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INTRODUCTION TO THE HIDDEN VARIABLE QUESTION

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1. MOTIVATION

Theoretical physicists live in a classical world, looking out into a quantum mechanical world. The latter we describe only subjectively, in terms of procedures and results in our classical domain. This subjective description is effected by means of quantum mechanical state functions ψ , which characterize the classical conditioning of quantum mechanical systems and permit predictions about subsequent events at the classical level. The classical world of course is described quite directly - "as it is". We could specify for example the actual positions $\Lambda_1, \Lambda_2, \ldots$ of material bodies, such as the switches defining experimental conditions and the pointers, or print, defining experimental results. Thus in contemporary theory the most complete description of the state of the world as a whole, or of any part of it extending into our classical domain, is of the form

$$(\Lambda_1, \Lambda_2, ----, \psi) \qquad (1.1)$$

with both classical variables and one or more quantum mechanical wave functions.

Now nobody knows just where the boundary between the classical and quantum domains is situated. Most feel that experimental switch settings and pointer readings are on this side. But some would think the boundary nearer, other would think it farther, and many would prefer not to think about it. In fact, the matter is of very little importance in practice. This is because of the immense difference in scale between things for which quantum mechanical description is numerically essential and those ordinarily perceptible by human beings. Nevertheless, the movability of the boundary is of only approximate validity; demonstrations of it depend on neglecting numbers which are small, but not zero, which might tend to zero for infinitely large systems, but are only very small for real finite systems. A theory founded in this way on arguments of manifestly approximate character, however good the approximation, is surely of provisional nature. It seems legitimate to speculate on how the theory might evolve. But of course no one is obliged to join in such speculation.

A possibility is that we find exactly where the boundary lies. More plausible to me is that we will find that there is no boundary. It is hard for me to envisage intelligible discourse about a world with no classical part - no base of given events, be they only mental events in a single consciousness, to be correlated. On the other hand, it is easy to imagine that the classical domain could be extended to cover the whole. The wave functions could prove to be a provisional or incomplete description of the quantum mechanical part, of which an objective account would become possible. It is this possibility, of a homogeneous account of the world, which is for me the chief motivation of the study of the so-called "hidden variable" possibility.

A second motivation is connected with the statistical character of quantum mechanical predictions. Once the incompleteness of the wave function description is suspected, it can be conjectured that the seemingly random statistical fluctuations are determined by the extra "hidden" variables - "hidden" because at this stage we can only conjecture their existence and certainly cannot control them. Analogously, the description of Brownian motion for example might first have been developed in a purely statistical way, the statistics becoming intelligible later with the hypothesis of the molecular constitution of fluids, this hypothesis then pointing to previously unimagined experimental possibilities, the exploitation of which made the hypothesis entirely convincing. For me the possibility of determinism is less compelling than the possibility of having one world instead of two. But, by requiring it, the programme becomes much better defined and more easy to come to grips with.

A third motivation is in the peculiar character of some quantum mechanical predictions, which seem almost to cry out for a hidden variable interpretation. This is the famous argument of Einstein, Podolsky and Rosen [1]. Consider the example, advanced by Bohm [2], of a pair of spin-$\frac{1}{2}$ particles formed somehow in the singlet spin state and then moving freely in opposite directions. Measurements can be made, say by Stern-Gerlach magnets, on selected components of the spins $\vec{\sigma}_1$ and $\vec{\sigma}_2$. If measurement of $\vec{\sigma}_1 \cdot \hat{a}$, where $\hat{a}$ is some unit vector, yields the value $+1$, then, according to quantum mechanics, measurement of $\vec{\sigma}_2 \cdot \hat{a}$ must yield the value -1, and vice versa. Thus we can know in advance the result of measuring any component of $\vec{\sigma}_2$ by previously, and possibly at a very distant place, measuring the corresponding component of $\vec{\sigma}_1$. This strongly suggests that the outcomes of such measurements, along arbitrary directions, are actually determined in advance, by variables over which we have no control, but which are sufficiently revealed by the first measurement so that we can anticipate the result of the second. There need then be no temptation to regard the performance of one measurement as a causal influence on the result of the second, distant, measurement. The description of the situation could be manifestly "local". This idea seems at least to merit investigation.

