since $d_G(x, D-\{x\}) \le s$, any path of length at most s joining x to a vertex of $D-\{x\}$ in G contains the edge vu. Therefore $d_G(x,v) \neq d_G(x,u)$ for every $x \in D'$ and the sets $D_v = \{x \in D' : d_G(x,v) < d_G(x,u)\}$ and $D_u = \{x \in D' : d_G(x,u) < d_G(x,v)\}$ form a partition of D'. In addition, since vu is not an end-edge of G, $N_G[v] - \{u\}$ ($N_G[u]-\{v\}$, resp.) is not a subset of D if $D_v \neq \emptyset$ ($D_u \neq \emptyset$, resp.). Now if $D_v \neq \emptyset$ ($D_u \neq \emptyset$, resp.) and if v' is any vertex from $N_G[v] - (D \cup \{u\})$ (u' is any vertex from $N_G[u] - (D \cup \{v\})$, resp.), then the set $D \cup \{v'\}$ (if $D_u = \emptyset$), $D \cup \{u'\}$ (if $D_v = \emptyset$) or $D \cup \{v',u'\}$ is (r,s)-dominating in $G-vu$ and so $\gamma_{r,s}(G-vu) \le |D|+2 = \gamma_{r,s}(G)+2$.

Case 4. D is neither an $(r,-)$-set nor a $(-,s)$-set of $G-vu$. Then both the sets $V' = \{x \in V(G) - D : d_{G-vu}(x,D) > r\}$ and $D' = \{x \in D : d_{G-vu}(x, D-\{x\}) > s\}$ are nonempty. Certainly, for any x from V', every path of length at most r joining x to a vertex of D in G contains the edge vu. Similarly, if x belongs to D', then every path of length at most s joining x to a vertex of $D-\{x\}$ in G contains vu. In addition, since $V' \neq \emptyset$, we have $d_G(v,D) \neq d_G(u,D)$, say $d_G(v,D) < d_G(u,D)$ and let d be a vertex of D for which $d_G(v,D) = d_G(v,d)$. As in Case 3, the sets $D_v = \{x \in D' : d_G(x,v) < d_G(x,u)\}$ and $D_u = \{x \in D' : d_G(x,u) < d_G(x,v)\}$ form a partition of D'. The assumption $d_G(v,D) < d_G(u,D)$ easily implies that either $D_v = \emptyset$ or $D_v = \{d\}$. If $D_v = \emptyset$, then $D \cup \{u\}$ is an (r,s)-dominating set in $G-vu$. Finally, if $D_v \neq \emptyset$, then $N_G[v] - \{u\}$ is not a subset of D (since vu is not an end-edge of G) and for any $v' \in N_G[v] - (D \cup \{u\})$, the set $D \cup \{u,v'\}$ is (r,s)-dominating in $G-vu$. Thus in each case $\gamma_{r,s}(G-vu) \le |D|+2 = \gamma_{r,s}(G)+2$.

This completes the proof. □

Corollary 1. *Let G be a unicyclic graph and let r and s be positive integers. If $\gamma_{r,s}(G) = a$, then $\gamma_{r,s}(\mathcal{T}(G))$ is a subset of $\{a, a+1, a+2\}$.*

Proof. Let C be the unique cycle of G. Then $\mathcal{T}(G) = \{G-vu : vu \in E(C)\}$ and the result follows from Lemma 5. □

Lemma 6. *Let G be a unicyclic graph with $\gamma_{1,2}(G) = a$, and let $v'v$, vu and uu' be three consecutive edges on the unique cycle of G. If $\gamma_{1,2}(G-vu) > a$, then $\gamma_{1,2}(G-vv') \le a+1$ or $\gamma_{1,2}(G-uu') \le a+1$.*

Proof. Let D be a minimum $(1,2)$-dominating set of G. Then $D \cap \{v,u\} \neq \emptyset$ and $\{v, u, v', u'\}$ is not a subset of D; otherwise D would be a $(1,2)$-dominating set of

$G - vu$ which is impossible as $|D| = a < \gamma_{1,2}(G - vu)$. We consider two possibilities.

