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An upper bound of the basis number of the lexicographic product of graphs

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Abstract

This paper is primarily concerned with giving an upper bound of the basis number of the lexicographic product of graphs.

1 Introduction

Unless otherwise specified all graphs considered here are finite, undirected, simple and connected. Our terminologies and notations will be standard except as indicated. Let G be a graph and $e_1, e_2, \ldots, e_{|E(G)|}$ be an enumeration of its edges. Then any subgraph S of $E(G)$ corresponds to a $(0,1)$-vector $(\zeta_1, \zeta_2, \ldots, \zeta_{|E(G)|}) \in (Z_2)^{|E(G)|}$ with $\zeta_i = 1$ if $e_i \in S$ and $\zeta_i = 0$ if $e_i \notin S$. Let $\mathcal{C}(G)$, called the *cycle space*, be the subspace of $(Z_2)^{|E(G)|}$ generated by the vectors corresponding to the cycles in G. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is well known that if r is the number of components of G, then $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$.

A basis of $\mathcal{C}(G)$ is called *d-fold* if each edge of G occurs in at most d of the cycles in the basis. The *basis number* of G, $b(G)$, is the smallest non-negative integer number d such that $\mathcal{C}(G)$ has a d-fold basis. The *required basis* of $\mathcal{C}(G)$ is a basis which is $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\mathcal{B}' = \{\varphi(c) | c \in \mathcal{B}\}$ is called the *corresponding (required) basis* of $\mathcal{B}$ in H. The *fold of an edge* in a given basis $\mathcal{B}$ is the number of cycles of $\mathcal{B}$ containing this edge. The first use of the basis number of a graph was the theorem of MacLane when he classified graphs into planar and non planar with respect to $b(G)$. In fact, MacLane proved that a graph G is planar if and only if $b(G) \leq 2$. Formally, the basis number was introduced by Schmeichel when he proved that there are graphs with arbitrary large basis numbers. Moreover, Schmeichel proved that $b(K_n) \leq 3$.

Let G_1 and G_2 be two graphs. The *direct product* $G = G_1 \wedge G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)\}$. The *lexicographic product* $G = G_1[G_2]$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ or $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ $\}$. The *cartesian product* $G = G_1 \times G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G_1)$ and $u_2 = v_2$ or $u_2v_2 \in E(G_2)$ and $u_1 = v_1\}$.

The basis number of the lexicographic product graphs was studied by Schmeichel [7] and Ali [1] who proved the following results.

Theorem 1.1 *(Schmeichel) For each* $n \geq 5, b(P_2[N_n]) \leq 4$ *where* N_n *is a null graph with* n *vertices.*

In fact, Schemeichel proved the more general case when he proved that $b(K_{n,m}) \leq 4$. Ali [1] proved the following result:

Theorem 1.2 *(Ali) For each* $n, m \geq 5, b(K_n[N_m]) \leq 3 + 2b(K_n)$

The direct product was studied by Jaradat [4] who proved the following results. Moreover, Jaradat classified trees with respect to the basis number of their direct product with paths of order greater than or equal to 5

Theorem 1.3 *(Jaradat) For each bipartite graphs* G *and* H*,* $b(G \wedge H) \leq 5 + b(G) + b(H)$.

Theorem 1.4 *(Jaradat) For each bipartite graph* G *and cycle* C_m*,* $b(G \wedge C_m) \leq 3 + b(G)$.

Alsardary [2] gave the following result:

Theorem 1.5 *(Alsardary) For every* $n \geq 2$ *and* $d \geq 1$*, we have* $b(K_n^d) \leq 2d + 1$ *where* K_n^d *is the cartesian product of* d *copies of* K_n*.*

The results cited above give rise to the following problem whether similar results hold for the lexicographic product. More precisely, we have the following question.

Problem. Does there exist an upper bound of the basis number of the lexicographic product of two graphs with respect to the basis number of the factors?

In this paper we focus our attention on obtaining a complete solution to this problem. The method employed in this paper is based in part on ideas of Ali [1], Jaradat [4] and Schmeichel [7]. Since trees do not have a uniform form, we state the following result of Jaradat which gives an appropriate decomposition for any tree and will be useful in our work.

