Bounds on the Number of Γ-Operation Edge Substitutions required to Transform a Maximal Planar Graph into Another

by

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ABSTRACT

A maximal planar graph (MPG) is a planar graph to which no more edges can be added without destroying the planarity of the MPG. Each face of a plane embedding of an MPG is triangular and hence they are often referred to as plane triangulations. The diagonal operation used by Ore and others and the Γ-operation developed by Al-Hakim each remove one edge of an MPG and replace it with another to produce a new MPG. The diagonal operation removes an edge, creating a quadrilateral face, and replaces it with the other diagonal of the quadrilateral. This diagonal may be an edge of the original MPG, in which case the edge to be replaced is said to be braced by its diagonal and no diagonal operation can be carried out. The Γ-operation is an extension of the diagonal operation which can act on any edge of an MPG. These operations have been used in heuristics for facilities layout planning.

It has been shown by Ore and by Eggleton and Al-Hakim and by Ning, that any MPG can be transformed into any other, with the same vertex set, by a sequence of diagonal operations. As the diagonal operation is a special case of the Γ-operation this is also true for the Γ-operation . This paper examines the number of operations required to transform an MPG into a very similar one. Theorem 1 shows that even when each of the two MPG's have three edges which are not in the other (an *edge-difference* of three), the sequence of Γ-operations can be made arbitrarily large for a sufficiently large vertex set. An edge-difference of at least three is necessary for a long sequence. Theorem 2 shows that when the edge-difference is only two, the transformation can always be completed with two or three Γ-operations. Theorem 3 shows that the Γ-operation is the best possible single edge replacement operation as an MPG can be transformed into any other with one different edge by one Γ-operation. Theorem 4 illustrates the advantage that the Γ-operation gives compared to the diagonal operation with a pair of graphs which require only one Γ-operation but many diagonal operations for the transformation. The implications of these results for the layout heuristics are also discussed.

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INTRODUCTION

A maximal planar graph (MPG) is a planar graph to which no more edges can be added without destroying its planarity. A plane embedding of such a graph is often referred to as a plane triangulation, since all faces are triangular. All embeddings of a particular MPG on a sphere are topologically equivalent. Different plane embeddings are merely different stereographic projections of this sphere embedding or its reflection.

If G is a maximal planar graph with vertex set V and edge set E the Euler Polyhedron Formula gives $|E| = 3|V| - 6$, and the number of faces $= 2|V| - 4$.

An application of the transformation of MPG's is found in facilities layout planning. An MPG can be used to represent the adjacency of facilities in a layout. Each vertex represents a facility and two vertices are joined by an edge if they are adjacent in the layout. Each pair of facilities is given a weight which represents the benefit of making them adjacent in the layout. The adjacency problem for the layout planner is to construct an MPG with the maximum sum of edge weights. This problem is fully explained and has been shown to be NP-complete by Giffin [8]. Many heuristics have been developed to obtain good solutions in reasonable time. These fall into two categories, construction heuristics and improvement heuristics. Construction techniques begin with a small graph, usually a triangle or tetrahedron, and add vertices one at a time. Improvement techniques begin with a constructed solution and attempt to increase the total benefit by exchanging some of the edges of the MPG for edges of higher weight.

The diagonal operation and the Γ-operation are two edge substitution operations which have been used in improvement heuristics. The Γ-operation is explained in detail in the next section. The diagonal operation has been used by many authors and appears under this name in [5] and [9]. In [7] it is the first case of the α-operation, and [10] uses the terminology of [7]. The Γ-operation, developed by Al-Hakim in [1], is an extension of the diagonal operation which overcomes the difficulties posed by braced edges (see [1] or [6] for a full explanation of bracing). The Γ-operation has been used in two improvement heuristics in [2] and [3].

It has been conjectured in [7] and shown in [5], [9], [10] that any MPG can be transformed into any other MPG on the same vertex set by a sequence of diagonal operations. As the Γ-operation includes the diagonal operation as a special case, this is also true for Γ-operations. None of the proofs have attempted to specify the length of the shortest sequence.

This paper will show that even for very similar MPG's, the length of the shortest sequence of operations can be arbitrarily large. To measure the degree of similarity between two MPG's A and B, with the same vertex set and edge sets E_A and E_B respectively ($|E_A| = |E_B|$), we define their *edge-difference* as $|E_A| - |E_A \cap E_B|$.

Theorem 1 shows that even when the edge-difference is only three, the length of the shortest sequence of Γ-operations can be made large for a sufficiently large vertex set. This has a serious implication for the facilities layout problem. Even when the current best solution contains almost all of the edges of the optimal MPG it may be far away from the optimal in terms of the number of operations required.

It is reported in [1], without proof, that if two MPG's have edge-difference two, then at most three Γ-operations are needed for the transformation. This result is stated in Theorem 2. The proof of Theorem 2 is by an exhaustive list of cases and is very long, so only an outline of the proof is given. The complete proof can be obtained from the author.