Case 1. $\{v, u\} \subseteq D$. Then $\{v', u'\} - D \neq \emptyset$. Now it is easy to observe that if $v' \notin D$, then $D \cup \{v'\}$ is a $(1,2)$-dominating set of $G - vv'$ and so $\gamma_{1,2}(G - vv') \leq a + 1$. Similarly, $\gamma_{1,2}(G - uu') \leq a + 1$ if $u' \notin D$.

Case 2. $|\{v, u\} \cap D| = 1$, say $u \in D$ and $v \notin D$. If $N_G(v) \cap (D - \{u\}) = \emptyset$, then D is a $(1,2)$-dominating set of $G - vv'$ and $\gamma_{1,2}(G - vv') < a + 1$. Suppose that $N_G(v) \cap (D - \{u\}) \neq \emptyset$. If $u' \notin D$, then $D \cup \{u'\}$ is a $(1,2)$-dominating set of $G - uu'$ and $\gamma_{1,2}(G - uu') \leq a + 1$. Finally, if $u' \in D$, then, for any $u'' \in N_G(u') - \{u\}$, $D \cup \{u''\}$ is a $(1,2)$-dominating set of $G - uu'$ and so $\gamma_{1,2}(G - uu') \leq a + 1$. □

Lemma 7. *Let G be a unicyclic graph with $\gamma_{1,2}(G) = a$, and let $v'v$, vu and uu' be three consecutive edges on the unique cycle of G. If $\gamma_{1,2}(G - vv') = a = \gamma_{1,2}(G - uu')$, then $\gamma_{1,2}(G - vu) \leq a + 1$.*

Proof. Let G_v be the component of $G - \{vu, vv'\}$ that contains the vertex v. Similarly, let G_u be the component of $G - \{vu, uu'\}$ that contains u. Let $\mathcal{D}$, $\mathcal{D}_v$ and $\mathcal{D}_u$ denote the sets of all minimum $(1,2)$-dominating sets of the graphs G, $G - vv'$ and $G - uu'$, respectively. Since $\gamma_{1,2}(G - vv') = \gamma_{1,2}(G - uu') = \gamma_{1,2}(G) = a$, $\mathcal{D}_v \cup \mathcal{D}_u \subseteq \mathcal{D}$.

It is easy to observe that $\gamma_{1,2}(G - vu) \leq a + 1$ if $D \cap \{v, v'\} \neq \emptyset$ for some $D \in \mathcal{D}_v$ or $D' \cap \{u, u'\} \neq \emptyset$ for some $D' \in \mathcal{D}_u$. Thus assume that $D \cap \{v, v'\} = \emptyset$ for every $D \in \mathcal{D}_v$ and $D' \cap \{u, u'\} = \emptyset$ for every $D' \in \mathcal{D}_u$. Again it is no problem to observe that $\gamma_{1,2}(G - vu) \leq a + 1$ if $(N_G[v] - \{u\}) \cap D = \emptyset$ for some $D \in \mathcal{D}_v$ or $(N_G[u] - \{v\}) \cap D' = \emptyset$ for some $D' \in \mathcal{D}_u$. Now assume that $(N_G[v] - \{u\}) \cap D \neq \emptyset$ for every $D \in \mathcal{D}_v$ and $(N_G[u] - \{v\}) \cap D' \neq \emptyset$ for every $D' \in \mathcal{D}_u$. It is easy to see that $\gamma_{1,2}(G - vu) \leq a + 1$ if $|(N_G[v] - \{u\}) \cap D| \geq 2$ for some $D \in \mathcal{D}_v$ or $|(N_G[u] - \{v\}) \cap D'| \geq 2$ for some $D' \in \mathcal{D}_u$. Thus assume that $|(N_G[v] - \{u\}) \cap D| = 1$ and $|(N_G[u] - \{v\}) \cap D'| = 1$ for every $D \in \mathcal{D}_v$ and $D' \in \mathcal{D}_u$. For $D \in \mathcal{D}_v$ and $D' \in \mathcal{D}_u$, let $v(D)$ and $u(D')$ be the unique vertex of $(N_G[v] - \{u\}) \cap D$ and $(N_G[u] - \{v\}) \cap D'$, respectively. Certainly, $v(D)$ is a vertex of G_v and $u(D')$ is a vertex of G_u. Again it is easy to observe that if there exists $D \in \mathcal{D}_v$ such that $u \notin D$ or if there exists $D' \in \mathcal{D}_u$ such that $v \notin D'$, then $\gamma_{1,2}(G - vu) \leq a + 1$. Thus assume that u belongs to every $D \in \mathcal{D}_v$ and v belongs to every $D' \in \mathcal{D}_u$. If there exists $D \in \mathcal{D}_v$ and $z \in D - \{v(D), u\}$ such that $d_G(z, \{v(D), u\}) \leq 2$ or if there exists $D' \in \mathcal{D}_u$ and $z' \in D' - \{u(D'), v\}$ such that $d_G(z', \{u(D'), v\}) \leq 2$, then $\gamma_{1,2}(G - vu) \leq a + 1$. Finally assume that $d_G(x, \{v(D), u\}) > 2$ for every $D \in \mathcal{D}_v$ and every $x \in D - \{v(D), u\}$, and $d_G(y, \{u(D'), v\}) > 2$ for every $D' \in \mathcal{D}_u$ and every $y \in D' - \{u(D'), v\}$. Take any $D \in \mathcal{D}_v$ and $D' \in \mathcal{D}_u$. Let F be the component of $G - uu(D')$ that contains $u(D')$, and let H denote the subgraph $F - u(D')$. Take any $y \in N_G(u(D')) - \{u\}$ and let H_y be the component of H that contains y. Since $d_G(x, \{v(D), u\}) > 2$ for every $x \in D - \{v(D), u\}$, neither $u(D')$ nor y belongs to D. This and the minimality of D imply that the set $D_y = D \cap V(H_y)$ is a minimum $(1,2)$-dominating set of H_y. Now take any vertex t from $N_G(y) \cap D_y$ and consider the graph $H_y - y$. Since D' is a minimum $(1,2)$-dominating set of $G - uu'$ and no vertex of $N_G[y] - \{u(D')\}$ belongs to D', it must be $\gamma_{1,2}(H_y - y) < |D_y|$; otherwise

