AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **58(2)** (2014), Pages 292–303

GDDs with two associate classes and with three groups of sizes $1, n$ and n

Wannee Lapchinda Narong Punnim

Department of Mathematics
Srinakharinwirot University
Sukhumvit 23, Bangkok 10110
Thailand
gs522120002@swu.ac.th narongp@swu.ac.th

Nittiya Pabhapote

Department of Mathematics
University of the Thai Chamber of Commerce
Dindaeng, Bangkok 10400
Thailand
nittiya_pab@utcc.ac.th

Abstract

A group divisible design GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ is an ordered pair $(V, \mathcal{B})$ where V is an $(1 + n + n)$-set of symbols and $\mathcal{B}$ is a collection of 3-subsets (called blocks) of V satisfying the following properties: the $(1 + n + n)$-set is divided into 3 groups of sizes 1, n and n; each pair of symbols from the same group occurs in exactly λ_1 blocks in $\mathcal{B}$; and each pair of symbols from different groups occurs in exactly λ_2 blocks in $\mathcal{B}$. The spectrum of λ_1, λ_2, denoted by $\mathsf{Spec}(\lambda_1, \lambda_2)$, is defined by

$$\mathsf{Spec}(\lambda_1, \lambda_2) = \{n \in \mathbb{N} : a \text{ GDD}(v = 1 + n + n, 3, \lambda_1, \lambda_2) \text{ exists}\}.$$

We find the spectrum $\mathsf{Spec}(\lambda_1, \lambda_2)$ for all $\lambda_1 \geq \lambda_2$.

1 Introduction

A *balanced incomplete block design* BIBD(v, b, r, k, λ) is a set S of v elements together with a collection of b k-subsets of S, called *blocks*, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [6], [7], [12]).

Note that in a BIBD(v, b, r, k, λ), the parameters must satisfy the necessary conditions:

1. $vr = bk$ and
2. $\lambda(v-1) = r(k-1)$.

With these conditions a BIBD(v, b, r, k, λ) is usually written as BIBD(v, k, λ).

A *group divisible design* GDD$(v = v_1 + v_2 + \ldots + v_g, k, \lambda_1, \lambda_2)$ is a collection of k-subsets (called blocks) of a v-set of symbols, where the v-set is partitioned into g groups of sizes $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks. Elements occurring together in the same group are called *first associates*, and elements occurring in different groups are called *second associates*. The existence problem of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [1]. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (see [3] for a summary), and the designs here are called partially balanced incomplete block designs (PBIBDs) of group divisible type in [3]. The existence question for $k = 3$ has been solved by Sarvate, Fu and Rodger (see [6], [7]) when all groups are of the same size.

The existence problem of GDD$(v = v_1 + v_2 + \ldots + v_g, k, \lambda_1, \lambda_2)$, when the groups may have different size, is considered recently. Chaiyasena, et al. [2] have published a paper in this direction. In particular, they found all ordered pairs (n, λ) of positive integers such that a GDD$(v = 1 + n, 3, 1, \lambda)$ exists. Pabhapote and Punnim found in [13] all ordered triples (m, n, λ) of positive integers such that a GDD$(v = m + n, 3, \lambda, 1)$ exists. The existence problem of a GDD$(v = m + n, 3, \lambda_1, \lambda_2)$ is more difficult if $\lambda_1 < \lambda_2$. Punnim and Uiyyasathian found in [14] infinitely many ordered pairs (m, n) of positive integers such that a GDD$(v = m + n, 3, 1, 2)$ exists. Let $(V = X \cup Y, \mathcal{B})$ be a GDD$(v = m + n, 3, \lambda_1, \lambda_2)$, where X and Y are of cardinality m and n, respectively. Then $(V = X \cup Y, \mathcal{B})$ is called *gregarious* if for each block $B \in \mathcal{B}$, $B \cap X \neq \emptyset$ and $B \cap Y \neq \emptyset$. El-Zanati et al. found in [5] all ordered pairs (m, n) of positive integers such that a gregarious GDD$(v = m + n, 3, 1, 2)$ exists.

We now consider the problem of determining the existence of a GDD$(v = n_1 + n_2 + n_3, 3, \lambda_1, \lambda_2)$. Chaiyasena, et al. [2] published a paper in this direction for small values of n_1, n_2, n_3. In particular, for each $n \in \{2, 3, 4, 5, 6\}$ they found all ordered pairs (λ_1, λ_2) of positive integers such that a GDD$(v = 1 + 2 + n, 3, \lambda_1, \lambda_2)$ exists. Hurd and Sarvate found in [8] all ordered pairs (n, λ) of positive integers such that a GDD$(v = 1 + 1 + n, 3, 1, \lambda)$ exists. Later, Hurd and Sarvate found in [9] all ordered pairs (n, λ) of positive integers such that a GDD$(v = 1+1+n, 3, \lambda, 1)$ exists. Recently, Hurd and Sarvate found in [10] all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 > \lambda_2$, such that a GDD$(v = 1 + 2 + n, 3, \lambda_1, \lambda_2)$ exists. More recently, Lapchinda, et al. found in [11] all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 < \lambda_2$, such that a GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists. It is now reasonable to consider the problem of determining all ordered triples $(n, \lambda_1, \lambda_2)$ of positive integers, with $\lambda_1 \geq \lambda_2$, such that a GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists. The problem is equivalent to finding the *spectrum* which is defined as follows: Let λ_1, λ_2 be positive integers.