This result also shows that an edge-difference of three is necessary for a long sequence. The distinction between cases with edge-difference of three or more and those of one or two comes from the 3-edge-connected (see [4]) nature of MPG's. Consider the subgraph of edges common to both of a pair of MPG's. If they have an edge-difference of three or more the subgraph of common edges may be disconnected. These disconnected pieces may be arranged differently in each of the MPG's. Hence a transformation may involve the rearrangement of large blocks of common edges.

Al-Hakim also reports in [1], again without proof, that for any pair of MPG's with edge difference one, a single Γ-operation will make the transformation. This result is stated in Theorem 3, and a proof by an exhaustive list of cases is provided. Hence the Γ-operation is the best possible single edge replacement method. Any method which seeks to improve upon the Γ-operation must be able to replace a number of edges in a single iteration.

Foulds and Robinson [7] suggested such a method. They showed that a sequence of α-operations and β-operations would transform any MPG into another with the same vertex set. The first case of the α-operation is the diagonal operation, the second case is equivalent to two consecutive diagonal operations. The β-operation removes a vertex of degree three and inserts it into another face, replacing three edges with three new edges in one step. Either of the MPG's of Theorem 2 can be transformed into the other by just one β-operation. However, it seems unlikely that β-operation will provide a shortened sequence for cases where the subgraph of edges common to both MPG's is disconnected and the components are not isolated vertices. These ideas are under further investigation.

Theorem 4 illustrates the significance of the third case of the Γ-operation. A single Case (iii) Γ-operation may perform a transformation which would require an arbitrarily large number of diagonal operations.

THE Γ-OPERATION

Each edge *ab* of an MPG borders two faces *abc* and *abd*. The vertices *c* and *d* do not depend on the embedding. The *diagonal* of *ab* is *cd*. The edge *cd* may or may not appear in the graph. If it does, *ab* is said to be *braced* by *cd*.

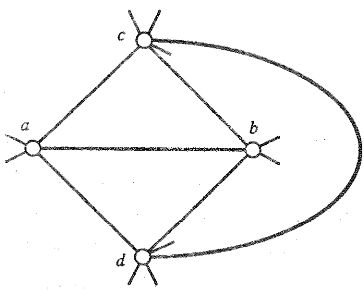

Figure 0.1 The edge *ab* is braced by *cd*.

The Γ-operation developed by Al-Hakim [1] is an extension of the diagonal operation. It acts on braced edges as well as unbraced ones. It has been used in two different improvement heuristics in [2] and [3]. The following description also appears in [3]. For a more formal treatment of the definition see [1].

There are three cases of the operation. Let *ab* be the edge which is to be replaced.

Case (i) The edge *ab* is unbraced. This case is the same as the diagonal operation, in which *ab* is replaced by its diagonal *cd*.

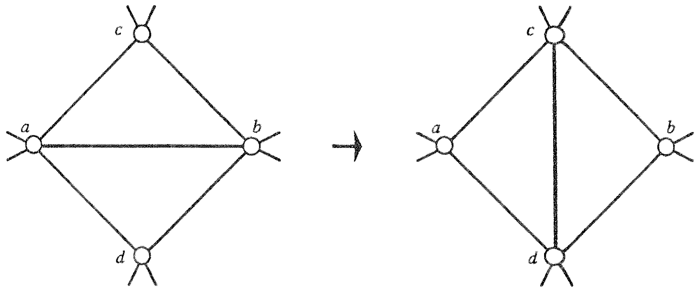

Figure 0.2 The Case (i) Γ-operation $ab \rightarrow cd$.

Case (ii) The edge *ab* is braced by *cd*. The diagonal of *cd* is of the form *ae* or *be*. The edge *ab* is replaced by the diagonal of *cd*, which is either *ae* or *be*. In Figure 0.3 the diagonal of *cd* is *ae* and *ab* is replaced by *ae*.

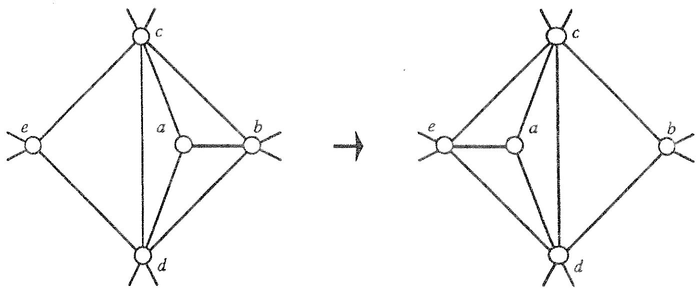

Figure 0.3 The Case (ii) Γ-operation $ab \rightarrow ae$.

Case (iii) *ab* is braced by *cd* which has diagonal say *ef* where both *e* and *f* are distinct from *a* and *b*.

The graph may have two essentially different configurations. In each one the edge *ab* may be replaced by one of three edges. There is a path in the graph from *a* to one of *e* or *f* which does not pass through *b*, *c* or *d*. If the path is between *a* and *e* then there is also a path between *b* and *f*. In this case *ab* may be replaced by any one of *ef* or *af* or *be* as shown in Figure 0.4. The shaded area in Figure 0.4 indicates an arbitrary subgraph which contains the path between *a* and *e*. Otherwise, such paths exist between *a* and *f* and between *b* and *e*, and *ab* may be replaced by *ef* or *ae* or *bf*.