$\overline{D}' = (D' - V(H_y - y)) \cup D_y$ containing t would be a minimum (1,2)-dominating set of $G - uu'$ and t would be at distance two from $u(\overline{D}') = u(D')$ which is impossible. Let C_y be a minimum (1,2)-dominating set of $H_y - y$. Then $C_y \cup \{y\}$ is a minimum (1,2)-dominating set of H_y and so $\overline{D} = (D - D_y) \cup (C_y \cup \{y\})$ is a minimum (1,2)-dominating set of $G - vv'$. But now the vertex y of $\overline{D} - \{v(\overline{D}), u\}$ is at distance two from u. This contradicts our assumption; therefore we must reject the assumption that $d_G(x, \{v(D), u\}) > 2$ for every $D \in \mathcal{D}_v$ and $x \in D - \{v(D), u\}$, and $d_G(y, \{u(D'), v\}) > 2$ for every $D' \in \mathcal{D}_u$ and $y \in D' - \{u(D'), v\}$. In all other cases, as we have already observed, $\gamma_{1,2}(G - vu) \leq a + 1$. This completes the proof. □

Corollary 2. *Let G be a unicyclic graph with $\gamma_{1,2}(G) = a$. If $v'v$, vu and uu' are three consecutive edges on the unique cycle of G and $\gamma_{1,2}(G - vu) = a + 2$, then $\gamma_{1,2}(G - vv') = a + 1$ or $\gamma_{1,2}(G - uu') = a + 1$.*

