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A bound on the order of a graph when both the graph and its complement are contraction-critically k-connected

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Abstract

An edge of a k-connected graph is said to be k-contractible if the contraction of the edge results in a k-connected graph. A k-connected graph with no k-contractible edge is called contraction-critically k-connected. For $k \geq 9^3$, we prove that if G is a graph such that both G and its complement $\bar{G}$ are contraction-critically k-connected, then $|V(G)| < k^{5/3}/3 + 3k^{3/2}$.

1 Introduction

In this paper, we consider only finite, undirected, simple graphs with no loops and no multiple edges.

Let k be an integer with $k \geq 2$. An edge e of a k-connected graph G is said to be k-contractible if the contraction of e results in a k-connected graph. If a k-connected graph G does not have a k-contractible edge, then G is said to be contraction-critically k-connected. For a graph G, we let $\bar{G}$ denote the complement of G.

It is known that for $k = 2, 3$, the complete graph of order $k + 1$ is the only contraction-critically k-connected graph (Tutte [4]), and a characterization of contraction-critically 4-connected graphs was obtained by Fontet [2] and independently by Martinov [3]. For $k \geq 5$, J. Akiyama et al. [1] considered graphs G for which both G and $\bar{G}$ are contraction-critically k-connected, and proved that such graphs have order less than $k^{5/3} + 4k^{3/2}$. Also in [1], for each k with $k \geq 2 \cdot 10^6$, a graph G of order greater than $3k^{5/3}/32 - 13k^{4/3}/64$ such that both G and $\bar{G}$ are contraction-critically k-connected was constructed. Thus the exponent 5/3 in the upper bound is best possible. The purpose of this paper is to improve the coefficient 1 of the term $k^{5/3}$ to 1/3 which, as we shall explain below, is likely to be best possible.

Theorem *Let k be an integer with $k \geq 9^3$, and let G be a graph such that both G and $\bar{G}$ are contraction-critically k-connected. Then*

$$|V(G)| < k^{5/3}/3 + 3k^{3/2}.$$

Judging from the argument in the proof of the Theorem (see Section 3), it is likely that there exist graphs G for which equality holds asymptotically in both Subcase II-(i) and Subcase II-(ii), i.e., graphs G such that $|X| = k/3+o(k)$, $|Z| = k^{4/3}/4+o(k^{4/3})$ and $|W| = k^{5/3}/3 + o(k^{5/3})$, where X, Z and W are as in the proof of the Theorem (though we have been unable to construct such graphs). Thus we make the following conjecture.

Conjecture. *Let n_k denote the maximum order of a graph G such that both G and $\bar{G}$ are contraction-critically k-connected. Then we have $n_k = k^{5/3}/3 + o(k^{5/3})$.*

We conclude this section with some more definitions. Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, we let $N_G(x)$ denote the neighborhood of x and, for $S \subseteq V(G)$, we let $N_G(S) = (\cup_{x\in S} N_G(x)) - S$. A subset S of $V(G)$ is said to be a cutset of G if $G - S$ is not connected. A cutset S is said to be an i-cutset if $|S| = i$. For $S \subseteq V(G)$, we let $G[S]$ denote the subgraph induced by S in G. For $A, B \subseteq V(G)$ with $A \cap B = \emptyset$, we let $E_G(A, B)$ denote the set of edges of G joining a vertex in A and a vertex in B. For $A \subseteq V(G)$ and an edge $e = uv$ of G with $u, v \in V(G) - A$, we say that A covers e in G if $u, v \in N_G(A)$. A vertex x is often identified with the set $\{x\}$; for example, if B is a subset of $V(G)$ with $x \notin B$, then we write $E_G(x, B)$ for $E_G(\{x\}, B)$.