Then the spectrum of λ_1, λ_2, denoted by $\mathsf{Spec}(\lambda_1, \lambda_2)$, is defined by

$$\mathsf{Spec}(\lambda_1, \lambda_2) = \{n \in \mathbb{N} : a \text{ GDD}(v = 1 + n + n, 3, \lambda_1, \lambda_2) \text{ exists}\}.$$

We find the spectrum $\mathsf{Spec}(\lambda_1, \lambda_2)$ for $\lambda_1 \geq \lambda_2$ in all situations. In order to solve the problem it may be easier to describe the problem in terms of, so-called, graph decomposition.

Let G and H be multigraphs. A *G-decomposition* of H is a partition of the edges of H such that each element of the partition induces a copy of G. We write $G \mid H$ if there exists a G-decomposition of H. Let λK_v denote the multigraph on v vertices in which each pair of distinct vertices is joined by λ edges. Let G_1 and G_2 be vertex disjoint graphs. Then $G_1 \vee_\lambda G_2$ is the graph obtained from the union of G_1 and G_2 and by joining each vertex in G_1 to each vertex in G_2 with λ edges. Let G_1, G_2, G_3 be pairwise vertex disjoint multigraphs. Then $G_1 \vee_\lambda G_2 \vee_\lambda G_3$ can be defined as $(G_1 \vee_\lambda G_2) \vee_\lambda G_3$. Thus the existence of a GDD$(v = n_1+n_2+n_3, 3, \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$. In this graph theoretic setting, edges joining vertices in the same group are referred to as *pure* edges, whereas edges joining vertices in different groups are called *mixed* edges.

The graph $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ is of order $n_1 + n_2 + n_3$ and size $\lambda_1[\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2}] + \lambda_2(n_1 n_2 + n_1 n_3 + n_2 n_3)$. It contains n_1 vertices of degree $\lambda_1(n_1 - 1) + \lambda_2(n_2 + n_3)$, n_2 vertices of degree $\lambda_1(n_2 - 1) + \lambda_2(n_1 + n_3)$, and n_3 vertices of degree $\lambda_1(n_3 - 1) + \lambda_2(n_1 + n_2)$.

Thus the existence of a K_3-decomposition of $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ implies the following conditions:

$$\begin{aligned}
\lambda_1[\tbinom{n_1}{2} + \tbinom{n_2}{2} + \tbinom{n_3}{2}] + \lambda_2(n_1 n_2 + n_1 n_3 + n_2 n_3) &\equiv 0 (\text{mod } 3)\\
\lambda_1(n_1 - 1) + \lambda_2(n_2 + n_3) &\equiv 0 \ (\text{mod } 2)\\
\lambda_1(n_2 - 1) + \lambda_2(n_1 + n_3) &\equiv 0 \ (\text{mod } 2)\\
\lambda_1(n_3 - 1) + \lambda_2(n_1 + n_2) &\equiv 0 \ (\text{mod } 2)
\end{aligned}$$

By putting $n_1 = 1$ and $n_2 = n_3 = n$, we get

$$\begin{aligned}
F(\lambda_1, \lambda_2) = \lambda_1 n(n-1) + \lambda_2 n(n+2) &\equiv 0 (\text{mod } 3) \quad \cdots \quad (1)\\
G(\lambda_1, \lambda_2) = \lambda_1(n-1) + \lambda_2(n+1) &\equiv 0 \ (\text{mod } 2) \quad \cdots \quad (2)
\end{aligned}$$

Note that $F(\lambda_1, \lambda_2) - F(\lambda_1 - 1, \lambda_2 + 1) = -3n \equiv 0 \pmod 3$, and $G(\lambda_1, \lambda_2) - G(\lambda_1 - 1, \lambda_2+1) = -2 \equiv 0 \pmod 2$. This means that n is a solution of $F(\lambda_1, \lambda_2) \equiv 0 \pmod 3$ and $G(\lambda_1, \lambda_2) \equiv 0 \pmod 2$ if and only if n is a solution of $F(\lambda_1-1, \lambda_2+1) \equiv 0 \pmod 3$ and $G(\lambda_1 - 1, \lambda_2 + 1) \equiv 0 \pmod 2$. Thus, it is enough to solve for n only for a fixed λ_2 and for all $\lambda_1 \equiv 0, 1, \ldots, 5 \pmod 6$. The following results are obtained.