We will find, in fact, that no local deterministic hidden variable theory can reproduce all the experimental predictions of quantum mechanics. This opens the possibility of bringing the question into the experimental domain, by trying to approximate as well as possible the idealized situations in which local hidden variables and quantum mechanics cannot agree. However, before coming to this, we must clear the ground by some remarks on various mathematical investigations that have been made on the possibility of hidden variables in quantum mechanics without any reference to locality.

2. THE ABSENCE OF DISPERSION FREE STATES IN VARIOUS FORMALISMS DERIVED FROM QUANTUM MECHANICS

Consider first the usual Heisenberg uncertainty principle. It says that for quantum mechanical states the predictions for measurements for at least one of a pair of conjugate variables must be statistically uncertain. Thus no quantum mechanical state can be "dispersion free" for every observable. It follows that if a hidden variable account is possible, in which the results of all observations are fully determined, each quantum mechanical state must correspond to an ensemble of states each with different values of the hidden variables. Only these component states will be dispersion free. So one way to formulate the hidden variable problem is as a search for a formalism permitting such dispersion free states.

An early, and very celebrated, example of such an investigation was that of von Neumann [3]. He observed that in quantum mechanics an observable whose operator is a linear combination of operators for other observables

$$A = \beta B + \gamma C$$

has for expectation value the corresponding linear combination of expectation values :

$$\langle A \rangle = \beta \langle B \rangle + \gamma \langle C \rangle \tag{2.1}$$

He considered more general schemes in which this particular feature was preserved. Now for the hypothetical dispersion free states there is no distinction between expectation values and eigenvalues - for each such state must yield with certainty a particular one of the possible results for any measurement. But eigenvalues are not additive. Consider for example components of spin for a particle of spin-$\frac{1}{2}$. The operator for the component along the direction half way between x and y axes is

$$(\sigma_x + \sigma_y)/\sqrt{2}$$

whose eigenvalues ± 1 are certainly not the corresponding linear combinations

$$(\pm 1 \pm 1)/\sqrt{2}$$

of eigenvalues of σ_x and σ_y. Thus the requirement of additive expectation values excludes the possibility of dispersion free states. Von Neumann concluded that a hidden variable interpretation is not possible for quantum mechanics : "it is therefore not, as is often assumed, a question of reinterpretation of quantum mechanics - the present system of quantum mechanics would have to be objectively false in order that another description of the elementary process than the statistical one be possible".

It seems therefore that von Neumann considered the additivity (2.1) more as an obvious axiom than as a possible postulate. But consider what it means in terms of the actual physical situation. Measurements of the three quantities

$$\sigma_x \qquad \sigma_y \qquad (\sigma_x + \sigma_y)/\sqrt{2}$$

require three different orientations of the Stern-Gerlach magnet, and cannot be performed simultaneously. It is just this which makes intelligible the non-additivity of the eigenvalues - the values observed in specific instances. It is by no means a question of simply measuring different components of a pre-existing vector, but rather of observing different products of different physical procedures. That the statistical averages should then turn out to be additive is really a quite remarkable feature of quantum mechanical states, which could not be guessed a priori. It is by no means a "law of thought" and there is no a priori reason to exclude the possibility of states for which it is false. It can be objected that although the additivity of expectation values is not a law of thought, it is after all experimentally true. Yes, but what we are now investigating is precisely the hypothesis that the states presented to us by nature are in fact mixtures of component states which we cannot (for the present) prepare individually. The component states need only have such properties that ensembles of them have the statistical properties of observed states.

It has subsequently been shown that in various other mathematical schemes, derived from quantum mechanics, dispersion free states are not possible [4]. The persistence in these schemes of a kind of uncertainty principle is of course useful and interesting to people working with those schemes. However, the importance of these results, for the question that we are concerned with, is easily exaggerated. The postulates often have great intrinsic appeal to those approaching quantum mechanics in an abstract way. Translated into assumptions about the behaviour of actual physical equipment, they are again seen to be of a far from trivial or inevitable nature [4].

On the other hand, if no restrictions whatever are imposed on the hidden variables, or on the dispersion free states, it is trivially clear that such schemes can be found to account for any experimental results whatever. Ad hoc schemes of this kind are devised every day when experimental physicists, to optimize the design of their equipment, simulate the expected results by deterministic computer programmes drawing on a table of random numbers. Such schemes, from our present point of view, are not very interesting. Certainly what Einstein wanted was a comprehensive account of physical processes evolving continuously and locally in ordinary space and time. We proceed now to describe a very instructive attempt in that direction.