Let now G be a k-connected graph of order at least $k + 2$. A nonempty subset A of $V(G)$ is called a k-fragment of G if $|N_G(A)| = k$ and $V(G) - A - N_G(A) \neq \emptyset$. Thus if A is a k-fragment and if we let $A' = V(G) - A - N_G(A)$, then $N_G(A)$ is a k-cutset and A' is also a k-fragment with $N_G(A') = N_G(A)$. Note also that an edge e of G is k-contractible if and only if e is not covered by any of the k-fragments of G.

2 Preliminary Results

Throughout the rest of this paper, let k be an integer with $k \geq 4$. The first three lemmas are proved in [1; Lemmas 2.1 through 2.3].

Lemma 2.1. *Let G be a k-connected graph of order at least $k+2$. Let $A_1, A_2, \cdots, A_s$ be k-fragments of G, and set $L = A_1 \cup A_2 \cup \cdots \cup A_s$. Then for each $x \in L$, $|E_G(x, V(G) - L)| \leq k$.*

Lemma 2.2. *Let G be a contraction-critically k-connected graph of order at least $k+2$. Choose k-fragments $A_1, A_2, \cdots, A_p$ covering all edges of G so that $(|A_1|, |A_2|, \cdots, |A_p|)$ is lexicographically minimum. Let $1 \leq i < j \leq p$. Then the following hold.*

(i) *We have $A_i \cap A_j = \emptyset$ or $A_i \subseteq A_j$.*
(ii) *If $|A_i| \geq k + 1$ and $A_i \cap A_j = \emptyset$, then $E_G(A_i, A_j) = \emptyset$.*

For a real number x, we let $\binom{x}{2} = x(x-1)/2$.

Lemma 2.3. *Let G, $A_1, A_2, \cdots A_p$ be as in Lemma 2.2. Let $1 \le s \le p$, and set $L = A_1 \cup A_2 \cup \cdots \cup A_s$. Let m denote the number of those edges of $G - L$ which are covered by some A_i $(1 \le i \le s)$. Then $m \le |L| \binom{k}{2}$.*

We now prove a numerical result.

Lemma 2.4. *Let μ be an integer with $1 \le \mu \le k$. Let $l_1, \cdots, l_t$ be integers such that $k - \mu \le l_j \le k$ for each $1 \le j \le t$, and write $l_1 + \cdots + l_t = (k-\mu)t + \lambda$. Then $\binom{l_1}{2} + \cdots + \binom{l_t}{2} \le (\lambda/\mu)\binom{k}{2} + (t - \lambda/\mu)\binom{k-\mu}{2}$.*

Proof. For each $1 \le i \le t$, write $l_i = k - \mu + \mu x_i$ $(0 \le x_i \le 1)$. Since $\binom{x}{2}$ is a convex function, we have $\binom{l_i}{2} \le x_i \binom{k}{2} + (1 - x_i)\binom{k-\mu}{2}$ for each $1 \le i \le t$. Hence $\sum_{1 \le i \le t} \binom{l_i}{2} \le \sum_{1 \le i \le t}(x_i \binom{k}{2} + (1-x_i)\binom{k-\mu}{2}) = (\lambda/\mu)\binom{k}{2} + (t - \lambda/\mu)\binom{k-\mu}{2}$. □

We need the following refinements of Lemma 2.3.

Lemma 2.5. *Let G, $A_1, \cdots, A_s$, L be as in Lemma 2.3, and let X, W be subsets of $V(G-L)$ such that $X \cup W = V(G-L)$, $X \cap W = \emptyset$, and $1 \le |X| \le k$. Let λ be an integer with $0 \le \lambda \le |X||L|$, and suppose that $|E_G(L, X)| \ge |L||X| - \lambda$. Let m denote the number of those edges in $E(G[W])$ which are covered by some A_i with $1 \le i \le s$. Then $m \le (\lambda/|X|)\binom{k}{2} + (|L| - \lambda/|X|)\binom{k-|X|}{2}$.*

