Rectangle and Square Coverings of Tolerance Spaces and their Direct Product

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Abstract. This article is a sequel to the paper "Blocks of the Direct Product of Tolerance Relations" [7]. The square cover number of the direct product of tolerance spaces and the rectangle cover number of the direct product of formal contexts is treated. Furthermore, we compare rectangle and square covers of tolerance spaces.

Keywords: tolerance relation, formal concept analysis, direct product, factor analysis, rectangle cover, square cover.

1 Introduction

A *tolerance relation* or simply a *tolerance* is a reflexive and symmetric binary relation τ on a non-empty finite set V. The pair $(V, \tau) =: \mathbb{T}$ is called *tolerance space*. An introduction to tolerance spaces together with applications can be found in [10] and [11].

For a tolerance τ on V, a non-empty subset $S \subseteq V$ induces a *square* in τ if $S \times S \subseteq \tau$. If S is maximal with respect to set inclusion, then $S \times S$ defines a *maximal square*.

The set of all maximal squares of $\mathbb{T}$ is denoted by $\mathrm{Sq}(\mathbb{T})$ and determines the tolerance τ, that is $\tau = \bigcup \mathrm{Sq}(\mathbb{T})$. But often not all squares are necessary to cover τ. This motivates the definition of the *square cover number*, $\mathrm{sc}(\mathbb{T})$, of a tolerance space $\mathbb{T}$, as the minimal number of maximal squares necessary to cover τ.

$$\mathrm{sc}(\mathbb{T}) := \min\{k \mid \exists\, \mathcal{S} \subseteq \mathrm{Sq}(\mathbb{T}),\ \tau = \bigcup \mathcal{S},\ |\mathcal{S}| = k\}. \quad (1)$$

In [7] the direct product (defined in Section 2) of tolerance spaces was treated by means of formal concept analysis, which lead to the conjecture:

Conjecture 1. *Let* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *be tolerance spaces. For their direct product* $\mathbb{T}_1 \check{\times} \mathbb{T}_2$ *it holds that* $\mathrm{sc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) = \mathrm{sc}(\mathbb{T}_1) + \mathrm{sc}(\mathbb{T}_2)$.

When we analysed Conjecture 1, it turned out that it is not valid in general. Still, we will provide a sufficient condition for this conjecture to hold (Section 5). The meta framework for this will be formal concept analysis, introduced in

Section 2, together with some tools from graph theory (Section 4). Additionally, we will treat the rectangle cover number of the direct product of formal contexts in Section 3 and Section 6 provides example classes of tolerance spaces for which the square cover number and rectangle cover number are equal. Lastly, Section 7 analyses a construction principle for tolerance spaces, which is based on formal contexts.

2 Formal Concept Analysis

In this section, we will provide the definitions and facts from formal concept analysis (see [5]) that will be used in the sequel.

A *formal context* (or in short *context*) is a triple $\mathbb{K} = (G, M, I)$, where the *incidence* $I \subseteq G \times M$ is a binary relation. For $A \subseteq G$ and $B \subseteq M$, we define two derivation operators:

$$A^I := \{m \in M | \forall a \in A : (a, m) \in I\} = \bigcap_{a \in A} \{a\}^I,$$

$$B_I := \{g \in G | \forall b \in B : (g, b) \in I\} = \bigcap_{b \in B} \{b\}_I.$$

If $A^I = B$ and $B_I = A$, the pair (A, B) is called a *formal concept* (or in short *concept*) with *extent* A and *intent* B. The set of all formal concepts of $\mathbb{K}$ is denoted by $\mathfrak{B}(\mathbb{K})$ and defines the *concept lattice* $\underline{\mathfrak{B}}(\mathbb{K})$, via the order $(A_1, B_1) \leq (A_2, B_2) :\Leftrightarrow A_1 \subseteq A_2$. The *complementary context* is defined as $\mathbb{K}^c = (G, M, I^c) := (G, M, (G \times M) \setminus I)$ and the *dual context* as $\mathbb{K}^d := (M, G, I^{-1})$, with the *inverse relation* $I^{-1} := \{(m, g) \in M \times G \mid (g, m) \in I\}$.

Let $\dot\cup$ denote the disjoint union of sets. We define four binary operations on contexts $\mathbb{K}_1 = (G_1, M_1, I_1)$ and $\mathbb{K}_2 = (G_2, M_2, I_2)$.