Theorem 1.1 *If $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$, then λ_1, λ_2 and n are related* mod 6 *as in the following table.*

λ_1 \ λ_2	0	1	2	3	4	5
0	all n	1, 3	0, 1, 3, 4	1, 3, 5	0, 1, 3, 4	1, 3
1	1,3	0, 1, 3, 4	1, 3, 5	0, 1, 3, 4	1, 3	all n
2	0, 1, 3, 4	1, 3, 5	0, 1, 3, 4	1, 3	all n	1, 3
3	1, 3, 5	0, 1, 3, 4	1, 3	all n	1, 3	0, 1, 3, 4
4	0, 1, 3, 4	1, 3	all n	1, 3	0, 1, 3, 4	1, 3, 5
5	1, 3	all n	1, 3	0, 1, 3, 4	1, 3, 5	0, 1, 3, 4

The definition of GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ along with the existence of BIBD$(n, 3, 6)$ for all $n \geq 3$ if GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists and $n \geq 3$, then for any positive integer i, GDD$(v = 1 + n + n, 3, \lambda_1 + 6i, \lambda_2)$ exists. This means that λ_1 can be arbitrarily large.

2 Preliminary

We review some known results concerning triple designs that will be used in the sequel, most of which are taken from [12]. We will also prove some results that are needed for proving the main theorem.

A BIBD$(v, 3, 1)$ is usually called a *Steiner triple system* and is denoted by STS(v). Let $(V, \mathcal{B})$ be an STS(v). Then the number of triples $b = |\mathcal{B}| = v(v-1)/6$.

The following results on the existence of λ-fold triple systems are well known (see, e.g., [12]).

Theorem 2.1 *Let n be a positive integer. Then a* BIBD$(n, 3, \lambda)$ *exists if and only if λ and n are in one of the following cases:*

(a) $\lambda \equiv 0 \pmod 6$ *and* $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and* $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ *and* $n \equiv 0$ *or* $1 \pmod 3$, *and*

(d) $\lambda \equiv 3 \pmod 6$ *and n is odd.*

Let $(V, \mathcal{B})$ be an STS(v). An *automorphism* of an $(V, \mathcal{B})$ is a bijection $\alpha : V \to V$ such that $t = \{x, y, z\} \in \mathcal{B}$ if and only if $t\alpha = \{x\alpha, y\alpha, z\alpha\} \in \mathcal{B}$. An STS$(v)$ is *cyclic* if it has an automorphism that is a permutation consisting of a single cycle of length v. It is natural to ask, for which integers v does there exist a cyclic STS(v)? This question can be answered by solving Heffter's Difference Problems posed by L. Heffter in 1896 (see page 32 of [12]).

For any integer v, a *difference triple* of $\{1, 2, 3, \ldots, v-1\}$ is a subset $\{x, y, z\}$ of three distinct elements of $\{1, 2, \ldots, v-1\}$ such that $x + y \equiv \pm z (\mathrm{mod}\ v)$.

Heffter's Difference Problems:

1. Let $v = 6k + 1$. Is it possible to partition the set $\{1, 2, \dots, (v-1)/2\}$ into difference triples?

2. Let $v = 6k + 3$. Is it possible to partition the set $\{1, 2, \dots, (v-1)/2\} \setminus \{v/3\}$ into difference triples?

If $\{x, y, z\}$ is a difference triple (so $x + y \equiv \pm z (\mathrm{mod}\ v)$), we define the corresponding *base block* to be the triple $\{0, x, x + y\}$.

Peltesohn solved both of Heffter's Difference Problems in 1939 (see page 33 of [12]) as stated in the following theorem.

Theorem 2.2 *For all* $v \equiv 1$ or $3 (\mathrm{mod}\ 6)$*,* $v \neq 9$*, there exists a cyclic* STS(v)*.*

Let K_v be the complete graph of order v with $\mathbb{Z}_v = \{0, 1, 2, \dots, v-1\}$ as its vertex set. The *length* of an edge xy, denoted by $\ell(x, y)$, is defined by

$$\ell(x, y) = \min\{|x - y|, v - |x - y|\}.$$

A *factor* of a graph G is a spanning subgraph. An r*-factor* of a graph is a spanning r-regular subgraph, and an r*-factorization* is a partition of the edges of the graph into disjoint r-factors. A graph G is said to be r*-factorable* if it admits an r-factorization. In particular, a 1-factor is a *perfect matching,* and a 1-factorization of an r-regular graph G is a set of 1-factors which partition the edge set of G.

The following observations are useful.

Let K_v be the complete graph of order v with $\mathbb{Z}_v = \{0, 1, 2, \dots, v-1\}$ as its vertex set.