Proof. Let $A_{i_1}, A_{i_2}, \cdots, A_{i_t}$ be maximal members among $A_1, A_2, \cdots, A_s$. Then by Lemma 2.2 (i), $A_{i_h} \cap A_{i_j} = \emptyset$ for any h, j with $h \ne j$. Also $L = \cup_{1 \le j \le t} A_{i_j}$, and hence $t \le |L|$. Now if an edge e of $G[W]$ is covered by A_i $(1 \le i \le s)$, then letting j be the index such that $A_i \subseteq A_{i_j}$, we see that e is covered by A_{i_j}. Thus m is equal to the number of edges of $E[W]$ covered by some A_{i_j}. For each $1 \le j \le t$, let $l_j = |N_G(A_{i_j}) - X|$. Then for each j, the number of edges of $G[W]$ covered by A_{i_j} is at most $\binom{|N_G(A_{i_j}) \cap W|}{2} \le \binom{l_j}{2}$. Hence $m \le \sum_{1 \le j \le t}\binom{l_j}{2}$. On the other hand, for each j, we have $l_j = k - |N_G(A_{i_j}) \cap X|$ because A_{i_j} is a k-fragment, and hence $k - |X| \le l_j \le k$. Write $\sum_{1 \le j \le t} l_j = (k - |X|)t + \lambda'$. Then by Lemma 2.4, $\sum_{1 \le j \le t}\binom{l_j}{2} \le (\lambda'/|X|)\binom{k}{2} + (t - \lambda'/|X|)\binom{k-|X|}{2}$. Further for each j, $|X| - |N_G(A_{i_j}) \cap X| \le |N_{\bar{G}}(A_{i_j}) \cap X| \le |E_{\bar{G}}(A_{i_j}, X)| = |A_{i_j}||X| - |E_G(A_{i_j}, X)|$, and hence $l_j = k - |N_G(A_{i_j}) \cap X| = (k - |X|) + (|X| - |N_G(A_{i_j}) \cap X|) \le (k - |X|) + |A_{i_j}||X| - |E_G(A_{i_j}, X)|$. Therefore $\sum_{1 \le j \le t} l_j \le (k - |X|)t + \sum_{1 \le j \le t}(|A_{i_j}||X| - |E_G(A_{i_j}, X)|) = (k - |X|)t + (|L||X| - |E_G(L, X)|)$. Since $|L||X| - |E_G(L, X)| \le \lambda$ by assumption, this implies $\sum_{1 \le j \le t} l_j \le (k - |X|)t + \lambda$, and hence $\lambda' \le \lambda$. Since $\binom{k-|X|}{2} < \binom{k}{2}$, this clealy implies

$(\lambda'/|X|)\binom{k}{2}+(t-\lambda'/|X|)\binom{k-|X|}{2} \le (\lambda/|X|)\binom{k}{2}+(t-\lambda/|X|)\binom{k-|X|}{2}$. Since $t \le |L|$, we now obtain $m \le \sum_{1\le j\le t}\binom{l_j}{2} \le (\lambda'/|X|)\binom{k}{2}+(t-\lambda'/|X|)\binom{k-|X|}{2} \le (\lambda/|X|)\binom{k}{2}+(t-\lambda/|X|)\binom{k-|X|}{2} \le (\lambda/|X|)\binom{k}{2}+(|L|-\lambda/|X|)\binom{k-|X|}{2}$. □

Lemma 2.6. *Let G, $A_1, \cdots, A_s$, L be as in Lemma 2.3, and let W be a subset of $V(G-L)$. Let λ be an integer, and suppose that $|E_G(L,W)| \le \lambda$. Let m denote the number of those edges in $E(G[W])$ which are covered by some A_i with $1 \le i \le s$. Then $m \le (\lambda/k)\binom{k}{2}$.*