Proof. Assume on the contrary that $\gamma_{1,2}(G - vv') \neq a + 1$ and $\gamma_{1,2}(G - uu') \neq a + 1$. Then it follows from Lemmas 6 and 7 that $\min\{\gamma_{1,2}(G - vv'), \gamma_{1,2}(G - uu')\} = a$ and $\max\{\gamma_{1,2}(G - vv'), \gamma_{1,2}(G - uu')\} = a + 2$, say $\gamma_{1,2}(G - vv') = a$ and $\gamma_{1,2}(G - uu') = a + 2$. Let D be any minimum (1,2)-dominating set of $G - vv'$. Then D is a minimum (1,2)-dominating set of G. Since $\gamma_{1,2}(G - vu) = \gamma_{1,2}(G - uu') = a + 2 > a = |D|$, D is neither a (1,2)-dominating set of $G - vu$ nor a (1,2)-dominating set of $G - uu'$. This implies that $D \cap \{v, u\} \neq \emptyset$, $D \cap \{u, u'\} \neq \emptyset$ and neither $\{v, u\}$ nor $\{u, u'\}$ is a subset of D. It is easy to observe that $D \cap \{v, u, u'\} \neq \{v, u'\}$; otherwise $D \cup \{u\}$ would be (1,2)-dominating in $G - vu$. Consequently, $D \cap \{v, u, u'\} = \{u\}$. Let x be any vertex of $D - \{u\}$ for which $d_G(u, x) \leq 2$. We must have $d_G(u, x) = 2$; otherwise $D \cup \{v\}$ and $D \cup \{u'\}$ would be (1,2)-dominating in $G - vu$ and in $G - uu'$, respectively, which is impossible. Thus, let x' be a common neighbour of u and x. It is easy to observe that neither $x' = v$ nor $x' = u'$; for if $x' = v$ ($x' = u'$, resp.), then $D \cup \{u'\}$ ($D \cup \{v\}$, resp.) would be (1,2)-dominating in $G - uu'$ ($G - vu$, resp.) which is impossible. This implies that neither x' nor x belongs to the unique cycle of G. But now $D \cup \{v\}$ ($D \cup \{u'\}$, resp.) is (1,2)-dominating in $G - vu$ ($G - uu'$, resp.) which again is impossible. Therefore we must reject the assumption that $\gamma_{1,2}(G - vv') \neq a + 1$ and $\gamma_{1,2}(G - uu') \neq a + 1$. This completes the proof. □

We are now ready to prove that $\gamma_{1,2}$ interpolates over every connected graph of order at least two.

Theorem 2. *The $(1,2)$-domination number $\gamma_{1,2}$ is an interpolating function.*

Proof. By Lemma 1, it suffices to show that $\gamma_{1,2}$ interpolates over every unicyclic graph. Let G be a unicyclic graph with $\gamma_{1,2}(G) = a$. Then the set $A = \{\gamma_{1,2}(T) : T \in \mathcal{T}(G)\}$ is a subset of $\{a, a + 1, a + 2\}$ (by Corollary 1) and $a \in A$ (by Lemma 2). Certainly, it follows from Corollary 2 that $a + 1 \in A$ if $a + 2 \in A$. This proves that $\gamma_{1,2}$ interpolates over G. □

We conclude with a result that describes the interpolating character of the (r, s)-domination number for 2-connected graphs. In the proof we will use the following property of 2-connected graphs. Lovász [9, p. 269] and later Harary, Mokken and Plantholt [3] proved that if G is a 2-connected graph, then any spanning tree T of G can be transformed into any spanning tree T^* of G through a sequence $T_0 = T, T_1, \ldots, T_n = T^*$ of spanning trees of G, called a *sequence of end edge-exchanges* transforming T into T^*, such that for every $k = 0, 1, \ldots, n-1$, $T_{k+1} = T_k + f_k - e_k$ where e_k and f_k are end edges in T_k and T_{k+1}, respectively.

Theorem 3. *For any positive integers r and s, the (r, s)-domination number $\gamma_{r,s}$ interpolates over every 2-connected graph.*

Proof. Assume G is a 2-connected graph, and let m and M be respectively the smallest and largest integer of $\gamma_{r,s}(\mathcal{T}(G))$. Let T_0 and T^* be spanning trees of G with $\gamma_{r,s}(T_0) = m$ and $\gamma_{r,s}(T^*) = M$. Since G is 2-connected, there exists a sequence of end edge-exchanges $T_0, T_1, \ldots, T_n = T^*$ transforming T_0 into T^*. To prove that $\gamma_{r,s}(\mathcal{T}(G))$ is an integer interval, we need only show that each step of the end edge-exchange may increase the value of $\gamma_{r,s}$ by at most one, that is $\gamma_{r,s}(T_{k+1}) \leq \gamma_{r,s}(T_k) + 1$ for $k = 0, 1, \ldots, n-1$, which, in turn, implies that the sequence $(\gamma_{r,s}(T_0), \gamma_{r,s}(T_1), \ldots, \gamma_{r,s}(T_n))$ contains $(m, m+1, \ldots, M)$ as a subsequence and this proves that $\gamma_{r,s}(\mathcal{T}(G)) = \{m, m+1, \ldots, M\}$.