1. $\ell(x, y) = \ell(y, x)$ and for each integer i, $\ell(x + i, y + i) = \ell(x, y)$, where " + " is taken modulo v.

2. Let i be an integer with $1 \leq i < \frac{v}{2}$. Then the set of edges of K_v of length i forms a 2-factor of K_v.

3. If $v = 2m$, then the set of edges of K_v of length m forms a 1-factor of K_v.

4. It is well known that K_v is 1-factorable if v is even while K_v is 2-factorable if v is odd. Since a union of k 1-factors of K_v is a k-factor of K_v, it follows that if v is even, then K_v is k-factorable if and only if $k \mid v - 1$.

5. A union of a disjoint k-factor and an h-factor of K_v forms a $(k + h)$-factor of K_v.

Let v be an integer of the form $6k + 4$. Then an STS(v) does not exist. By using an idea similar to Heffter's Difference Problems, we obtain the following theorem.

Theorem 2.3 *Let k be a positive integer and $n = 6k + 4$. Then there exists $t \in \{1, 2, \ldots, 3k + 1\}$ such that $\{1, 2, \ldots, 3k + 1\} \setminus \{t\}$ can be partitioned into k difference triples.*

Proof. Let k be a positive integer and $n = 6k + 4$. We prove the result by constructing difference triples directly according to k as follows.

We start with $k = 1$. It is easy to see that $\{1, 2, 3\}$ forms a difference triple, thus, in this case, we may choose $t = 4$.

For $k = 2$, the set $\{1, 2, \ldots, 3k + 1\} = \{1, 2, \ldots, 7\}$. Since $\{1, 3, 4\}, \{2, 5, 7\}$ are two disjoint difference triples, we choose $t = 6$ for $k = 2$.

For $k = 3$, the set $\{1, 2, \ldots, 3k+1\} = \{1, 2, \ldots, 10\}$. Since $\{1, 4, 5\}, \{2, 8, 10\}, \{3, 6, 9\}$ are three pairwise disjoint difference triples, we choose $t = 7$ for $k = 3$.

For $k = 4$, it is clear that $\{1, 5, 6\}, \{2, 8, 10\}, \{3, 9, 12\}, \{4, 7, 11\}$ are four pairwise disjoint difference triples. Thus, in this case, we can choose $t = 13$.

We now suppose that $k \geq 5$.

If $k = 2r + 1$ for some integer $r \geq 2$, then $\{1, 2, \ldots, 3k + 1\} \setminus \{3r + 4\}$ can be partitioned into k difference triples as follows:

$\{1, 2r + 2, 2r + 3\}, \{2r + 1, 2r + 4, 4r + 5\}, \{2r, 3r + 5, 5r + 5\},$

$\{2s + 1, 3r + 4 - s, 3r + 5 + s\}$ for $1 \leq s \leq r - 1$,

$\{2s, 5r + 5 - s, 5r + 5 + s\}$ for $1 \leq s \leq r - 1$.

If $k = 2r$ for some integer $r \geq 3$, then $\{1, 2, \ldots, 3k+1\} \setminus \{5r+3\}$ can be partitioned into k difference triples as follows:

$\{1, 2r + 1, 2r + 2\}, \{2r, 2r + 3, 4r + 3\}, \{2r - 1, 3r + 3, 5r + 2\},$

$\{2s, 3r + 3 - s, 3r + 3 + s\}$ for $1 \leq s \leq r - 1$,

$\{2s + 1, 5r + 2 - s, 5r + 3 + s\}$ for $1 \leq s \leq r - 2$. □

Let $n = 6k + 4$. Then $M = \{\{j, j + 3k + 2\} : j = 1, 2, \ldots, 3k + 1\}$ is a 1-factor of K_n and $H = \{\{j, j + t\} : j \in \mathbb{Z}_n\}$ is a 2-factor of K_n, where t is the removal element as mentioned in Theorem 2.3. The following result can be obtained as a direct consequence of Theorem 2.3.

Theorem 2.4 *Let k be a positive integer. If $n = 6k + 4$. Then there exist a 1-factor M and a 2-factor H of K_n such that $K_3 \mid (K_n \setminus (M \cup H))$.* □

The following notation will be used throughout the paper for our constructions.

1. Let $G = \langle V(G), E(G)\rangle$ and $H = \langle V(H), E(H)\rangle$ be two vertex disjoint simple graphs. If $e = uv \in E(G)$ and $a \in V(H)$, then we use $a + e$ for the triple $\{a, u, v\}$. If $\emptyset \neq X \subseteq E(G)$, then we use $a + X$ for the collection of triples $a + e$ for all $e \in X$.