3. A SIMPLE EXAMPLE

Consider the simple hidden variable picture of elementary wave mechanics advanced originally by de Broglie [5] and subsequently clarified by Bohm [6]. Take the case of a single particle of spin-$\frac{1}{2}$ moving in a magnetic field $\vec{H}$. The Schrödinger equation is

$$i \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left\{ \frac{1}{2m} \left(\frac{1}{i} \frac{\partial}{\partial \vec{r}} \right)^2 + \mu \vec{\sigma} \cdot \vec{H} \right\} \psi(\vec{r}, t) \tag{3.1}$$

where the wave function ψ is a two-component Pauli spinor. Let us supplement this quantum mechanical picture by an additional (hidden) variable $\vec{\lambda}$, a single three-vector, which evolves as a function of time according to the law

$$\frac{d\vec{\lambda}}{dt} = \frac{\vec{j}_\psi(\vec{\lambda}, t)}{\rho_\psi(\vec{\lambda}, t)} \tag{3.2}$$

where $\vec{j}$ and ρ are probability currents and densities calculated in the usual way

$$\vec{j}_\psi(\vec{r}, t) = \frac{1}{2} \operatorname{Im} \psi^*(\vec{r}, t) \frac{\partial}{\partial \vec{r}} \psi(\vec{r}, t)$$

$$\rho_\psi(\vec{r}, t) = \psi^*(\vec{r}, t)\, \psi(\vec{r}, t)$$

With summation over suppressed spinor indices understood. It is supposed that the quantum mechanical state specified by the wave function ψ corresponds to an ensemble of states $(\vec{\lambda}, \psi)$ in which the $\vec{\lambda}$'s occur with probability density $\rho(\vec{\lambda}, t)$ such that

$$\rho(\vec{\lambda}, t) = \rho_\psi(\vec{\lambda}, t)$$

It is easy to see that if the distribution ρ of $\vec{\lambda}$ is equal to ρ_ψ in this way at some initial time, then in virtue of the equations of motion (3.1) and (3.2) it remains so at later times.

The fundamental interpretative rule of the model is just that $\vec{\lambda}(t)$ is the real position of the particle at time t, and that observation of position will yield this value. Thus the quantum statistics of position measurements, the probability density ρ_ψ, is recovered immediately. But many other measurements reduce to measurements of position. For example, to "measure the spin component σ_x" the particle is allowed to pass through a Stern-Gerlach magnet and we see whether it is deflected up or down, i.e., we observe position at a subsequent time. Thus the quantum statistics of spin measurements is also reproduced, and so on.

This scheme is readily generalized to many particle systems, within the framework of non-relativistic wave mechanics. The wave function is now in the 3n dimensional configuration space

$$\psi(\vec{r}_1, \vec{r}_2, \text{---}\ \ , t)$$

and the Schrödinger equation can contain interactions between the particles. The hidden variables are n vectors

$$\vec{\lambda}_1, \vec{\lambda}_2, \text{---}$$

moving according to

$$(d\vec{\lambda}_m/dt) = \vec{J}_{m\psi}(\vec{\lambda}_1, \vec{\lambda}_2, ---, t) / \rho_\psi(\vec{\lambda}_1, \vec{\lambda}_2, ---, t)$$
$$\rho_\psi(\vec{\lambda}_1, \vec{\lambda}_2, ---, t) = |\psi(\vec{\lambda}_1, \vec{\lambda}_2, ---, t)|^2$$
$$\vec{J}_{m\psi}(\vec{\lambda}_1, \vec{\lambda}_2, ---, t) = (1/2)\ g_m\ \psi^*(\partial/\partial\vec{\lambda}_m)\psi|_{\vec{r}=\vec{\lambda}}$$

Again the ensemble corresponding to the quantum mechanical state has the $\vec{\lambda}$'s initially distributed with probability density $|\psi|^2$ in the 3n dimensional space, and this remains so in virtue of the equations of motion. Thus the quantum statistics of position measurements, and of any procedure ending up in a position measurement (be it only the observation of a pointer reading) can be reproduced.

What happens to the hidden variables during and after the measurement is a delicate matter. Note only that a prerequisite for a specification of what happens to the hidden variables would be a specification of what happens to the wave function. But it is just at this point that the notoriously vague "reduction of the wave packet" intervenes, at some ill-defined time, and we come up against the ambiguities of the usual theory, which for the moment we aim only to reinterpret rather than to replace. It would indeed be very interesting to go beyond this point. But we will not make the attempt here, for we will find a very striking difficulty at the level to which the scheme has been developed already. Before coming to this, a number of instructive features of the scheme are worth indicating.