Exchanging the labels a and b in Figure 0.4 will produce the appropriate diagrams for this configuration.

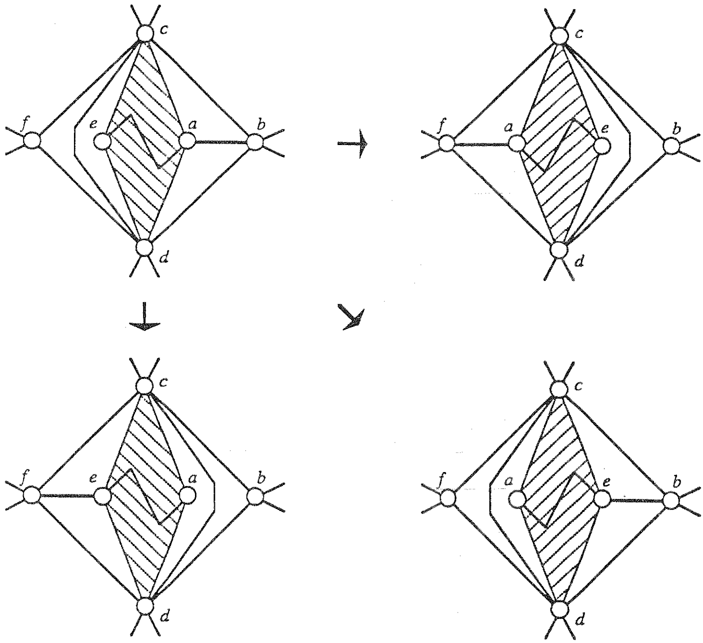

Figure 0.4 The Case (iii) Γ-operations on ab.

THE RESULTS

THEOREM 1

For any positive integer k, there exists a pair of maximal planar graphs, on the same set of $3k+1$ vertices, with edge-difference three, such that to transform one into the other requires $2k$-1 Γ-operations.

PROOF

Given any positve integer k, we construct two maximal planar graphs A and B with edge-difference three as follows:-

Arrange $3k$ vertices on a sphere, at the intersections of k latitudes and three meridians. Connect the three vertices on each latitude to each other. Connect each latitude to the adjacent one by a zigzag. That is, connect adjacent vertices on each meridian, and connect each vertex except those on the most southern meridian to the closest vertex to the south-east. Make two such spheres and consider these edges to be the common edges of A and B. To A add the $3k+1$st vertex at the north pole, and connect it to each vertex on the northernmost latitude. To B add the $3k+1$st vertex at the south pole, and connect it to each vertex on the southernmost latitude. The graphs A and B have an edge-difference of three. Plane embeddings of a pair of such a graphs with four latitudes are shown in Figures 1.1 and 1.2 (for clarity vertex v_i is labelled i in both figures).

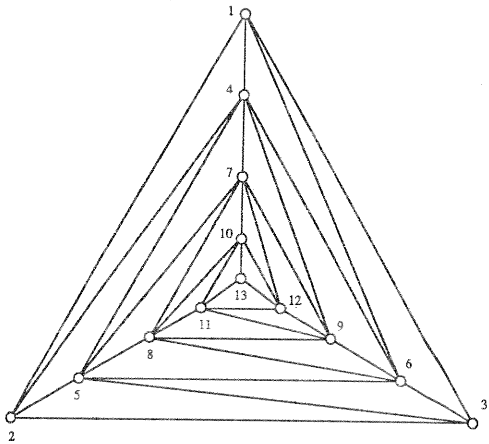

Figure 1.1 A plane embedding of A for $k = 4$.

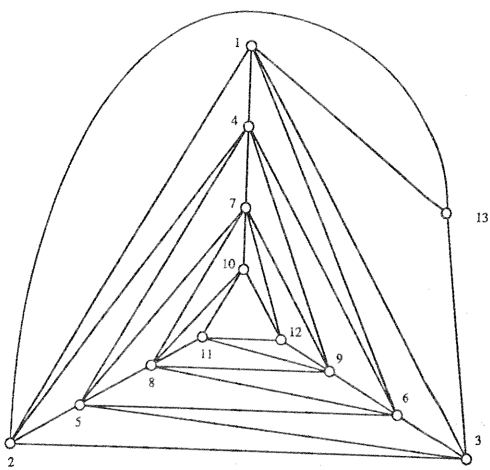

Figure 1.2 A plane embedding of B for $k = 4$.