Proposition 1.1 *(Jaradat) Let T be a tree of order ≥ 2. Then T can be decomposed into an edge-disjoint union of subgraphs S_i, $i = 1, 2, \ldots, r$ for some integer r, such that, the following holds:*
(i) For each $i \geq 1$, S_i is either a star or a path of order 2 and S_1 is a path.
(ii) For each $v \in V(T)$, if $d_T(v) \geq 2$, then v belongs to exactly two of the subgraphs S_i, and if $deg_T(v) = 1$, then v belongs to only one of them.
(iii) $V(S_i) \cap (\cup_{j=1}^{i-1} V(S_j)) = v_1^{(i)}$ where $v_1^{(i)} \in V(S_i)$ such that $deg_{S_i}(v_1^{(i)}) = Max_{v \in V(S_i)}\, deg_{S_i}(v)$ with $deg_{\cup_{j=1}^{i-1} S_j}(v_1^{(i)}) = 1$ for each $i = 2, 3, \ldots, r$, and $v_1^{(i)} \neq v_1^{(j)}$ for each $i \neq j$.

In the rest of this paper $f_B(e)$ stands for the number of cycles in B containing e and $E(B) = \cup_{b \in B} E(b)$ where $B \subseteq \mathcal{C}(G)$. $\mathcal{B}_G$ and $\mathcal{B}_H$ denote the required basis of G and H, respectively.

2 Main Results

In this section we give an upper bound of the basis number of the lexicographic product of two graphs. Let M be a null graph with vertex set $\{a_1, a_2, \ldots, a_n\}$ and let $P_2 = uv$ be a path of order 2. Then

$$\mathcal{B} = \{(u, a_j)(v, a_l)(u, a_{j+1})(v, a_{l+1})(u, a_j) : 1 \leq j, l \leq n-1\}.$$

is the Schemeichel's 4-fold basis of $\mathcal{C}(P_2[M])$ (see Theorem 2.4 in [7]). Moreover, (1) if $e = (u, a_1)(v, a_n)$ or $e = (u, a_n)(v, a_1)$ or $e = (u, a_1)(v, a_1)$ or $e = (u, a_n)(v, a_n)$, then $f_{\mathcal{B}}(e) = 1$. (2) If $e = (u, a_1)(v, a_l)$ or $(u, a_j)(v, a_1)$ or $(u, a_n)(v, a_l)$ or $(u, a_j)(v, a_n)$, then $f_{\mathcal{B}}(e) \leq 2$. (3) if $e \in E(P_2[M])$ and is not of the above form, then $f_{\mathcal{B}}(e) \leq 4$.

The graph $P_2[T]$ contains the graph $P_2\left[N_{|V(T)|}\right]$ as a subgraph where $N_{|V(T)|}$ is the null graph with the vertex set $V(T)$. One can see that $V(P_2[T]) = V\left(P_2\left[N_{|V(T)|}\right]\right)$ and $E(P_2[T]) = E\left(P_2\left[N_{|V(T)|}\right]\right) \cup E(M)$ where $M = (u \times T) \cup (v \times T)$. Moreover, $P_2\left[N_{|V(T)|}\right]$ is isomorphic to $K_{|V(T)|,|V(T)|}$. Note that dim $\mathcal{C}(P_2[T]) =$ dim $\mathcal{C}\left(P_2\left[N_{|V(T)|}\right]\right) + 2|E(T)|$.