One such feature is this. We have here a picture in which although the wave has two components, the particle has only position $\vec{\lambda}$. The particle does not "spin", although the experimental phenomena associated with spin are reproduced. Thus the picture resulting from a hidden variable account of quantum mechanics need not very much resemble the traditional classical picture that the researcher may, secretly, have been keeping in mind. The electron need not turn out to be a small spinning yellow sphere.

A second way in which the scheme is instructive is in the explicit picture of the very essential role of the apparatus. The result of a "spin measurement", for example, depends in a very complicated way on the initial position $\vec{\lambda}$ of the particle and on the strength and geometry of the magnetic field. Thus the result of the measurement does not actually tell us about some property previously possessed by the system, but about something which has come into being in the combination of system and apparatus. Of course, the vital role of the complete physical set-up we learnt long ago, especially from Bohr. When it is forgotten, it is more easy to expect that the results of the observations should satisfy some simple algebraic relations and to feel that these relations should be preserved even by the hypothetical dispersion free states of which quantum mechanical states may be composed. The model illustates how the algebraic relations valid for the statistical ensembles, which are the quantum mechanical states, may be built up in a rather complicated way. Thus the contemplation of this simple model could have a liberalizing effect on mathematical investigators.

Finally, this simple scheme is also instructive in the following way. Even if the infamous boundary, between classical and quantum worlds, should not go away, but rather become better defined as the theory evolves, it seems to me that some classical variables will remain essential (they may describe "macroscopic" objects, or they may be finally restricted to apply only to my sense data.) Moreover, it seems to me that the present "quantum theory of measurement" in which the quantum and classical levels interact only fitfully during highly idealized "measurements" should be replaced by an interaction of a continuous, if variable, character. The equations (3.1) and (3.2) of the simple scheme form a sort of prototype of a master equation of the world in which classical variables are continuously influenced by a quantum mechanical state.

4. A DIFFICULTY

The difficulty is this. Looking at (3.2) one sees that the behaviour of a given variable $\vec{\lambda}_1$ is determined not only by the conditions in the immediate neighbourhood (in ordinary three-space) but also by what is happening at all the other positions $\vec{\lambda}_2, \vec{\lambda}_3, \ldots$. That is to say, that although the system of equations is "local" in an obvious sense in the 3n dimensional space, it is not at all local in ordinary three-space. As applied to the Einstein-Podolsky-Rosen situation, we find that this scheme provides an explicit causal mechanism by which operations on one of the two measuring devices can influence the response of the distant device. This is quite the reverse of the resolution hoped for by EPR, who envisaged that the first device could serve only to reveal the character of the information already stored in space, and propagating in an undisturbed way towards the other equipment.

The question then arises : can we not find another hidden variable scheme with the desired local character ? It can be shown that this is not possible [7),8),9)]. The demonstration moreover is in no way restricted to the context of non-relativistic wave mechanics, but depends only on the existence of separated systems highly correlated with respect to quantities such as spin.

Consider again for example the system of two spin-$\frac{1}{2}$ particles. Suppose they have been prepared somehow in such a state that they then move in different directions towards two measuring devices, and that these devices measure spin components along directions $\hat{a}$ and $\hat{b}$ respectively. Suppose that the hypothetical complete description of the initial state is in terms of hidden variables λ with probability distribution $\rho(\lambda)$ for the given quantum mechanical state. The result A $(=\pm 1)$ of the first measurement can clearly depend on λ and on the setting $\hat{a}$ of the first instrument. Similarly, B can depend on λ and $\hat{b}$. But our notion of locality requires that A does not depend on $\hat{b}$, nor B on $\hat{a}$. We then ask if the mean value $P(\hat{a},\hat{b})$ of the product AB, i.e.,

$$P(\hat{a},\hat{b}) = \int d\lambda\, \rho(\lambda)\, A(\hat{a},\lambda)\, B(\hat{b},\lambda) \tag{4.1}$$

can equal the quantum mechanical prediction.