The only possible Γ-operations on the edges incident with v_{3k+1} are Case (ii) Γ-operations which have the effect of moving v_{3k+1} across one of the edges of the northernmost latitude into an adjacent face. The vertex still has degree 3 and hence the only Γ-operations available are Case (ii) which have the effect of moving v_{3k+1} into an adjacent face. The length of the shortest path of adjacent faces from the northernmost latitude to the southernmost latitude is $2k$-2. Hence transforming A into B requires $2k$-1 Case (ii) Γ-operations. □

Theorem 1 has important implications for layout heuristics. Consider that graph $\mathfrak{B}$ of Theorem 1 is the optimal solution to an adjacency problem, and that graph $\mathfrak{A}$ is the current best known solution. As $\mathfrak{A}$ and $\mathfrak{B}$ have all but three edges in common graph $\mathfrak{A}$ will also be high in weight. The operations required to transform this solution into the optimal are likely to produce a decrease in the total weight of the graph in the intermediate stages. In other words, graph $\mathfrak{A}$ is likely to be a local optimum, and it may be many steps away from the global optimum graph $\mathfrak{B}$.

Hence any heuristic which uses Γ-operations must have some facility for negative steps, to overcome the difficulties caused by local optima.

Graph A of Theorem 1 can be transformed into graph B by one β-operation (see [7]). Use of the β-operation in conjunction with the Γ-operation may not remove the possibility of an arbitrarily long sequence. Similar pairs of graphs can be constructed with a common subgraph in place of v_{3k+1}. This would require an edge-difference greater than three, but the edge-difference need not be arbitrary. The β-operation can only act on vertices of degree three and the common subgraphs can be chosen to limit its use. The properties of a combined sequence are under investigation by the author.

THEOREM 2

(Stated without proof in [1])

If two maximal planar graphs have edge-difference two, then one can be transformed into the other by two or three Γ-operations.

PROOF OUTLINE

A large number of subcases need to be considered in this proof. To make it easier to follow, the subcases are numbered and a decision tree dealing with them is shown in

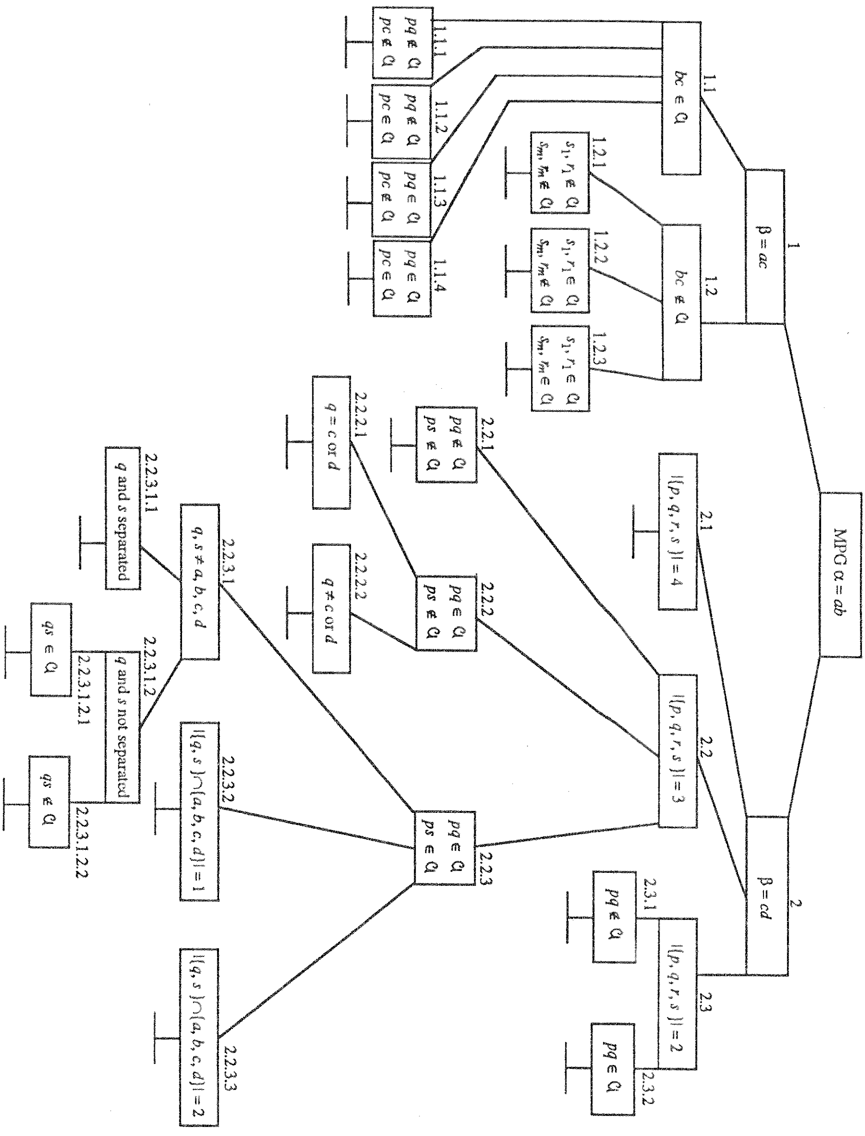

Figure 2.1 Decision tree.

Figure 2.1. Only the first two levels of the decision tree are described here. Full details of the proof can be obtained from the author.

Let the two maximal planar graphs be $\mathfrak{A}$ and $\mathfrak{B}$. Let α_1 and α_2 be the edges of $\mathfrak{A}$ which are not in $\mathfrak{B}$. Embed the graph $\mathfrak{A}$ on a sphere. Call this embedding A.