Lemma 2.1. *Let T be a tree and $P_2 = uv$ be a path of order 2. Then $b(P_2[T]) \leq 4$. Moreover, if $|V(T)| \geq 14$, the equality holds.*

Proof. Let $T = \bigcup_{i=1}^{k} S_i$ as in Proposition 1.1. Let $V(S_i) = \left\{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \ldots, v_{n_i}^{(i)}\right\}$ where $\deg_{S_i}(v_1^{(i)}) = n_i - 1$. Define $\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}^{**}$ where $\mathcal{B}^*$ is the Schmeichel 's 4-fold basis of the subspace $\mathcal{C}\left(P_2\left[N_{|V(T)|}\right]\right)$ and $\mathcal{B}^{**} = \bigcup_{i=1}^{k} \mathcal{B}_i$ where $\mathcal{B}_i = \mathcal{B}_{iu} \cup \mathcal{B}_{iv}$ and

$$\mathcal{B}_{iu} = \left\{a_{ju}^{(i)} = (u, v_1^{(1)})(v, v_j^{(i)})(v, v_1^{(i)})(v, v_{j-1}^{(i)})(u, v_1^{(1)}) : j = 3, \ldots, n_i\right\} \cup$$
$$\left\{a_{2u}^{(i)} = (u, v_{n_k}^{(k)})(v, v_2^{(i)})(v, v_1^{(i)})(u, v_{n_k}^{(k)})\right\},$$
$$\mathcal{B}_{iv} = \left\{a_{jv}^{(i)} = (v, v_1^{(1)})(u, v_j^{(i)})(u, v_1^{(i)})(u, v_{j-1}^{(i)})(v, v_1^{(1)}) : j = 3, \ldots, n_i\right\} \cup$$

$\left\{a_{2v}^{(i)}=(v,v_{n_k}^{(k)})(u,v_2^{(i)})(u,v_1^{(i)})(v,v_{n_k}^{(k)})\right\}$.
We now proceed by induction on n_i to show that $\mathcal{B}_{iu}$ is linearly independent for each $i=1,2,\ldots,k$. If $n_i=2$, then $\mathcal{B}_{iu}$ consists of one cycle $a_{2u}^{(i)}$ and so it is linearly independent. Assume $n_i\geq 3$ and it is true for less than n_i. Note that $\mathcal{B}_{iu}=\{a_{ju}^{(i)}\}_{j=2}^{n_i-1}\cup\{a_{n_iu}^{(i)}\}$. By the inductive step, both of $\{a_{ju}^{(i)}\}_{j=2}^{n_i-1}$ and $\{a_{n_iu}^{(i)}\}$ are linearly independent. Since $a_{n_iu}^{(i)}$ contains the edge $(v,v_{n_i}^{(i)})(v,v_1^{(i)})$ which is not in any cycle of $\{a_{ju}^{(i)}\}_{j=2}^{n_i-1}$, as a result $\mathcal{B}_{iu}=\{a_{ju}^{(i)}\}_{j=2}^{n_i}$ is linearly independent. One can see that any linear combination of $\mathcal{B}_{iu}=\{a_{ju}^{(i)}\}_{j=2}^{n_i}$ must contain an edge of $E(v\times S_i)$. Moreover, $E(v\times S_i)\cap E(v\times S_j)=\phi$ for each $i\neq j$. Thus, $\cup_{i=1}^{k}\mathcal{B}_{iu}$ is linearly independent. Similarly, one can show that $\cup_{i=1}^{k}\mathcal{B}_{iv}$ is linearly independent. By remarking that

$$E(\cup_{i=1}^{k}\mathcal{B}_{iu})\cap E(\cup_{i=1}^{k}\mathcal{B}_{iv})=\left\{(u,v_{n_k}^{(k)})(v,v_{n_k}^{(k)}),(u,v_1^{(1)})(v,v_{n_k}^{(k)}),(u,v_{n_k}^{(k)})(v,v_1^{(1)})\right\}$$

which is an edge set of a path, we have that any linear combination of cycles of $\cup_{i=1}^{k}\mathcal{B}_{iu}$ must contain at least one edge which is not in any linear combination of cycles of $\cup_{i=1}^{k}\mathcal{B}_{iv}$. Therefore, $\mathcal{B}^{**}$ is linearly independent. Also, every linear combination of cycles of $\mathcal{B}^{**}$ contains at least one edge of $E(v\times S_i)\cup E(v\times S_j)$ which is not in any cycle of $P_2\left[N_{|V(T)|}\right]$. Thus, $\mathcal{B}$ is linearly independent. To this end,

$$\sum_{i=1}^{k}n_i=|V(T)|+k-1,$$

And so,

$$\begin{aligned}|\mathcal{B}^{**}| &= 2\sum_{i=1}^{k}(n_i-1)\\ &= 2|V(T)|-2\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{B}| &= |\mathcal{B}^{*}|+|\mathcal{B}^{**}|\\ &= (|V(T)|^2-2|V(T)|+1)+(2|V(T)|-2)\\ &= |V(T)|^2-1\\ &= \dim\,\mathcal{C}\left(P_2\left[T\right]\right).\end{aligned}$$