2. Let V be a v-set. We use $K(V)$ for the complete graph K_v on the vertex set V.

3. Let V be a v-set. A BIBD$(V, 3, \lambda)$ can be defined as

$$\text{BIBD}(V, 3, \lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \text{BIBD}(v, 3, \lambda)\}.$$

4. Let X and Y be disjoint sets of cardinality m and n, respectively. We define a GDD$(X, Y; \lambda_1, \lambda_2)$ as

$$\text{GDD}(X, Y; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y; \mathcal{B}) \text{ is a } \text{GDD}(v = m + n, 3, \lambda_1, \lambda_2)\}.$$

5. Let X, Y and Z be three pairwise disjoint sets of cardinality n_1, n_2 and n_3, respectively. We define a GDD$(X, Y, Z; \lambda_1, \lambda_2)$ as

$$\text{GDD}(X, Y, Z; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y, Z; \mathcal{B}) \text{ is a GDD}(v = n_1 + n_2 + n_3, 3, \lambda_1, \lambda_2)\}.$$

6. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus "$\cup$" in our context will be used for the union of multisets.

7. Finally, if we have a set X, the cardinality of X is denoted by $|X|$.

3 Sufficiency

We prove in this section that the necessary conditions given in Theorem 1.1 become sufficient by constructing a GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ corresponding to (λ_1, λ_2) given in the table. The problem of determining (λ_1, λ_2) such that a GDD$(v = 1+2+2, 3, \lambda_1, \lambda_2)$ exists was completely solved in [2]. Thus for GDD$(v = 1+n+n, 3, \lambda_1, \lambda_2)$ we understand that $n \geq 3$. As we will construct GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$, we will use in this section X, Y, Z for sets of sizes $1, n, n$, respectively. The following observations are useful.

1. GDD$(v = 1 + n + n, 3, \lambda, \lambda)$ exists if and only if BIBD$(2n + 1, 3, \lambda)$ exists.

2. $\mathsf{Spec}(\lambda, \lambda)$ can be obtained by applying results of Theorem 2.1 and we can characterize $\mathsf{Spec}(\lambda, \lambda)$ according to $\lambda \pmod 6$ as

   (a) Since $2n+1$ is odd, it follows that $n \in \mathsf{Spec}(\lambda, \lambda)$ for all $\lambda \equiv 0$ or $3 \pmod 6$.

   (b) If $\lambda \equiv 1, 2, 4$ or $5 \pmod 6$, then $n \in \mathsf{Spec}(\lambda, \lambda)$ if and only if $n \equiv 0$ or $1 \pmod 3$.

3. Let $\langle X, Y, Z; \mathcal{B} \rangle$ be a GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$. Then for each positive integer i, $\langle X, Y, Z; i\mathcal{B} \rangle$ is a GDD$(v = 1 + n + n, 3, i\lambda_1, i\lambda_2)$, where $i\mathcal{B}$ is the union of i copies of $\mathcal{B}$. Thus, if $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$, then $n \in \mathsf{Spec}(i\lambda_1, i\lambda_2)$.

4. If $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ and for each pair of non-negative integers (i, j) with $i \geq j$, then $n \in \mathsf{Spec}(\lambda_1 + 6i, \lambda_2 + 6j)$.

5. If a BIBD$(n, 3, \lambda_1)$ exists and a BIBD$(2n + 1, 3, \lambda_2)$ exists, then a GDD$(v = 1 + n + n, 3, \lambda_1 + \lambda_2, \lambda_2)$ exists.

With these observations and Theorem 1.1 we have the following results.

Theorem 3.1 *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$ and $\lambda_1 \equiv \lambda_2 \pmod 6$. Then*

1. *for all $n \geq 3$, $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ if and only if $\lambda_1 \equiv 0$ or $3 \pmod 6$,*
2. *for all $n \geq 3$ and $n \not\equiv 2 \pmod 3$, $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ if and only if $\lambda_1 \equiv 1, 2, 4$ or $5 \pmod 6$.*

Theorem 3.1 confirms that all entries in the main diagonal of the table are sufficient.

Theorem 3.2 *Let λ_1 and λ_2 be positive integers such that $\lambda_1 \geq \lambda_2$. If $n \equiv 1$ or $3 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$.*

Theorem 3.2 shows that the necessary conditions for $n \equiv 1$ or $3 \pmod 6$ appearing in every entry of the table become sufficient.