Case 1. Edges α_1 and α_2 have a common vertex, say vertex a.
Suppose α_1 is ab and α_2 is ac.

Case 1.1 The edge bc exists in $\mathfrak{A}$.
Locate the vertices p and q such that $p \neq c$, $q \neq b$ and abp and acq are faces of A. Form A' by re-embedding A on the sphere with equator $apbcqa$ and edges α_1 and α_2 in the northern hemisphere, as shown in Figure 2.2. The edge pq may or may not be in $\mathfrak{A}$. In either case, neither or one (but not both) of the edges pc or qb may be in $\mathfrak{A}$. Assume WLOG that if one of these edges exists in $\mathfrak{A}$ that it is pc. Consider the four subcases, where neither edge, pq only, pc only or both of the edges are in $\mathfrak{A}$. It can now be shown directly that the only possible embeddings of $\mathfrak{B}$ can be obtained by two or three Γ-operations on A'.

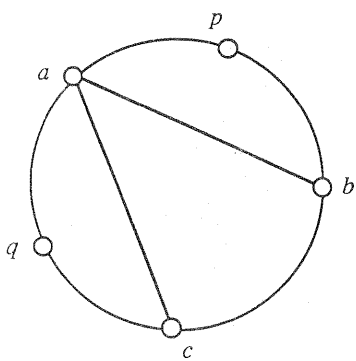

Figure 2.2 The northern hemisphere of A'.

Case 1.2 Edges α_1 and α_2 have a common vertex but the edge bc does not exist in $\mathfrak{A}$.

Form A' by embedding A on the sphere with vertex a as the north pole, all vertices which are connected to a on the equator, and only edges incident with a in the northern hemisphere, as shown in Figure 2.3. Neither, one or both of the edges s_1, r_1 and s_n, r_m may be in A'. If one or both of these edges are in A' a new embedding A" is formed. In each of the three subcases it can be shown directly that two or three operations will transform A' (or A") into any embedding of $\mathfrak{B}$.

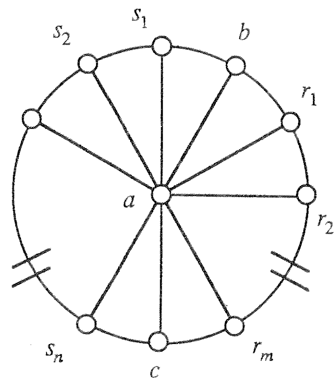

Figure 2.3 The northern hemisphere of A' when bc is not in A.

Case 2. Edges α_1 and α_2 have no common vertex.

Assume $\alpha_1 = ab$ and $\alpha_2 = cd$. Locate vertices p, q, r, and s such that $p \neq q$ and $r \neq s$ and abp and abq and cdr and cds are faces of A. There are three subcases to consider, when p, q, r and s represent four, three or two distinct vertices.

Case 2.1 Vertices p, q, r and s are all distinct.

Form A' by re-embedding A on the sphere with edges ab and cd on the equator, none of p, q, r, s on the equator, and p in the northern hemisphere. Then either r or s is in the northern hemisphere. Assume WLOG that it is r. Form A" by re-embedding A' on the sphere with equator $aqbt_1t_2...t_ncsdu_1u_2...u_ma$ and vertices p and r in the northern hemisphere. The northern hemisphere of A" is shown in Figure 2.4. Form B by embedding $\mathfrak{B}$ on a sphere with the same equator and southern hemisphere as A". There is only one configuration of the northern hemisphere of B, which can be obtained from A" by the Case (i) Γ-operations $ab \to pq$ and $cd \to rs$.

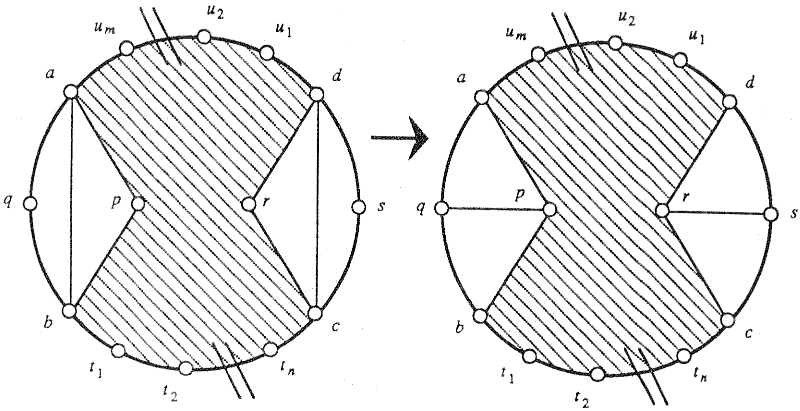

Figure 2.4 The northern hemispheres of A" and B when p, q, r and s are all distinct.

Case 2.2 Three of p, q, r and s are distinct vertices.

Assume WLOG that $p = r$. There are three subcases, when neither, one or both of the edges pq and ps are in $\mathfrak{A}$.