Therefore, $\mathcal{B}$ is a basis for $\mathcal{C}\left(P_2\left[T\right]\right)$. We now show that $\mathcal{B}$ is a 4-fold basis. Let $e\in E(P_2\left[T\right])$. (1) If $e\in E(u\times T)\cup E(v\times T)$, then $f_{\mathcal{B}^*}(e)=0$ and $f_{\mathcal{B}^{**}}(e)\leq 2$. (2) If $e\in M=\left\{(u,v_1^{(1)})(v,v_j^{(i)}),(v,v_1^{(1)})(u,v_j^{(i)}):i=1,2,\ldots,k \text{ and } j=1,2,\ 3,\ldots,n_i\right\}\cup\left\{(u,v_{n_k}^{(k)})(v,v_j^{(i)}),\quad (v,v_{n_k}^{(k)})(u,v_j^{(i)}),:i=1,2,\ldots,k \text{ and } j=1,2,3,\ldots,n_i\right\}$, then $f_{\mathcal{B}^*}(e)\leq 2$ and $f_{\mathcal{B}^{**}}(e)\leq 2$. (3) If $e\in E\left(P_2\left[N_{|V(T)|}\right]\right)-\left\{M\cup\left\{(v,v_{n_k}^{(k)})(u,v_{n_k}^{(k)})\right\}\right\}$, then $f_{\mathcal{B}^*}(e)\leq 4$ and $f_{\mathcal{B}^{**}}(e)=0$. Hence, $b\left(P_2\left[T\right]\right)\leq 4$. To complete the proof, we eliminate any possibility for $\mathcal{C}\left(P_2\left[T\right]\right)$ to have a 3-fold basis when $|V(T)|\geq 14$.

Suppose that $\mathcal{C}(P_2[T])$ contains a 3-fold basis $\mathcal{B}$, we show that such a basis does not exist. We consider the following cases:

Case 1. Suppose that $\mathcal{B}$ consists of only 3-cycles. Then $|\mathcal{B}| \le 6|E(T)|$ because any 3-cycle must contain an edge of $(u \times T) \cup (v \times T)$ and the fold of any of these edges is at most 3. Since $|\mathcal{B}| = |V(T)|^2 - 2|V(T)| + 2|E(T)| + 1$, as a result $|V(T)|^2 - 2|V(T)| + 2|E(T)| + 1 \le 6|E(T)|$. Thus, $|V(T)|^2 - 6|V(T)| + 5 \le 0$, which implies $(|V(T)| - 5)(|V(T)| - 1) \le 0$, which does not hold if $|V(T)| \ge 6$.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4. Then $4\left(|V(T)|^2 - 2|V(T)| + 2|E(T)| + 1\right) \le 3\left(|V(T)|^2 + 2|E(T)|\right)$. Therefore, $|V(T)|^2 - 6|V(T)| + 2 \le 0$ which does not hold if $|V(T)| \ge 6$.