Let n be a positive integer. Then there exists $i \in \{0, 1, \ldots, 5\}$ such that $n \equiv i \pmod 6$. We say that n and (λ_1, λ_2) are *related* if i is an entry in (λ_1, λ_2) position in the table. Let

$$F(n \equiv i \pmod 6) = \{(\lambda_1, \lambda_2) : n \text{ and } (\lambda_1, \lambda_2) \text{ are related}\}.$$

Let $n \equiv 0$ or $4 \pmod 6$. We can see in the table that $F(n \equiv 0 \pmod 6) = F(n \equiv 4 \pmod 6)$ and they are equal to $\{(i, i) : i \in \{0, 1, \ldots, 5\}\} \cup \{(2, 0), (0, 2), (0, 4), (4, 0), (1, 3), (3, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 5), (5, 3)\}$. Since $n \equiv 0$ or $4 \pmod 6$, it follows that $2n + 1 \equiv 1$ or $3 \pmod 6$ and hence BIBD$(2n + 1, 3, 1)$ and BIBD$(n, 3, 2)$ exist. Thus, it is clear that if GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists, then GDD$(v = 1 + n + n, 3, \lambda_1 + 1, \lambda_2 + 1)$ and GDD$(v = 1 + n + n, 3, \lambda_1 + 2, \lambda_2)$ exist. We use

$$(a, b) \Rightarrow (a + 1, b + 1)$$

to denote that if GDD$(v = 1 + n + n, 3, a, b)$ exists, then GDD$(v = 1 + n + n, 3, a + 1, b + 1)$ exists and we use

$$\begin{array}{c}(a, b)\\ \Downarrow \\ (a + 2, b)\end{array}$$

to denote that if GDD$(v = 1+n+n, 3, a, b)$ exists, then GDD$(v = 1+n+n, 3, a+2, b)$ exists. The following diagram shows that if $n \equiv 0$ or $4 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all $(\lambda_1, \lambda_2) \in \mathcal{F}(n \equiv 0 \pmod 6)$.

$$\begin{array}{ccccccccccc}
(1,1) & \Rightarrow & (2,2) & \Rightarrow & (3,3) & \Rightarrow & (4,4) & \Rightarrow & (5,5) & \Rightarrow & (6,6) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(3,1) & \Rightarrow & (4,2) & \Rightarrow & (5,3) & \Rightarrow & (6,4) & \Rightarrow & (7,5) & \Rightarrow & (8,6) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(5,1) & \Rightarrow & (6,2) & \Rightarrow & (7,3) & \Rightarrow & (8,4) & \Rightarrow & (9,5) & \Rightarrow & (10,6) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(7,1) & \Rightarrow & (8,2) & \Rightarrow & (9,3) & \Rightarrow & (10,4) & \Rightarrow & (11,5) & \Rightarrow & (12,6)
\end{array}$$

Let $n \equiv 2 \pmod 6$. Since $|X \cup Y| = |X \cup Z| \equiv 3 \pmod 6$, it follows, by Theorem 2.1, that $\text{BIBD}(X \cup Y, 3, 2)$ and $\text{BIBD}(X \cup Z, 3, 2)$ are not empty. We now choose $\mathcal{B}_1 \in \text{BIBD}(X \cup Y, 3, 2)$ and $\mathcal{B}_2 \in \text{BIBD}(X \cup Z, 3, 2)$. Since $Y \cup Z$ is a set of size $12k + 4$, it follows, by Theorem 2.1, that there exists $\mathcal{B}_3 \in \text{BIBD}(Y \cup Z, 3, 2)$. We now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. It can be easily check that $(X, Y, Z; \mathcal{B})$ forms a $\text{GDD}(v = 1 + n + n, 3, 4, 2)$.

Let n be an integer with $n \equiv 2 \pmod 6$. Let X, Y and Z be pairwise disjoint sets of cardinality $1, n$ and n, respectively. Since $|X \cup Y| = |X \cup Z| \equiv 3 \pmod 6$, it follows, by Theorem 2.1, that $\text{BIBD}(X \cup Y, 3, 1)$ and $\text{BIBD}(X \cup Z, 3, 1)$ are not empty. We now choose $\mathcal{B}_1 \in \text{BIBD}(X \cup Y, 3, 1)$ and $\mathcal{B}_2 \in \text{BIBD}(X \cup Z, 3, 1)$. It was shown in [13] that $\text{GDD}(Y, Z; 4, 1) \neq \emptyset$, so we choose $\mathcal{B}_3 \in \text{GDD}(Y, Z; 4, 1)$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. It can be easily checked that $(X, Y, Z; \mathcal{B})$ forms a $\text{GDD}(v = 1 + n + n, 3, 5, 1)$.