When neither of the edges are in A' there is only one possible embedding B of $\mathfrak{B}$. The Case (i) Γ-operations $ab \rightarrow pq$ and $cd \rightarrow ps$ will transform A' into this embedding, as shown in Figure 2.5.

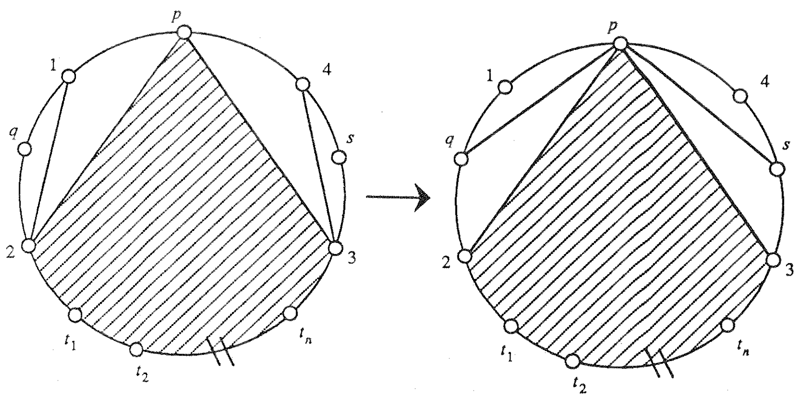

Figure 2.5 The northern hemispheres of A" (left) and B (right) when $p = r$ and neither pq nor ps are in A.

If one or both of the edges pq and ps are in A' then further consider the relative positions of vertices a, b, c, d, p, q, r and s. This leads to eight new embeddings A", which are represented in Figure 2.1 by subcases 2.2.1, 2.2.2.1, 2.2.2.2, 2.2.3.1.1, 2.2.3.1.2.1, 2.2.3.1.2.2, 2.2.3.2 and 2.2.3.3. The numerous possible embeddings of $\mathfrak{B}$ can all be obtained from the corresponding A" by two or three Γ-operations. Case 2.2.3.3 is interesting in that for this configuration of $\mathfrak{A}$ the MPG $\mathfrak{B}$ does not exist. Figure 2.6 shows such an MPG. The external face adp could also contain common edges. It is not possible to construct an MPG $\mathfrak{B}$ with all the edges of this $\mathfrak{A}$ except ab and cd.

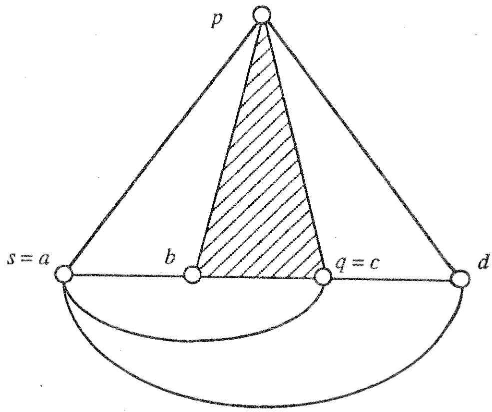

Figure 2.6 An MPG $\mathfrak{A}$ for which no $\mathfrak{B}$ exists.

Case 2.3 Only two of p, q, r and s are distinct vertices.
Assume WLOG that $p = r$ and $q = s$. There are two subcases, when pq is not an edge of A and when it is. In each of these there is a number of possible embeddings of $\mathfrak{B}$. It can be shown directly that all of these can be obtained from the corresponding A by two or three Γ-operations. $\square$

This result confirms that an edge-difference of three is a necessary condition for Theorem 1 to hold.

THEOREM 3

(Stated without proof in [1])

If two maximal planar graphs have edge-difference one then one can be transformed into the other by a single Γ-operation.

PROOF

Let the two maximal planar graphs be $\mathfrak{A}$ and $\mathfrak{B}$. Let α be the edge of $\mathfrak{A}$ which is not in $\mathfrak{B}$. Label the vertices of $\mathfrak{A}$ in such a way that α connects vertices a and b. Embed the graph $\mathfrak{A}$ on a sphere. Call this embedding A. Label the vertices of A in such a way that abs and abt are faces of A.

Case 1 The edge st is not an edge of A.

Form A' by re-embedding A on the sphere so that the equator is $asbta$ and the edge ab is alone in the northern hemisphere. Form B by embedding $\mathfrak{B}$ on a sphere such that the equator is $asbta$ and the southern hemisphere is the same as that of A'. The northern hemisphere of B does not contain ab but B is maximal planar so B must contain the edge st, as shown in Figure 3.1. Hence $\mathfrak{A}$ can be transformed into $\mathfrak{B}$ by the Case (i) Γ-operation $ab \to st$.

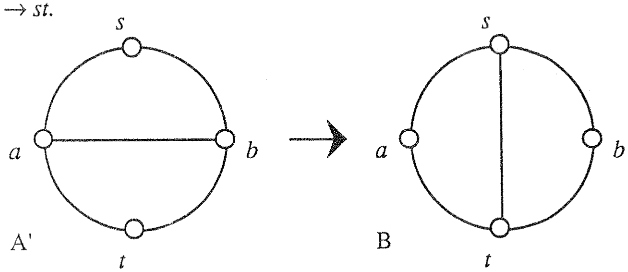

Figure 3.1 The northern hemispheres of A' and B.