Let D be any minimum (r, s)-dominating set in T_k and suppose that $T_{k+1} = T_k + wv - vu$, where v is an end vertex of T_k (and of T_{k+1}) and u is the unique neighbour of v in T_k. Since $T_k \neq K_2$, the minimality of D implies that the set $N_{T_k}[u]$ is not a subset of D and therefore we may assume that $v \notin D$; otherwise $D' = (D - \{v\}) \cup \{u\}$ (if $u \notin D$) or $D'' = (D - \{v\}) \cup \{x\}$ for any $x \in N_{T_k}[u] - D$ (if $u \in D$) is a minimum (r, s)-dominating set in T_k, no one of them contains v and we could replace D by D' or D''. Since $T_{k+1} = T_k + wv - vu$ and D is (r, s)-dominating in T_k, we have $d_{T_{k+1}}(y, D - \{y\}) = d_{T_k}(y, D - \{y\}) \leq s$ for any $y \in D$, $d_{T_{k+1}}(v, D) = d_{T_k}(w, D) + 1 \leq r + 1$ and $d_{T_{k+1}}(x, D) = d_{T_k}(x, D) \leq r$ for any $x \in V(G) - (D \cup \{v\})$. Thus, if $d_{T_k}(w, D) \leq r - 1$, then D is an (r, s)-dominating set in T_{k+1} and therefore $\gamma_{r,s}(T_{k+1}) \leq |D| < \gamma_{r,s}(T_k) + 1$. On the other hand, if $d_{T_k}(w, D) = r$, then let t be a vertex of D for which $d_{T_k}(w, t) = r$ and let t' be the unique neighbour of t which belongs to the $t - w$ path in T_k. Then $D \cup \{t'\}$ is an (r, s)-dominating set in T_{k+1} and again $\gamma_{r,s}(T_{k+1}) \leq |D \cup \{t'\}| = \gamma_{r,s}(T_k) + 1$. □

From Theorem 3, we immediately have the following corollary proved in [13].

Corollary 3. *The total domination number γ_t interpolates over any 2-connected graph.*

Problem. If $r \geq 2$, does $\gamma_{r,2r}$ interpolate over every connected graph?

References

[1] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, *Total domination in graphs*, Networks 10 (1980) 211–219.

[2] F. Harary, *Maximum versus minimum invariants for graphs*, J. Graph Theory 7 (1983) 275–284.

[3] F. Harary, R. J. Mokken and M. J. Plantholt, *Interpolation theorem for diameters of spanning trees*, IEEE Trans. Circuits Syst. 30 (1983) 429–432.

[4] F. Harary and M. J. Plantholt, *Classification of interpolation theorems for spanning trees and other families of spanning subgraphs*, J. Graph Theory 13 (1989) 703–712.

[5] F. Harary and S. Schuster, *Interpolation theorems for invariants of spanning trees of a given graph: covering numbers*, Congr. Numer. 64 (1988) 215–219.

[6] F. Harary and S. Schuster, *Interpolation theorems for the independence and domination numbers of spanning trees*, Ann. Discrete Math. 41 (1989) 221–228.

[7] F. Harary, S. Schuster and P. D. Vestergaard, *Interpolation theorems for the invariants of spanning trees of a given graph: edge-covering*, Congr. Numer. 59 (1987) 107–114.

[8] M. A. Henning, O. R. Oellermann and H. C. Swart, *Bounds on distance domination parameters*, J. Combin. Inform. System Sci. 16 (1991) 11–18.

[9] L. Lovász, *Combinatorial Problems and Exercises*, Akadémiai Kiadó, Budapest 1979.

[10] A. Meir and J. W. Moon, *Relation between packing and covering numbers of a tree*, Pacific J. Math. 61 (1975) 225–233.

[11] Z. Mo and K. Williams, *(r, s)-domination in graphs and directed graphs*, Ars Combin. 29 (1990) 129–141.

[12] J. Topp, *Domination, independence and irredundance in graphs*, Diss. Math. 342 (1995) 99 pp.

[13] J. Topp and P. D. Vestergaard, *Interpolation theorems for domination numbers of a graph*, submitted.

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