Let $n \equiv 2 \pmod 6$. We can see in the table that $F(n \equiv 2 \pmod 6)$ is equal to $\{(0,0), (5,1), (4,2), (3,3), (2,4), (1,5)\}$. Since $n \equiv 2 \pmod 6$, it follows that $2n+1 \equiv 5 \pmod 6$ and hence $\text{BIBD}(2n + 1, 3, 3)$ exist. Thus, it is clear that if $\text{GDD}(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists, then $\text{GDD}(v = 1 + n + n, 3, \lambda_1 + 3, \lambda_2 + 3)$ exist. We use

$$(a, b) \Rightarrow (a + 3, b + 3)$$

to denote that if $\text{GDD}(v = 1+n+n, 3, a, b)$ exists, then $\text{GDD}(v = 1+n+n, 3, a+3, b+3)$ exists. The following diagram shows that if $n \equiv 2 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all $(\lambda_1, \lambda_2) \in \mathcal{F}(n \equiv 2 \pmod 6)$. Note that $n \in \mathsf{Spec}(0, 0)$ and $n \in \mathsf{Spec}(3, 3)$ has been proved in Theorem 3.1.

$$\begin{array}{c}
(4,2) \Rightarrow (7,5) = (1,5) \\
(5,1) \Rightarrow (8,4) = (2,4)
\end{array}$$

Let n be an integer such that $n \equiv 5 \pmod 6$. We first observe the following construction. Let $n = 5$. Put $X = \{x\}$, $Y = \{1, 2, 3, 4, y\}$, and $Z = \{a, b, c, d, z\}$, and $Y' = Y \setminus \{y\}$ and $Z' = Z \setminus \{z\}$. Since $|X \cup Y' \cup Z'| = 9$, it follows that $\text{BIBD}(X \cup Y' \cup Z', 3, 1) \neq \emptyset$. We choose $\mathcal{B}_1 \in \text{BIBD}(X \cup Y' \cup Z', 3, 1)$. Define

$\mathcal{B}_2 = \{\{1, 2, y\}, \{2, 3, y\}, \{3, 4, y\}, \{4, 1, y\}\}$,

$\mathcal{B}_3 = \{\{a, b, z\}, \{b, c, z\}, \{c, d, z\}, \{d, a, z\}\}$,

$\mathcal{B}_4 = \{\{1, 3, z\}, \{2, 4, z\}\}$, and

$\mathcal{B}_5 = \{\{a, c, y\}, \{b, d, y\}\}$.

Choose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \{\{x, y, z\}\}$. It can be easily checked that $(X, Y, Z; \mathcal{B})$ forms a GDD$(v = 1 + 5 + 5, 3, 2, 1)$.

Now suppose that $n \equiv 5 \pmod 6$ and $n = 6k + 5 \geq 11$. Let X, Y and Z be pairwise disjoint sets of cardinality $1, n$ and n, respectively. Choose $y \in Y$ and $z \in Z$ and put $Y' = Y \setminus \{y\}$ and $Z' = Z \setminus \{z\}$. Since $|Y'| = |Z'| = 6k + 4$, it follows, by Theorem 2.4, that there exist a perfect matching M_1 and a 2-factor H_1 of $K(Y')$ such that $K_3 \mid (K(Y') \setminus (M_1 \cup H_1))$. Similarly, there exist a perfect matching M_2 and a 2-factor H_2 of $K(Z')$ such that $K_3 \mid (K(Z') \setminus (M_2 \cup H_2))$. Let $\mathcal{T}_1$ be a set of triples in $K(Y') \setminus (M_1 \cup H_1)$, and $\mathcal{T}_2$ be a set of triples in $K(Z') \setminus (M_2 \cup H_2)$ as described in Theorem 2.4. Let $\mathcal{B}_1 = y + H_1$, $\mathcal{B}_2 = z + M_1$, $\mathcal{B}_3 = z + H_2$, $\mathcal{B}_4 = y + M_2$. Since $X \cup Y' \cup Z'$ is a $(12k + 9)$-set, it follows, by Theorem 2.1, that BIBD$(X \cup Y' \cup Z', 3, 1) \neq \emptyset$. We choose $\mathcal{B}_5 \in$ BIBD$(X' \cup Y' \cup Z, 3, 1)$. Let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{\{x, y, z\}\}$. It can be checked that $(X, Y, Z; \mathcal{B})$ forms a GDD$(v = 1 + n + n, 3, 2, 1)$. Therefore, GDD$(v = 1 + n + n, 3, 2, 1)$ exists and GDD$(v = 1 + n + n, 3, 2i, i)$ exists as well for all positive integers i. In particular, GDD$(v = 1 + n + n, 3, 4, 2)$ exists

Let $n \equiv 5 \pmod 6$. We can see in the table that $F(n \equiv 5 \pmod 6))$ is equal to $\{(0, 0), (3, 0), (2, 1)), (5, 1), (1, 2), (4, 2), (0, 3), (3, 3), (2, 4), (5, 4), (1, 5), (4, 5)\}$. Since $n \equiv 5 \pmod 6$, it follows that $2n + 1 \equiv 5 \pmod 6$ and hence BIBD$(2n + 1, 3, 3)$ and BIBD$(n, 3, 3)$ exist. Thus, it is clear that if GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ exists, then GDD$(v = 1 + n + n, 3, \lambda_1 + 3, \lambda_2 + 3)$ and GDD$(v = 1 + n + n, 3, \lambda_1 + 3, \lambda_2)$ exist. Note that $n \in \mathsf{Spec}(0, 0)$ and $n \in \mathsf{Spec}(3, 3)$ has been proved in Theorem 3.1. We use