Case 2 The edge st is an edge of A.

Locate the vertices p and q such that stp and stq are faces of A. If $\{p, q\}=\{a, b\}$ then $\mathfrak{A}$ is the complete graph on four vertices as shown in Figure 3.2. Hence it is not possible for this to occur as $\mathfrak{A}$ and $\mathfrak{B}$ have edge-difference one.

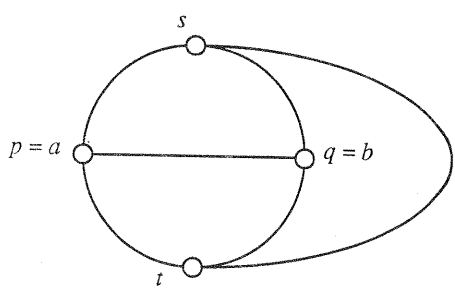

Figure 3.2 A plane embedding of the complete graph on four vertices.

It is possible that one of the vertices p or q is one of the vertices a or b. In this case assume without loss of generality(WLOG) that $p = b$. If a, b, p and q are all distinct, assume WLOG that there is a path between p and b which does not pass through $b, s, t,$ or q. In either case, form A' by re-embedding A on the sphere with $asqta$ as the equator and p and b in the northern hemisphere. Form B by embedding $\mathfrak{B}$ with the same equator and southern hemisphere as A'. Figure 3.3 shows the northern hemisphere of A and the three possible northern hemispheres of B for the case when $p \neq b$. Each of these can be obtained from A by the one of the three Case (iii) Γ-operations $ab \to ap$ or bq or pq. Figure 3.4 shows the northern hemisphere of A and the only possible northern hemisphere of B for the case when $p = b$. In this case A can be transformed into B by the Case (ii) Γ-operation $ab \to bq$.

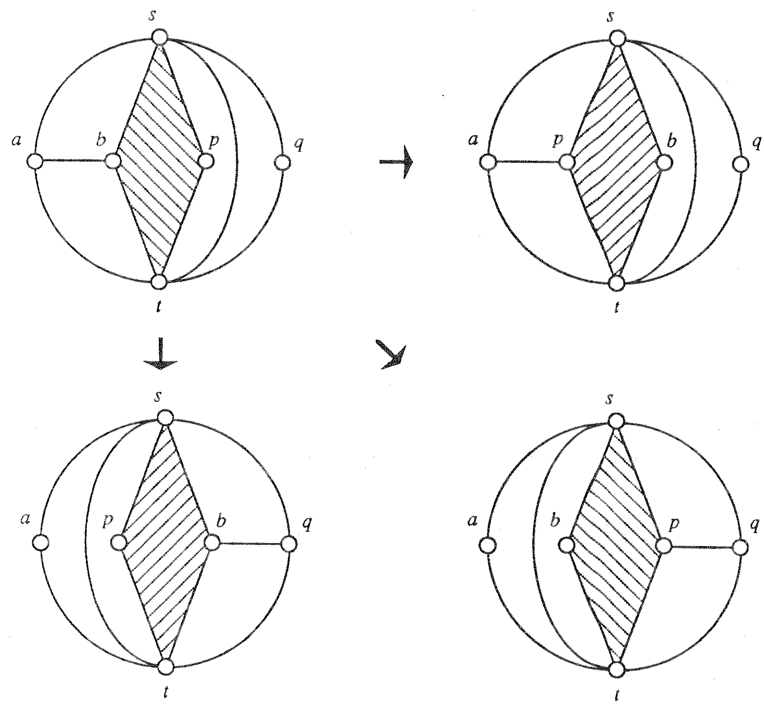

Figure 3.3 When $p \neq b$, the three configurations of B can be obtained by the three Case (iii) Γ-operations $ab \to ap$ or bq or pq.

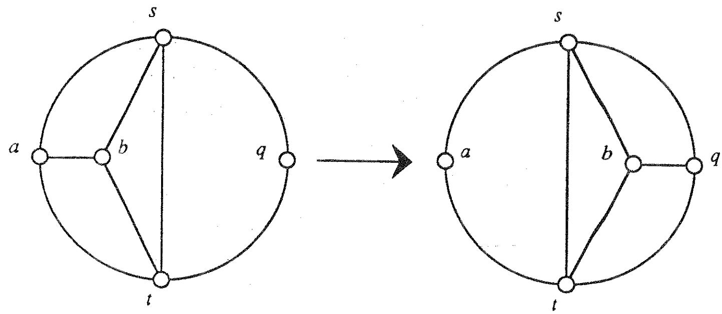

Figure 3.4 When $p = b$, A' is transformed into B by the Case (ii) Γ-operation $ab \rightarrow bq$.$\square$

Take an MPG $\mathfrak{A}$ say, and select one of its edges. We can generate all possible MPG's with all the edges of $\mathfrak{A}$ except the one we have specified by applying the appropriate Γ-operation to that edge. Hence any MPG has one or three neighbours with edge-difference one for each of its edges.