Actually we can, and should, be somewhat more general. The instruments themselves could contain hidden variables [10)] which could influence the results. If we average first over these instrument variables, we obtain the representation

$$P(\hat{a},\hat{b}) = \int d\lambda\, \rho(\lambda)\, \bar{A}(\hat{a},\lambda)\, \bar{B}(\hat{b},\lambda) \tag{4.2}$$

where the averages $\bar{A}$ and $\bar{B}$ will be independent of $\hat{b}$ and $\hat{a}$, respectively, if the corresponding distributions of instrument variables are independent of $\hat{b}$ and $\hat{a}$, respectively, although of course they may depend on $\hat{a}$ and $\hat{b}$, respectively. Instead of

$$A = \pm 1 \qquad\qquad B = \pm 1 \tag{4.3}$$

we now have

$$|\bar{A}| \leqslant 1 \qquad |\bar{B}| \leqslant 1 \tag{4.4}$$

and this suffices to derive an interesting restriction on P.

In practice, there will be some occasions on which one or both instruments simply fail to register either way. One might then[11] count A and/or B as zero in defining P, $\bar{A}$, and $\bar{B}$; (4.4) remains true and the following reasoning remains valid.

Let $\hat{a}'$ and $\hat{b}'$ be alternative settings of the instruments. Then

$$\begin{aligned} P(\hat{a},\hat{b}) &- P(\hat{a},\hat{b}') \\ &= \int d\lambda\, \rho(\lambda)\left[\bar{A}(\hat{a},\lambda)\bar{B}(\hat{b},\lambda) - \bar{A}(\hat{a},\lambda)\bar{B}(\hat{b}',\lambda)\right] \\ &= \int d\lambda\, \rho(\lambda)\left[\bar{A}(\hat{a},\lambda)\bar{B}(\hat{b},\lambda)\left(1 \pm \bar{A}(\hat{a}',\lambda)\bar{B}(\hat{b}',\lambda)\right)\right] \\ &\quad - \int d\lambda\, \rho(\lambda)\left[\bar{A}(\hat{a},\lambda)\bar{B}(\hat{b}',\lambda)\left(1 \pm \bar{A}(\hat{a}',\lambda)\bar{B}(\hat{b},\lambda)\right)\right] \end{aligned}$$

Then using (4.4)

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| \leqslant \left\{ \begin{aligned} &\int d\lambda\, \rho(\lambda)\left(1 \pm \bar{A}(\hat{a}',\lambda)\bar{B}(\hat{b}',\lambda)\right) \\ + &\int d\lambda\, \rho(\lambda)\left(1 \pm \bar{A}(\hat{a}',\lambda)\bar{B}(\hat{b},\lambda)\right) \end{aligned} \right.$$

or

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| \leqslant 2 \pm \left(P(\hat{a}',\hat{b}') + P(\hat{a}',\hat{b})\right)$$

or more symmetrically

$$|P(\hat{a},\hat{b}) - P(\hat{a},\hat{b}')| + |P(\hat{a}',\hat{b}') + P(\hat{a}',b)| \leqslant 2 \tag{4.5}$$

With $\hat{a}' = \hat{b}'$ and assuming

$$P(\hat{b}', \hat{b}') = -1 \tag{4.6}$$

equation (4.5) yields

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| \leq 1 + P(\hat{b}', b) \tag{4.7}$$

This is the original form of the result [7]. Note that to realize (4.6) it is necessary that the equality sign holds in (4.4), i.e., for this case the possibility of the results depending on hidden variables in the instruments can be excluded from the beginning [12].

The more general relation (4.5) (essentially) was first written by Clauser, Holt, Horne and Shimony [8] for the restricted representation (4.1).

Suppose now, for example, that the system was in the singlet state of the two spins. Then quantum mechanically $P(\hat{a}, \hat{b})$ is given by the expectation value in that state

$$\langle \vec{\sigma}_1 \cdot \hat{a} \quad \vec{\sigma}_2 \cdot \hat{b} \rangle = - \hat{a} \cdot \hat{b} \tag{4.8}$$

This function has the property (4.6), but does not at all satisfy (4.7). With $P(\hat{a}, \hat{b}) = -\hat{a} \cdot \hat{b}$ one finds, for example, that for small angle between $\hat{b}$ and $\hat{b}'$ the left-hand side of (4.7) is in general of first order in this angle, while the right-hand side is only of second order. Thus the quantum mechanical result cannot be reproduced by a hidden variable theory which is local in the way described.