Case 3. Suppose that $\mathcal{B}$ consists of s cycles of length 3 and t cycles of length greater than or equal to 4. As in Case 1, $s \le 6|E(T)|$. Since $|E(P_2[T])| = |V(T)|^2 + 2|E(T)|$ and the fold of every edge of $P_2[T]$ in $\mathcal{B}$ is at most 3 and $3s$ edges are used to form the s 3-cycles, as a result $t \le \left\lfloor \left(3\left(|V(T)|^2 + 2|E(T)|\right) - 3s\right)/4 \right\rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Thus $\dim \mathcal{C}(P_2[T]) = |V(T)|^2 - 2|V(T)| + 2|E(T)| + 1 = s + t \le s + \left\lfloor \left(3\left(|V(T)|^2 + 2|E(T)|\right) - 3s\right)/4 \right\rfloor$, which implies that $|V(T)|^2 - 1 \le \left(s + 3|V(T)|^2 + 6|V(T)| - 6\right)/4$. Therefore, $4|V(T)|^2 - 4 \le 6|E(T)| + 3|V(T)|^2 + 6|V(T)| - 6$. And so $|V(T)|^2 - 4 \le 3|V(T)|^2 + 12|V(T)| - 12$. To this end we have $|V(T)|^2 - 12|V(T)| - 16 \le 0$, which does not hold if $|V(T)| \ge 14$. The proof is completed.

The graph $T_1[T_2]$ consists of $|E(T_1)|$ copies of $P_2\left[N_{|V(T_2)|}\right]$ and $|V(T_1)|$ copies of T_2. In the following result we give an upper bound of the basis number of the lexicographic product of two trees independent of their orders.

Lemma 2.2. *For each pair of trees T_1 and T_2, we have $b(T_1[T_2]) \le Max\{2\Delta(T_1), 4\}$.*

Proof. Let $E(T_1) = \left\{P_2^{(1)} = a_1b_1, P_2^{(2)} = a_2b_2, \ldots, P_2^{(|E(T_1)|)} = a_{|E(T_1)|}b_{|E(T_1)|}\right\}$ be the edge set of T_1. Now, for each $i = 1, 2, \ldots, |E(T_1)|$, define $\mathcal{B}_i$ to be the basis of $P_2^{(i)}[T_2]$ as in Lemma 2.1. Moreover, set $\mathcal{B} = \bigcup_{i=1}^{|E(T_1)|} \mathcal{B}_i$. We now show that $\mathcal{B}$ is a linearly independent set. Note that, $V(P_2^{(i)}) \cap V(P_2^{(j)})$ is either an empty set or it contains only one vertex, say a_i. Therefore,

$$E(P_2^{(i)}[T_2]) \cap E(P_2^{(j)}[T_2]) = \begin{cases} \phi, & \text{if } V(P_2^{(i)}) \cap V(P_2^{(j)}) = \phi, \\ E(a_i \times T_2) & \text{if } V(P_2^{(i)}) \cap V(P_2^{(j)}) = \{a_i\}. \end{cases} \qquad (1)$$

Suppose that $\sum_{r=1}^{s_1} c_{1r} + \sum_{r=1}^{s_2} c_{2r} + \cdots + \sum_{r=1}^{s_{|E(T_1)|}} c_{|E(T_1)|r} = 0 \pmod 2$ where $c_{ir} \in \mathcal{B}_i$. Without loss of generality, we may assume that $s_1 \neq 0$. Then $\sum_{r=1}^{s_1} c_{1r} = \sum_{r=1}^{s_2} c_{2r} + \cdots + \sum_{r=1}^{s_{|E(T_1)|}} c_{|E(T_1)|r} \pmod 2$. Hence, $E(\bigoplus_{r=1}^{s_1} c_{1r}) = E(\bigoplus_{r=1}^{s_1} c_{2r} \oplus \cdots \bigoplus_{r=1}^{s_{|E(T_1)|}} c_{|E(T_1)|r})$ where $\oplus$ is the ring sum. Therefore, by (1) $E(\bigoplus_{r=1}^{s_1} c_{1r})$ is a subgraph of the forest $(a_1 \times T_2) \cup (b_1 \times T_2)$, which contradicts the fact that $E(\bigoplus_{r=1}^{s_1} c_{1r})$ is a cycle or an edge-disjoint union of cycles. Since

$$\begin{aligned} |\mathcal{B}| &= \sum_{i=1}^{|E(T_1)|} |\mathcal{B}_i| \\ &= \sum_{i=1}^{|E(T_1)|} (|V(T_2)|^2 - 1) \\ &= |V(T_2)|^2 \, |E(T_1)| - |E(T_1)| \\ &= \dim \mathcal{C}(T_1 \, [T_2]), \end{aligned}$$

$\mathcal{B}$ is a basis for $\mathcal{C}(T_1 \, [T_2])$. It is an easy task to show that $\mathcal{B}$ satisfies the bound which is stated in the theorem. The proof is completed.

