On minimum possible volumes of strong Steiner trades

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Abstract

In this note we investigate the minimum possible volumes for strong Steiner trades (SST). We prove that a $(v, q+1, 2)$ SST must have at least q^2 blocks if q is even and $q^2 + q$ blocks if q is odd. We construct a $(v, q+1, 2)$ SST of volume q^2 for every q a power of two, and a $(v, q+1, 2)$ SST of volume $q^2 + q$, for every q such that $q + 1$ is a power of two. A construction of $(q^2+q+1, q+1, 2)$ SSTs of volume q^2+q+1 is also given for every prime power q. Combinations of these constructions are then used to construct further SSTs. We also show that when the bound for q even is achieved the elements of the trade are the duals of affine planes.

1 Introduction

A $(v, k, 2)$ *trade* $T = \{T_1, T_2\}$ of volume m consists of two disjoint collections T_1 and T_2, each containing m k-subsets (blocks) of some set V, such that all pairs from V occur in exactly the same number of blocks of T_1 as of T_2. If all pairs from V occur in either zero or one block of T_1, then the trade is called *Steiner*. (Note that there may exist elements of V which occur in no block of T_1.) The set of elements of V contained in T_1 is denoted by $F(T_1)$ or $F(T)$. We also note that the number of blocks of T_1 containing the element $x \in F(T)$ is the same as the number of blocks of T_2 containing the x. We denote this number by r_x.

A $(v, k, 2)$ Steiner trade $T = \{T_1, T_2\}$ is called *strong* if any block of T_1 intersects any block of T_2 in at most two elements. We denote a $(v, k, 2)$ strong Steiner trade by $(v, k, 2)$ SST. The requirement that any two blocks have at most one pair in common is well-known as the *orthogonality* or *super-simple* property. The *spectrum* of $(v, k, 2)$ Steiner trades is the unique set of integers such that a $(v, k, 2)$ Steiner trade exists if and only if its volume is in the spectrum. In [6, 7, 9] the spectrum of

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$(k, 2)$ Steiner trades is completely settled (in these papers the number of elements is not considered). When $k = 3$ any $(v, 3, 2)$ Steiner trade is also strong by definition. So strong trades are of interest for $k \geq 4$. Adams, Bryant and Khodkar in [1] show that a $(v, 4, 2)$ SST of volume $v(v-1)/12$ exists for all $v \equiv 1, 4 \pmod{12}$, $v \geqslant 13$. Using probabilistic arguments, Caro and Yuster in [3] have recently shown that for any two fixed integers k and μ, there exists $N = N(k, \mu)$ such that for every $v > N$, if a $(v, k, 1)$ BIBD exists then there are μ distinct $(v, k, 1)$ BIBDs such that any distinct pair of these BIBDs yields a $(v, k, 2)$ SST. Moreover, they proved [4] explicitly that there exists a finite set of positive integers $M(k, \mu)$ such that for every positive integer $m \notin M(k, \mu)$ there exist μ distinct $(v, k, 1)$ BIBDs such that any distinct pair of these BIBDs yields a $(v, k, 2)$ SST of volume m.

In this note we investigate the smallest positive integer which is not in $M(k, 2)$. Indeed we prove:

- A $(v, q+1, 2)$ SST has at least q^2 blocks if q is even and $q^2 + q$ blocks if q is odd.
- There exists a $(q^2 + q, q + 1, 2)$ SST of volume q^2 for every q a power of 2, and a $((q+1)^2, q+1, 2)$ SST of volume $q^2 + q$, for every q such that $q+1$ is a power of 2.
- There exists a $(q^2 + q + 1, q + 1, 2)$ SST of volume $q^2 + q + 1$ for every prime power q.
- If q is a power of 2 then there exists a $(v, q+1, 2)$ SST of volume m for every $m \geq q^2(q^2 + q + 1)$.
- If q is a power of 2 and $q - 1$ is a prime power then there exists a $(v, q, 2)$ SST of volume m for every $m \geq (q^2 - q)(q^2 - q + 1)$.
- If $T = \{T_1, T_2\}$ is a $(v, q+1, 2)$ SST of volume q^2 then T_1 and T_2 are the duals of affine planes.

2 Results

We start this section with the following result which gives a lower bound on the volume of strong Steiner trades.