This result opens up the possibility of bringing the questions that we have been considering into the experimental area. Of course, the situation envisaged above is highly idealized. It is supposed that the system is initially in a known spin state, that the particles are known to proceed towards the instruments, and to be measured there with complete efficiency. The question then is whether the inevitable departures from this ideal situation can be kept sufficiently small in practice that the quantum mechanical prediction still violates the inequality (4.5).

In this connection other systems, for example the two-photon system [8] or the two-kaon system [13], may be more promising than that of two-spin-$\frac{1}{2}$ particles. A very serious study of the photon case will be reported to this meeting by Shimony. The experiment described by him, him, and now under way, is not sufficiently close to the ideal to be conclusive for a quite determined advocate of hidden variables. However for most a confirmation of the quantum mechanical predictions, which is only to be expected given the general success of quantum mechanics [14], would be a severe discouragement.

REFERENCES

1) A. Einstein, B. Podolsky and N. Rosen, Phys.Rev. 47, 777 (1935).

2) D. Bohm, Quantum Theory, Prentice Hall (1951).

3) J. von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer Verlag, Berlin (1932) - (English translation, Princeton University Press, 1955).

4) For an analysis of some of these schemes, see :

J.S. Bell, Revs.Modern Phys. 38, 447 (1966). This considers in particular the result of J.M. Jauch and C. Piron [Helv.Phys. Acta 36, 827 (1963)] and the stronger form of von Neumann's result consequent, as observed by Jauch, on the work of A.M. Gleason [J.Math.and Mech. 6, 885 (1957)]. This corollary of Gleason's work was subsequently set out by S. Kochen and E.P. Specker [J.Math.and Mech. 17, 59 (1967)]. Other impossibility proofs have been given by S.P. Gudder [Revs.Modern Phys. 40, 229 (1968)] and by B. Misra [Nuovo Cimento 47, 843 (1967)]; both of these authors remark on the limited nature of their results. On the question of impossibility proofs, see also D. Bohm and J. Bub [Revs.Modern Phys. 38, 453 (1966) and 40, 232 (1968)]; J.M. Jauch and C. Piron [Revs.Modern Phys. 40, 228 (1968)] and J.E. Turner [J.Math.Phys. 9, 1411 (1968)].

5) L. de Broglie gives a documented account of the early development in Louis de Broglie, Physicien et Penseur, Albin Michel, Paris (1953), p.465.

6) D. Bohm, Phys.Rev. 85, 166, 180 (1952). For hidden variable schemes see also the review of H. Friestadt [Nuovo Cimento Suppl. 5, 1 (1957)] and later work by D. Bohm and J. Bub [Revs.Modern Phys. 38, 470 (1966)] and S.P. Gudder [J.Math.Phys. 11, 431 (1970)].

7) J.S. Bell, Physics 1, 195 (1964).

8) J.F. Clauser, M.A. Horne, A. Shimony and R.A. Holt, Phys.Rev. Letters 26, 880 (1969).

9) E.P. Wigner, preprint (1969).

10) We speak here as if the instruments respond in a deterministic way when all variables, hidden or non-hidden, are given. Clearly (4.2) is appropriate also for <u>indeterminism</u> with a certain <u>local</u> character.

11) This is a suggestion of F.S. Crawford.

12) This was the procedure, as regards the ideal case (4.8), in Ref. 7). However in that reference the subsequent discussion of the non-ideal case started again from the restricted representation (4.1). This was quite arbitrary. But the reasoning of that section, used again here, goes through with the more general (4.2). In this connection, I am indebted to F.S. Crawford for a stimulating correspondence.

13) T.B. Day, Phys.Rev. 121, 1204 (1961);

D.R. Inglis, Revs.Modern Phys. 33, 1 (1961).

Note that the spontaneous decay times of the two kaons, because they cannot be set at the will of the experimenter, are not to be regarded as analogous to the settings $\hat{a}$ and $\hat{b}$ of the Stern-Gerlach magnets. The thicknesses of a pair of slabs of matter placed in the lines of flight would be more relevant. I am told by Professor B. d'Espagnat that the rapid decay of the short-lived kaon is a major obstacle to devising a critical experiment.

14) The helium atom, essentially a pair of spin-$\frac{1}{2}$ particles, is a system for which quantum mechanics is strikingly successful. See, for example, H.A. Bethe and E.E. Salpeter, Handbuch der Physik, Springer Verlag, Berlin, 35, 88 (1957).