In the rest of this paper T_G denotes a spanning tree of G with maximal degree as small as possible and $\Delta(T_G)$ denotes the maximal degree of T_G.

Lemma 2.3. *For each connected graph G and tree T, we have $b(G\,[T]) \leq Max$ $\{4, 2\Delta(G), 2 \ +b(G)\}$.*

Proof. Let $V(T) = \left\{a_1, a_2, \ldots, a_{|V(T)|}, \right\}$. We may assume that $a_{|V(T)|}$ is an end point of T which is different from $v_{n_k}^{(k)}$ as in Proposition 1.1. Note that $G\,[T] = T_G\,[T] \cup (\bigcup_{e \in E(G)-E(T_G)} e\left[N_{|V(T)|}\right])$. Let $\mathcal{B}^*$ be the basis of $T_G[T]$ as defined in Lemma 2.2. Let $\mathcal{B}^{**} = \bigcup\limits_{e \in E(G)-E(T_G)} \mathcal{B}_e$ where $\mathcal{B}_e$ is the basis of $e\,[T]$ as in Lemma 2.1 and $\mathcal{B}^{***}$ be the corresponding required basis of $\mathcal{B}_G$ in $G \times a_{|V(T)|}$. Now, set $\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}^{**} \cup \mathcal{B}^{***}$. Since each cycle of $\mathcal{B}^{***}$ contains at least one edge $e \in E(G \times a_{|V(T)|}) - E(T_G \times a_{|V(T)|})$, which is not in $\mathcal{B}^*$, we have that $\mathcal{B}^* \cup \mathcal{B}^{***}$ is linearly independent. By arguments similar to the one in Lemma 2.2, one can show that $\mathcal{B}^{**}$ is linearly independent. It is an easy task to see that for any edge $e = uw \in E(G) - E(T_G)$, we have that $E(uw\,[T]) \cap \left(E(T_G\,[T] \cup E(G \times a_{|V(T)|})\right) \subseteq \left\{E(u \times T) \cup E(w \times T) \cup (u, a_{|V(T)|})(w, a_{|V(T)|})\right\}$ which forms edges of a forest. Thus, if $\sum_{i=1}^{t} l_i = \sum_{i=1}^{s} c_i \pmod 2$ where $c_i \in \mathcal{B}^* \cup \mathcal{B}^{***}$ and $l_i \in \mathcal{B}^{**}$, then $l_1 \oplus l_2 \oplus \cdots \oplus l_t$ is a subgraph of the forest. This contradicts the fact that $l_1 \oplus l_2 \oplus \cdots \oplus l_t$ is a cycle or an edge disjoint union of cycles where $\oplus$ is the ring sum. Thus, $\mathcal{B}$ is linearly independent. Since

$$\begin{aligned} \dim \mathcal{C}(G) &= |E(G)| - |V(G)| + 1 \\ &= |E(G)| - |E(T_G)|, \end{aligned}$$

As a result,

$$\begin{aligned} |\mathcal{B}^{**}| &= \sum_{e \in E(G)-E(T_G)} |\mathcal{B}_e| \\ &= \sum_{e \in E(G)-E(T_G)} (|V(T)|^2 - 1) \\ &= \dim \mathcal{C}(G)(|V(T)|^2 - 1). \end{aligned}$$