Lemma 2.1 *Let $T = \{T_1, T_2\}$ be a $(v, q+1, 2)$ SST of volume m. Then $r_x \geq q$ for $x \in F(T)$ and $m \geq q^2$.*

Proof: Let $\{a_1, a_2, a_3, \ldots, a_{q+1}\} \in T_1$. Since each pair $\{a_1, a_j\}$, $2 \leq j \leq q+1$, must occur in a block of T_2 and no two of these pairs can occur in the same block (since the trade is strong) it follows that a_1 occurs in at least q blocks of T_2. So $r_x \geq q$ for all $x \in F(T)$. Now since the trade is Steiner and $r_x \geq q$ it follows that there must be at least q^2 blocks in T_1. So $m \geq q^2$. □

When q is odd the lower bound for the volume of $(v, q+1, 2)$ SSTs increases to $q^2 + q$.

Lemma 2.2 *Let q be odd. Then the volume of a $(v, q+1, 2)$ SST is at least $q^2 + q$.*

Proof: By Lemma 2.1, $r_x \geq q$ for all $x \in F(T)$. Suppose that $r_a = q$ for some $a \in F(T)$. Let the element a be contained in the blocks $B_1, B_2, \cdots, B_q$ of T_1 and in the blocks $C_1, C_2, \cdots, C_q$ of T_2. Define $X_{ij} = (B_i \cap C_j) \setminus \{a\}$ for all $1 \leq i, j \leq q$. Then $\sum_{j=1}^{q} |X_{ij}| = q$ for all $1 \leq i \leq q$. On the other hand, since the set $B_i \setminus \{a\}$ intersects the set $C_j \setminus \{a\}$ in at most one element it follows that $|X_{ij}| \leq 1$. Therefore, $|X_{ij}| = 1$ for all $1 \leq i, j \leq q$. So we can assume $X_{ij} = \{x_{ij}\}$ for $1 \leq i, j \leq q$. There are $q.\binom{q}{2}$ pairs of the form $\{x_{ir}, x_{is}\}$ which occur in the blocks $B_1, B_2, \cdots, B_q$. So they must occur in the blocks of T_2. A block of T_2 can have at most $\lfloor \frac{q}{2} \rfloor$ pairs of this form since the trade is Steiner. So T_2 has at least $q + (q.\binom{q}{2}) / \lfloor \frac{q}{2} \rfloor$ blocks. But

$$q + \left(q.\binom{q}{2}\right) \Big/ \left\lfloor \frac{q}{2} \right\rfloor = \begin{cases} q^2 & \text{if } q \text{ even} \\ q^2 + q & \text{if } q \text{ odd.} \end{cases}$$

Now suppose that $r_x \geq q + 1$ for all $x \in F(T)$. Then $|F(T)| \geq q.(q+1) + 1$. Since the block-size is $q + 1$ we must have

$$m \geq (q.(q+1) + 1)(q+1)/(q+1) = q^2 + q + 1.$$

This completes the proof. □

The following two theorems show that the lower bounds for the volumes of $(v, k, 2)$ SSTs, given in Lemmas 2.1 and 2.2, are sharp.

Theorem 2.3 *Let q be a power of 2. There exists a $(q^2 + q, q + 1, 2)$ SST of volume q^2.*

Proof: Let α be a primitive element of $GF[q] = \{a_0, a_1, a_2, \cdots, a_{q-1}\}$, with $a_0 = 0$ and $a_r = \alpha^{r-1}$ for $1 \leq r \leq q - 1$. Define

$$B_{(a_i,a_j)} = \{(a_i, -1)\} \cup \{(a_r a_i + a_j, r) \mid 0 \leq r \leq q-1\},$$

$T_1 = \{B_{(a_i,a_j)} \mid 0 \leq i, j \leq q-1\}$ and $V = \{(x, r) \mid x \in GF[q] \text{ and } -1 \leq r \leq q-1\}$. Then T_1 and V contain q^2 and $q^2 + q$ elements, respectively. Moreover, for any $x, y \in GF[q]$ and $-1 \leq r < s \leq q - 1$ the 2-subset $\{(x, r), (y, s)\}$ occurs precisely once in the blocks of T_1, namely in the block $B_{(a_i,a_j)}$, where

$$(a_i, a_j) = \begin{cases} (x, a_s x + y) & \text{if } r = -1; \text{ and} \\ \left(\dfrac{x+y}{a_r + a_s}, \dfrac{a_s x + a_r y}{a_r + a_s}\right) & \text{otherwise.} \end{cases}$$

Now define

$$C_{(a_i,a_j)} = \{(a_i, -1)\} \cup \{(a_r(a_i + a_r) + a_j, r) \mid 0 \leq r \leq q-1\},$$

and $T_2 = \{C_{(a_i,a_j)} \mid 0 \le i,j \le q-1\}$. For any $x, y \in GF[q]$ and $-1 \le r < s \le q-1$ the 2-subset $\{(x,r),(y,s)\}$ occurs precisely once in the blocks of T_2, namely in the block $C_{(a_i,a_j)}$, where

$$(a_i, a_j) = \begin{cases} (x, a_s(x+a_s)+y) & \text{if } r=-1; \text{ and} \\ \left(\dfrac{x+y}{a_r+a_s} + a_r + a_s, \dfrac{a_s x + a_r y}{a_r + a_s} + a_r a_s\right) & \text{otherwise.} \end{cases}$$