Proof. Let $A_{i_1}, A_{i_2}, \cdots, A_{i_t}$ be as in the proof of Lemma 2.5. Then m is equal to the number of edges of $G[W]$ covered by some A_{i_j}. For each $1 \le j \le t$, let $l_j = |N_G(A_{i_j}) \cap W|$. Then for each j, the number of edges of $G[W]$ covered by A_{i_j} is at most $\binom{l_j}{2}$. On the other hand, $0 \le l_j \le k$ for each j, and $\sum_{1\le j\le t} l_j \le \sum_{1\le j\le t}|E_G(A_{i_j}, W)| = |E_G(L,W)| \le \lambda$. Consequently, applying Lemma 2.4 with $\mu = k$, we obtain $m \le \sum_{1\le j\le t}\binom{l_j}{2} \le (\lambda/k)\binom{k}{2}$. □

Lemma 2.7. *Let G, $A_1, \cdots, A_s$, L be as in Lemma 2.3, and let Z, W be subsets of $V(G-L)$ such that $Z \cap W = \emptyset$. Let m denote the number of those edges in $E_G(Z,W)$ which are covered by some A_i with $1 \le i \le s$. Then $m \le |L|k^2/4$.*

Proof. Let $A_{i_1}, A_{i_2}, \cdots, A_{i_t}$ be as in the proof of Lemma 2.5. Then $t \le |L|$, and m is equal to the number of edges in $E_G(Z,W)$ covered by some A_{i_j}. For each j, the number of edges in $E_G(Z,W)$ covered by A_{i_j} is at most $|N_G(A_{i_j})\cap Z||N_G(A_{i_j})\cap W| \le |N_G(A_{i_j}) \cap Z|(k - |N_G(A_{i_j} \cap Z)|) \le k^2/4$. Hence $m \le tk^2/4 \le |L|k^2/4$. □

The following lemma is proved in [1; Lemma 2.4]

Lemma 2.8. *Let G be a graph with $|V(G)| > 3k$ such that both G and $\bar{G}$ are contraction-critically k-connected. Let A be a k-fragment of G and set $A' = V(G) - A - N_G(A)$, let B be a k-fragment of $\bar{G}$ and set $B' = V(G) - B - N_{\bar{G}}(B)$, and suppose that $|A'| \ge |A|$ and $|B'| \ge |B|$. Then $A \cap B = \emptyset$.*

3 Proof of the Theorem

Let k, G be as in the Theorem. We may assume $|V(G)| > 3k$. Choose k-fragments $A_1, A_2, \cdots A_p$ of G covering all edges of G so that $(|A_1|, |A_2|, \cdots |A_p|)$ is lexicographically minimum. Simillarly choose k-fragments $B_1, B_2, \cdots B_q$ of $\bar{G}$ covering all edges of $\bar{G}$ so that $(|B_1|, |B_2|, \cdots |B_q|)$ is lexicographically minimum. Set $X = \cup_{1\le i\le p} A_i$, $Y = \cup_{1\le j\le q} B_j$. By Lemma 2.8, $X \cap Y = \emptyset$. The following claim is proved in [1; Claim 2.6].

Claim 3.1. *$|X| \le 2k$ or $|Y| \le 2k$.*

By symmetry, we may assume $|X| \leq 2k$. Let r $(0 \leq r \leq q)$ be the index such that $|B_j| < k^{3/2}$ for all $1 \leq j \leq r$ and $|B_j| \geq k^{3/2}$ for all $r+1 \leq j \leq q$. Set $Z = \cup_{1 \leq j \leq r} B_j$ and $W = V(G) - X - Z$. The following three claims are proved in [1; Claims 2.7 through 2.9].

Claim 3.2. $B_{r+1} \subseteq B_{r+2} \subseteq \cdots \subseteq B_q$

Claim 3.3. *If $r < q$, then the number of those edges of $\bar{G}[W]$ which are covered by some B_j with $r+1 \leq j \leq q$ is at most $k(|(B_q - B_{r+1}) \cap W| + k/2)$.*

Claim 3.4. $|Z| < 2k^{3/2} + k$.