$$(a, b) \Rightarrow (a + 3, b + 3)$$

to denote that if GDD$(v = 1 + n + n, 3, a, b)$ exists, then GDD$(v = 1 + n + n, 3, a + 3, b + 3)$ exists and we use

$$\begin{array}{c} (a, b) \\ \Downarrow \\ (a + 3, b) \end{array}$$

to denote that if GDD$(v = 1+n+n, 3, a, b)$ exists, then GDD$(v = 1+n+n, 3, a+3, b)$ exists. The following diagram shows that if $n \equiv 5 \pmod 6$, then $n \in \mathsf{Spec}(\lambda_1, \lambda_2)$ for all $(\lambda_1, \lambda_2) \in \mathcal{F}(n \equiv 5 \pmod 6))$.

$$\begin{array}{ccc} (0, 0) & & (3, 3) \\ \Downarrow & & \Downarrow \\ (3, 0) & & (0, 3) \end{array}$$

$$\begin{array}{ccc} (2, 1) & \Rightarrow & (5, 4) \\ \Downarrow & & \Downarrow \\ (5, 1) & & (2, 4) \end{array}$$

$$\begin{array}{ccc} (4,2) & \Rightarrow & (1,5) \\ \Downarrow & & \Downarrow \\ (1,2) & & (4,5) \end{array}$$

Combining the results in this section we have the following main theorem.

Theorem 3.3 *Let n be an integer $n \geq 3$ and $\lambda_1 \geq \lambda_2$. Then* GDD$(v = 1 + n + n, 3, \lambda_1, \lambda_2)$ *exists if and only if*

$$\begin{aligned} \lambda_1 n(n-1) + \lambda_2 n(n+2) &\equiv 0 \pmod 3 \quad \textit{and} \\ \lambda_1(n-1) + \lambda_2(n+1) &\equiv 0 \pmod 2. \end{aligned}$$

Acknowledgements

N. Pabhapote is supported by the University of the Thai Chamber of Commerce, and N. Punnim is supported by the Thailand Research Fund.

References

[1] R.C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *J. Amer. Statist. Assoc.* **47** (1952), 151–184.

[2] A. Chaiyasena, S.P. Hurd, N. Punnim and D.G. Sarvate, Group divisible designs with two association classes, *J. Combin. Math. Combin. Comput.* **82** (1) (2012), 179–198.

[3] C.J. Colbourn and D.H. Dinitz (Eds), *Handbook of Combinatorial Designs*, 2nd ed., Chapman and Hall, CRC Press, Boca Raton, 2007.

[4] C.J. Colbourn and A. Rosa, *Triple Systems,* Clarendon Press, Oxford, 1999.

[5] S. I. El-Zanati, N. Punnim and C. A. Rodger, Gregarious GDDs with two associate classes, *Graphs Combin.* **26** (6) (2010), 775–780.

[6] H.L. Fu and C.A. Rodger, Group divisible designs with two associate classes: $n = 2$ or $m = 2$, *J. Combin. Theory Ser. A* **83** (1) (1998), 94–117.

[7] H.L. Fu, C.A. Rodger and D.G. Sarvate, The existence of group divisible designs with first and second associates, having block size 3, *Ars Combin.* **54** (2000), 33–50.

[8] S.P. Hurd and D.G. Sarvate, Group divisible designs with two association classes and with groups of sizes 1, 1, and n, *J. Combin. Math. Combin. Comput.* **75** (2010), 209–215.

[9] S.P. Hurd and D.G. Sarvate, Group association designs with two association classes and with two or three groups of size 1, *J. Combin. Math. Combin. Comput.* (2012), 179–198 .

[10] S.P. Hurd and D.G. Sarvate, Group divisible designs with three unequal groups and larger first index, *Discrete Math.* **311** (2011), 1851–1859.

[11] W. Lapchinda, N. Punnim and N. Pabhapote, GDDs with two associate classes with three groups of sizes $1, n, n$ and $\lambda_1 < \lambda_2$, *Lec. Notes Comp. Sc.* **8296** (2013), 101–109.

[12] C.C. Lindner and C.A. Rodger, *Design Theory*, 2nd ed., CRC Press, Boca Raton, 2009.

[13] N. Pabhapote and N. Punnim, Group divisible designs with two associate classes and $\lambda_2 = 1$, *Int. J. Math. Math. Sci.* Art. ID 148580 (2011), 10 pp.

[14] N. Punnim and C. Uiyyasathian, Group divisible designs with two associate classes and $(\lambda_1, \lambda_2) = (1, 2)$, *J. Combin. Math. Combin. Comput.* **82** (1) (2012), 117–130.

(Received 4 Mar 2013; revised 6 Nov 2013)