Therefore $T = \{T_1, T_2\}$ is a trade of volume q^2. Finally, we need to prove that any block of T_1 intersects any block of T_2 in at most two elements. Let $0 \le r < s < t \le q-1$. Suppose that $(a_r a_i + a_j, r)$, $(a_s a_i + a_j, s)$ and $(a_t a_i + a_j, t)$ are three elements of $B_{(a_i,a_j)}$ and $(a_r(a_m+a_r)+a_n, r)$, $(a_s(a_m+a_s)+a_n, s)$ and $(a_t(a_m+a_t)+a_n, t)$ are three elements of $C_{(a_m,a_n)}$ such that

$$\begin{aligned} (a_r a_i + a_j, r) &= (a_r(a_m+a_r)+a_n, r) \\ (a_s a_i + a_j, s) &= (a_s(a_m+a_s)+a_n, s) \\ (a_t a_i + a_j, t) &= (a_t(a_m+a_t)+a_n, t). \end{aligned}$$

From first and second equalities we obtain $a_i = a_m + a_r + a_s$ and $a_j = a_n + a_r a_s$. Substituting for a_i and a_j in the third equality leads to $a_t a_r = a_t^2$. So $a_t = 0$ or $a_r = a_t$, both of which are impossible. The following case also needs to be considered. Let $0 \le s < t \le q-1$. Suppose that $(a_i, -1)$, $(a_s a_i + a_j, s)$ and $(a_t a_i + a_j, t)$ are three elements of $B_{(a_i,a_j)}$ and $(a_m, -1)$, $(a_s(a_m+a_s)+a_n, s)$ and $(a_t(a_m+a_t)+a_n, t)$ are three elements of $C_{(a_m,a_n)}$ such that

$$\begin{aligned} (a_i, -1) &= (a_m, -1) \\ (a_s a_i + a_j, s) &= (a_s(a_m+a_s)+a_n, s) \\ (a_t a_i + a_j, t) &= (a_t(a_m+a_t)+a_n, t). \end{aligned}$$

From first equality we have $a_i = a_m$ and from second and third equalities we obtain $a_i = a_m + a_s + a_t$. So $a_s = a_t$ which is impossible. Therefore, $T = \{T_1, T_2\}$ is a $(q^2+q, q^2+1, 1)$ strong trade of volume q^2. □

In the Desarguesian plane $PG(2,q)$ of order q, q even, there is an easy representation of a strong trade. Consider the collection of q^2 (non-degenerate) conics in $PG(2,q)$

$$F_{bc} = \{\langle(x,y,z)\rangle : x^2 + by^2 + cz^2 + yz = 0\}$$

for $b, c \in GF(q)$. It is then easily verified that every line on the point $\langle(1,0,0)\rangle$ meets each of the F_{bc} in a unique point ($\langle(1,0,0)\rangle$ is the *nucleus* of each of the conics, see [5, p.165]). Let $\mathcal{B}$ be the set of lines of $PG(2,q)$ not on $\langle(1,0,0)\rangle$. Then a little algebra verifies that the set $\{\mathcal{B}, \{F_{bc} : b, c \in GF(q)\}\}$ is a $(q^2+q, q+1, 2)$ strong trade of volume q^2.

Theorem 2.4 *Let q be a power of two. There exists a $(q^2, q, 2)$ SST of volume $q^2 - q$.*

Proof: We use *oval derivation* to construct a strong trade as a subset of lines and conics in $PG(2,q)$. See [2] for details of oval derivation.

Choose a line l of $PG(2,q)$, and choose two distinct point P and N on l. Let $\mathcal{C}$ be the set of conics of $PG(2,q)$ with nucleus N and containing the point P. There are q^2-q such conics. Let $\mathcal{L}_{PN}$ be the sets of $2q$ lines that contain either P or N, but not both. Then it is well known that the incidence structure with points given by points of $PG(2,q)-\{l\}$, and lines $\mathcal{C}\cup\mathcal{L}_{PN}$ is a $(q^2,q,1)$ BIBD, i.e. is an affine plane of order q. This follows from the fact that a conic is determined uniquely by its nucleus and three further points such that the nucleus and the three points form a quadrangle.