Write $|X| = \alpha k$, $|Z| = \beta k^{4/3}$. Since $|X| \leq 2k$ by assumption, $\alpha \leq 2$.

Case I. $0 \leq \beta < 1/9$.

By Claim 3.3, the number of edges of $\bar{G}[W]$ covered by some B_j with $r+1 \leq j \leq q$ is at most $k(|W| + k/2)$ (note that this is true even if $r = q$). Also, applying Lemma 2.3 to $\bar{G}$, we see from the the assumption of Case I that the number of edges of $\bar{G}[W]$ covered by some B_j with $1 \leq j \leq r$ is at most $|Z|\binom{k}{2} < |Z|k^2/2 < k^{10/3}/18$. Hence $|E(\bar{G}[W])| < k(|W| + k/2) + k^{10/3}/18$. On the other hand, since $|X| \leq 2k$, $|E(G[W])| \leq |X|\binom{k}{2} < k^3$ by Lemma 2.3. Consequently $\binom{|W|}{2} = |E(\bar{G}[W])| + |E(G[W])| < k(|W| + k/2) + k^{10/3}/18 + k^3$. That is to say, $|W|^2 - (1 + 2k)|W| - k^{10/3}/9 - 2k^3 - k^2 < 0$, which implies $|W| < k^{5/3}/3 + 3k^{4/3} - 2k$ (note that $(k^{5/3}/3 + 3k^{4/3} - 2k)^2 - (1+2k)(k^{5/3}/3 + 3k^{4/3} - 2k) - k^{10/3}/9 - 2k^3 - k^2 = 7k^{8/3} - 18k^{7/3} + 7k^2 - k^{5/3}/3 - 3k^{4/3} + 2k > 0$). Therefore $|V(G)| = |W| + |Z| + |X| < (k^{5/3}/3 + 3k^{4/3} - 2k) + k^{4/3}/9 + 2k < k^{5/3}/3 + 3k^{3/2}$ by the assumption of Case I.

Case II. $\beta \geq 1/9$.

Since $k \geq 9^3$, we have $|Z| \geq k^{4/3}/9 \geq k$.

Subcase II-(i). $0 < \alpha < 1$, $\beta > 3\alpha/4$.

Applying Lemma 2.1 to $\bar{G}$, we get $|E_{\bar{G}}(Z, W)| \leq k|Z|$. On the other hand, $|E_G(Z, W)| \leq k^2|X|/4$ by Lemma 2.7. Consequently $|Z||W| = |E_{\bar{G}}(Z, W)| + |E_G(Z, W)| \leq k|Z| + k^2|X|/4$. Since $\beta > 3\alpha/4$ by the assumption of Subcase II-(i), this implies $|W| \leq k + k^2|X|/(4|Z|) = k + k^{5/3}\alpha/(4\beta) < k + k^{5/3}/3$. Therefore $|V(G)| = |W| + |Z| + |X| < (k + k^{5/3}/3) + (2k^{3/2} + k) + k < k^{5/3}/3 + 3k^{3/2}$ by Claim 3.4 and the assumption of Subcase II-(i).

Subcase II-(ii). $0 < \alpha < 1$, $\beta \leq 3\alpha/4$.

By Claim 3.3, the number of edges of $\bar{G}[W]$ covered by some B_j with $r+1 \leq j \leq q$ is at most $k(|W| + k/2)$. By Lemma 2.1, $|E_G(X, Z)| \leq k|X|$, and hence $|E_{\bar{G}}(Z, X)| \geq |Z||X| - k|X|$. Also recall that we have $k \leq |Z|$ by the assumption of Case II. Thus applying Lemma 2.5 to $\bar{G}$ with $L = Z$ and $\lambda = k|X|$, we see that the number of edges of $\bar{G}[W]$ covered by some B_j with $1 \leq j \leq r$ is at most