THEOREM 4

For any positive integer k, there exists a pair of maximal planar graphs, on the same set of $2k+6$ vertices, with edge-difference one, such that to transform one into the other requires more than k diagonal operations.

PROOF

The maximal planar graphs A and B shown in Figures 4.1 and 4.2 have edge-difference one, as $E_A \backslash E_B = \{ab\}$ and $E_B \backslash E_A = \{af\}$. Each of A and B has $2k+6$ vertices. Transforming A into B requires more than k diagonal operations.

If the edge af is added to A (destroying its planarity) it must cross at least $k+1$ edges of A. Re-embedding the graph A cannot reduce this crossing number.

The process of re-embedding corresponds to the stereographic projection of the plane onto a sphere (points "at infinity" are projected onto the north pole). The north pole is then positioned in another face of the graph and the graph is stereographically projected once more onto the plane. Hence if a different embedding exists, say A', which when af is added has a crossing number less than $k+1$, the graph A'+$\{af\}$ can be stereographically projected onto the sphere and the north pole repositioned in face dg_2g_3 and then projected onto the embedding of A as shown in Figure 4.1. If *af* crosses the face dg_2g_3 then the north pole can be positioned on either side of it, and *af* will pass through the external face of A. The edge *af* retains its crossings throughout the process of stereographic projection. But no path for *af* exists in A with less then k+1 crossings, hence no other embedding A' has less than k+1 crossings.

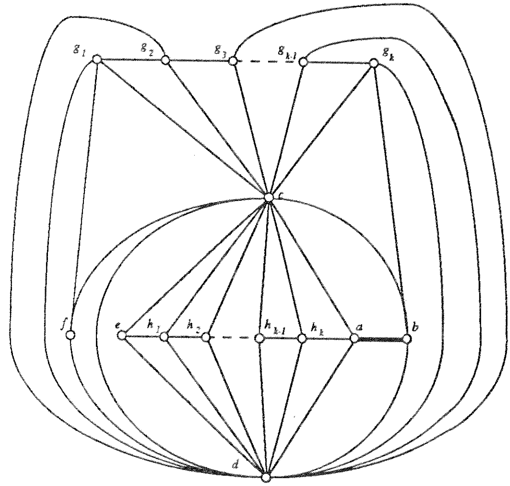

Figure 4.1 The graph A with exactly one edge *ab*, not in graph B.

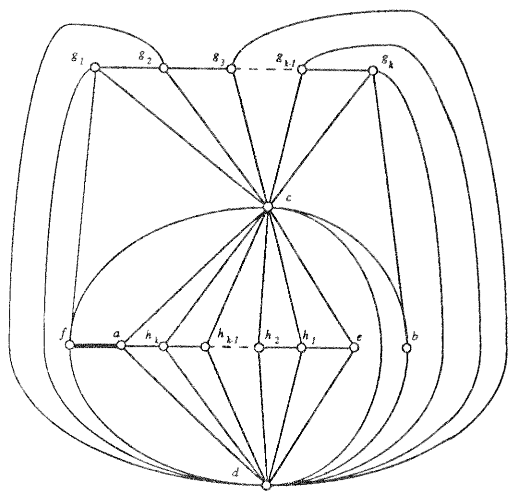

Figure 4.2 The graph B with exactly one edge *af*, not in graph A.

If af is to be introduced by a diagonal operation then k of the edges crossed by af must be moved out of the way first. One such sequence of moves is

1.	g_1c	$\rightarrow$	g_2f
2.	g_2c	$\rightarrow$	g_3f
...			
k.	g_kc	$\rightarrow$	bf
k+1.	bc	$\rightarrow$	af

Further diagonal operations are then required to remove ab and replace the edges g_ic for $1 \leq i \leq k$.

Thus for any positive integer k, a pair of MPG's with edge-difference one can be constructed such that more than k diagonal operations are required to transform one into the other. $\square$

The graph A of Theorem 4 can be transformed into graph B by one Case (iii) Γ-operation, but requires many diagonal operations. Hence the rotational features of the Γ-operation are worth the small amount of additional time and space required in an improvement heuristic.

CONCLUSIONS

The 3-edge-connected nature of MPG's explains the contrast between cases of edge-difference one or two (Theorems 2 and 3) and cases of edge-difference three or more (Theorem 1) as in the latter the subgraph of common edges may be disconnected. The components may then be arranged differently in the two MPG's, so that the transformation may involve the rearrangement of large components.

The properties of the Γ-operation identified in the four theorems are relevant to its use in facilities layout planning. Theorem 3 shows that the Γ-operation is the best possible way of moving toward an optimal graph by replacing only one edge at each step. Theorem 4 illustrates the superiority of the Γ-operation over the diagonal operation. Both of these results recommend the use of the operation in heuristics for the adjacency problem. However Theorem 1 suggests caution in its use, as such a heuristic must be able to escape from local optima. Various schemes to acheive this are under investigation. Theorem 1 also encourages the search for an alternative to the one-edge-at-a-time approach.

It must be remembered that this an NP-complete problem, so there must be some tradeoff between solution quality and efficiency considerations.

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