Let $\mathcal{L}$ be the set of q^2-q lines of $PG(2,q)-(\{l\}\cup\mathcal{L}_{PN})$. We claim that $\{\mathcal{L},\mathcal{C}\}$ is a $(q^2,q,2)$ SST of volume q^2-q on the point-set of $PG(2,q)-\{l\}$.

Since the point-set is $PG(2,q)-\{l\}$ every block of $\mathcal{L}$ or $\mathcal{C}$ has q points. Now $\mathcal{C}\cup\mathcal{L}_{PN}$ is a $(q^2,q,1)$ BIBD so the only pairs of points that are not contained in some block of $\mathcal{C}$ are those contained in some line of $\mathcal{L}_{PN}$. It follows immediately that a pair of points is contained in some block of $\mathcal{L}$ if and only if they are contained in some block of $\mathcal{C}$. Hence $\{\mathcal{L},\mathcal{C}\}$ is a trade. The fact that it is strong follows since in a projective plane a line meets a conic in at most two points. □

The following theorem constructs SSTs with more blocks than the lower bounds of Lemmas 2.2 and 2.1, in the case of q odd the number of blocks is only one greater than the bound of Lemma 2.2.

Theorem 2.5 *There exists a $(q^2+q+1,q+1,2)$ SST of volume q^2+q+1 for every prime power q.*

Proof: We use the results of Jungnickel and Vedder in [8]. Let D be an abelian difference set of size $q+1$ in a group G for a finite projective plane π, i.e. π has points given by the elements of G, and lines the cosets of D. Then it is easy to show that for any $y\in G$ the set $-D+y$ is a set of $q+1$ points, no three collinear in π, i.e. is an oval.

Further, $-D$ is also a difference set in G and so the set of ovals $\{-D+y : y\in G\}$ in π are the lines of a projective plane π' (with point set G). It follows that every pair of elements of G is contained in a unique line of π and a unique line of π'. Also, each line of π' meets any line of π in at most two points. Hence the lines of π and the lines of π' are a strong Steiner trade of volume q^2+q+1.

Abelian difference sets of size $q+1$ are known for all prime powers q and can be easily constructed using a Singer cycle in the Desarguesian projective plane of order q [5, Theorem 4.2.2]. □

Corollary 2.6 *Let q be a power of 2. Then there exists a $(v,q+1,2)$ SST of volume $rq^2+s(q^2+q+1)$ for $r,s\geq 0$. In particular, there exists a $(v,q+1,2)$ SST of volume m for every $m\geq q^2(q^2+q+1)$.*

Proof: First note that, using the method of Lemma 2.3 of [6], if there exists a $(v_i,k,2)$ SST of volume m_i, $i=1,2$, then there exists a $(v_1+v_2,k,2)$ SST of volume m_1+m_2. Now the result follows by Theorems 2.3 and 2.5. □

Similarly by Theorems 2.4 and 2.5 we have:

Corollary 2.7 *Let q be a power of 2 such that $q-1$ is also a prime power. Then there exists a $(v,q,2)$ SST of volume $r(q^2-q)+s(q^2-q+1)$ for $r,s \geq 0$. In particular, there exists a $(v,q,2)$ SST of volume m for every $m \geq (q^2-q)(q^2-q+1)$.*

We conclude by giving a structural result about SSTs of minimal size.

Lemma 2.8 *Let $T = \{T_1, T_2\}$ be a $(q^2+q, q+1, 2)$ SST of volume q^2. Then $\{F(T), T_1\}$ is the dual of an affine plane of order q, i.e. is the dual of a $(q^2, q, 1)$ BIBD.*

Proof: We need to show: (i) each block of T_1 has $q+1$ points; (ii) $r_x = q$ for all $x \in F(T)$; and (iii) every pair of blocks of T_1 intersects in a unique element.
(i) Follows immediately from the definition of a trade.
(ii) By Lemma 2.1 we have $r_x \geq q$. Now if there exists an element $a \in F(T)$ with $r_a \geq q+1$ then $|F(T)| \geq 1 + (q+1)q > q^2 + q$. This is a contradiction.
(iii) The total number of pairs of intersecting blocks in T_1 must equal

$$\binom{q}{2} . |F(T)| = (q(q-1)/2)(q^2+q) = q^2(q^2-1)/2 = \binom{q^2}{2}.$$

Therefore any two blocks of T_1 intersect in a unique element. □

This lemma shows that when the lower bound of Lemma 2.1 is achieved for q even, then the blocks of either of the elements of the trade must form the dual of an affine plane. In particular, if there do not exist non-prime power order projective planes, then the bound of Lemma 2.1 is only achievable for q a power of two. It would be interesting to have a similar structural result for q odd.

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