Note that

$$|\mathcal{B}^{***}| = \dim \mathcal{C}(G).$$

Thus,

$$\begin{aligned}
|\mathcal{B}| &= |\mathcal{B}^{*}| + |\mathcal{B}^{**}| + |\mathcal{B}^{***}| \\
&= |V(T)|^2 |E(T_G)| - |E(T_G)| + \dim \mathcal{C}(G)(|V(T)|^2 - 1) + \dim \mathcal{C}(G) \\
&= |V(T)|^2 (|E(T_G)| + \dim \mathcal{C}(G)) - (|E(T_G)| + \dim \mathcal{C}(G)) + \dim \mathcal{C}(G) \\
&= |V(T)|^2 |E(G)| - |E(T_G)| \\
&= |V(T)|^2 |E(G)| - |V(T_G)| + 1 \\
&= |V(T)|^2 |E(G)| - |V(G)| + 1 \\
&= \dim\ \mathcal{C}(G[T]).
\end{aligned}$$

Therefore, $\mathcal{B}$ is a basis. Let $e \in E(G[T])$. (1) If $e \in E(G \times a_{|V(T)|})$, then $f_{\mathcal{B}^* \cup \mathcal{B}^{**}}(e) \leq 2$ and $f_{\mathcal{B}^{***}}(e) \leq b(G)$. (2) If $e \in E(\cup_{v \in V(G)} E(v \times T))]$, then $f_{\mathcal{B}^* \cup \mathcal{B}^{**}}(e) \leq 2\Delta(G)$ and $f_{\mathcal{B}^{***}}(e) = 0$. (3) If $e \in E(e'[T]) - [E(G \times a_{|V(T)|}) \cup E(\cup_{v \in V(G)} E(v \times T))]$, then $f_{\mathcal{B}^* \cup \mathcal{B}^{**}}(e) \leq 4$ and $f_{\mathcal{B}^{***}}(e) = 0$ where $e' \in E(G)$. The proof is completed.

Theorem 2.4. *For each two connected graphs G and H, $b(G[H]) \leq \mathrm{Max}\{4, 2\Delta(G) + b(H), 2 + b(G)\}$.*

Proof. Let $\mathcal{B}^* = \mathcal{B}$ where $\mathcal{B}$ is the basis of $G[T_H]$ as defined in Lemma 2.3. Let $\mathcal{B}^{**} = \bigcup_{v \in V(G)} \mathcal{B}_v$ where $\mathcal{B}_v$ is the corresponding required basis of $\mathcal{B}_H$ in $v \times H$. $E(v \times H) \cap E(u \times H) = \phi$ for each $u \neq v$. Thus, $\mathcal{B}^{**}$ is linearly independent. Moreover, each cycle of $\mathcal{B}^{**}$ contains an edge of the form $E_{|V(G)|} \times (E(H) - E(T))$ which is not in any cycle of $\mathcal{B}^*$ where $E_{|V(G)|}$ is a null graph with vertex set $V(G)$. Thus $\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}^{**}$ is linearly independent. Since

$$\begin{aligned}
|\mathcal{B}| &= |\mathcal{B}^{*}| + |\mathcal{B}^{**}| \\
&= |V(T_H)|^2 |E(G)| - |V(G)| + 1 + |V(G)| \dim\ \mathcal{C}(H) \\
&= |V(T_H)|^2 |E(G)| + |V(G)||E(H)| - |V(G)||V(H)| + 1 \\
&= \dim\ \mathcal{C}(G[T]),
\end{aligned}$$

$\mathcal{B}$ is linearly independent basis. Now, we conclude the theorem by showing that $\mathcal{B}$ satisfies the fold which stated in the theorem. Let $e \in E(G[H])$. (1) If $e \in E(G[N_{|V(H)}])$, then $f_{\mathcal{B}^*}(e) \leq \mathrm{Max}\{4, 2 + b(G)\}$ and $f_{\mathcal{B}^{**}}(e) = 0$ (2) If $e \in \cup_{u \in E(G)} E(u \times H)$, then $f_{\mathcal{B}^*}(e) \leq 2\Delta(G)$ and $f_{\mathcal{B}^{**}}(e) \leq b(H)$. The proof is completed.

Now we give an example where the bound of the above Theorem is achieved. By specializing G and H in the above Theorem into a cycle and a path, respectively, and by using arguments similar to those three cases in Lemma 2.1 we obtain the following result.

Corollary 2.5. *For any cycle C and path P of order greater than or equal to 15, $b(C[P]) = 4$.*

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