$k\binom{k}{2}+(\beta k^{4/3}-k)\binom{(1-\alpha)k}{2} < k^3/2+(\beta k^{4/3}-k)(1-\alpha)^2k^2/2$. Hence $|E(\bar{G}[W])| < k(|W|+k/2)+k^3/2+(\beta k^{4/3}-k)(1-\alpha)^2k^2/2$. On the other hand, $|E(G[W])| \le \alpha k\binom{k}{2} < \alpha k^3/2$ by Lemma 2.3. Consequently $\binom{|W|}{2} = |E(\bar{G}[W])|+|E(G[W])| < k(|W|+k/2)+k^3/2+(\beta k^{4/3}-k)(1-\alpha)^2k^2/2+\alpha k^3/2$; that is to say, $|W|^2-(1+2k)|W|-\beta(1-\alpha)^2k^{10/3}-(3\alpha-\alpha^2)k^3-k^2 < 0$. Since $\beta(1-\alpha)^2 \le 3\alpha(1-\alpha)^2/4 \le 1/9$ and $3\alpha-\alpha^2 < 2$ by the assumption of Subcase II-(ii), this implies $|W|^2-(1+2k)|W|-k^{10/3}/9-2k^3-k^2 < 0$. As in Case I, this implies $|W| < k^{5/3}/3+3k^{4/3}-2k$. Therefore $|V(G)| = |W|+|Z|+|X| < (k^{5/3}/3+3k^{4/3}-2k)+(2k^{3/2}+k)+k \le k^{5/3}/3+3k^{3/2}$.

Subcase II-(iii). $1 \le \alpha \le 2$.

By Claim 3.3, the number of edges of $\bar{G}[W]$ covered by some B_j with $r+1 \le j \le q$ is at most $k(|W|+k/2)$. By Lemma 2.1, $|E_G(X,Z)| \le k|X|$, and hence $|E_{\bar{G}}(Z,X)| \ge |Z||X|-k|X|$. Applying Lemma 2.1 to $\bar{G}$, we also obtain $|E_{\bar{G}}(Z,X\cup W)| \le k|Z|$. Hence $|E_{\bar{G}}(Z,W)| \le k|Z|-(|Z||X|-k|X|) = k^2-(|X|-k)(|Z|-k)$. Since $|Z| \ge k$ by the assumption of Case II and $|X| \ge k$ by the assumption of Subcase II-(iii), this implies $|E_{\bar{G}}(Z,W)| \le k^2$. Thus applying Lemma 2.6 to $\bar{G}$ with $L = Z$ and $\lambda = k^2$, we see that the number of edges of $\bar{G}[W]$ covered by some B_j with $1 \le j \le r$ is at most $k\binom{k}{2} < k^3/2$. Hence $|E(\bar{G}[W])| < k(|W|+k/2)+k^3/2$. On the other hand, $|E(G[W])| \le \alpha k\binom{k}{2} < \alpha k^3/2$ by Lemma 2.3. Consequently $\binom{|W|}{2} = |E(\bar{G}[W])|+|E(G[W])| < k(|W|+k/2)+k^3/2+\alpha k^3/2$; that is to say, $|W|^2-(1+2k)|W|-(1+\alpha)k^3-k^2 < 0$. Since $\alpha \le 2$, this implies $|W|^2-(1+2k)|W|-3k^3-k^2 < 0$, and hence $|W| < 2k^{3/2}-3k$ (note that $(2k^{3/2}-3k)^2-(1+2k)(2k^{3/2}-3k)-3k^3-k^2 = k^3-16k^{5/2}+14k^2-2k^{3/2}+3k > 0$). Therefore $|V(G)| = |W|+|Z|+|X| < (2k^{3/2}-3k)+(2k^{3/2}+k)+2k \le k^{5/3}/3+3k^{3/2}$. This completes the proof of the Theorem.

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