The *direct product* $\mathbb{K}_1 \check\times \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, I_1 \check\times I_2)$ with

$$((g, h), (m, n)) \in I_1 \check\times I_2 \;:\Longleftrightarrow\; (g, m) \in I_1 \text{ or } (h, n) \in I_2,$$

the *cardinal product* $\mathbb{K}_1 \hat\times \mathbb{K}_2 := (G_1 \times G_2, M_1 \times M_2, I_1 \hat\times I_2)$ with

$$((g, h), (m, n)) \in I_1 \hat\times I_2 \;:\Longleftrightarrow\; (g, m) \in I_1 \text{ and } (h, n) \in I_2,$$

the *direct sum* $\mathbb{K}_1 \oplus \mathbb{K}_2 := (G_1 \dot\cup G_2, M_1 \dot\cup M_2, I_1 \dot\cup I_2 \dot\cup G_1 \times M_2 \dot\cup G_2 \times M_1)$,

and the *disjoint union* $\mathbb{K}_1 \dot\cup \mathbb{K}_2 := (G_1 \dot\cup G_2, M_1 \dot\cup M_2, I_1 \dot\cup I_2)$.

The two products fulfill De Morgan laws

$$(\mathbb{K}_1 \check{\times} \mathbb{K}_2)^c = \mathbb{K}_1^c \hat{\times} \mathbb{K}_2^c \quad \text{and} \quad (\mathbb{K}_1 \hat{\times} \mathbb{K}_2)^c = \mathbb{K}_1^c \check{\times} \mathbb{K}_2^c, \tag{2}$$

the relation $I_1 \check{\times} I_2$ can be expressed as

$$I_1 \check{\times} I_2 = (G_1 \times M_1) \hat{\times} I_2 \cup I_1 \hat{\times} (G_2 \times M_2), \tag{3}$$

and we will denote the incidence relation of the direct sum by $I_1 \oplus I_2$.

A context $\mathbb{K}$ is *crossed*, if the *adjacency matrix* $\mathcal{A}_I$ of its incidence I has at least one full row and one full column. If $\mathcal{A}_I$ has at least one empty row and one empty column, we say that $\mathbb{K}$ is *co-crossed*. In case of two crossed contexts, we can express the concept lattice of the cardinal product as the direct product (in terms of Universal Algebra) of each factors concept lattice (see [3]).

$$\underline{\mathfrak{B}}(\mathbb{K}_1 \hat{\times} \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2). \tag{4}$$

The concept lattice of the direct sum is isomorphic to the direct product of each components concept lattice too[1].

$$\underline{\mathfrak{B}}(\mathbb{K}_1 \oplus \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \times \underline{\mathfrak{B}}(\mathbb{K}_2). \tag{5}$$

Let $\mathbb{P} = (P, \leq_{\mathbb{P}}, 0_{\mathbb{P}}, 1_{\mathbb{P}})$ and $\mathbb{L} = (L, \leq_{\mathbb{L}}, 0_{\mathbb{L}}, 1_{\mathbb{L}})$ be bounded posets such that $P \cap L = \emptyset$. The poset $\mathbb{S} = (S, \leq, 0, 1)$, with $P^* := P \backslash \{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$, $L^* := L \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $S^* := P^* \cup L^*$, $S := S^* \cup \{0, 1\}$ and $\leq := \leq_{\mathbb{P}} \cup \leq_{\mathbb{L}} \cup \{0\} \times S \cup S \times \{1\}$, is called the *horizontal sum* of $(\mathbb{P}, \mathbb{L})$ and is denoted by $\mathbb{P} \hat{+} \mathbb{L} := \mathbb{S}$.

For the disjoint union of two contexts, the resultant concept lattice is the horizontal sum of each components concept lattice.

$$\underline{\mathfrak{B}}(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2) \cong \underline{\mathfrak{B}}(\mathbb{K}_1) \hat{+} \underline{\mathfrak{B}}(\mathbb{K}_2). \tag{6}$$

Next, since a concept (A, B) with non-empty sets A and B induces a *maximal rectangle* $A \times B$ in I, we define the *rectangle cover number* (see also [12]), $\mathrm{rc}(\mathbb{K})$, of a context $\mathbb{K}$ as

$$\mathrm{rc}(\mathbb{K}) := \min\{k \mid \exists\, \mathcal{F} \subseteq \mathfrak{B}(\mathbb{K}),\ I = \bigcup_{(A,B) \in \mathcal{F}} A \times B,\ |\mathcal{F}| = k\}. \tag{7}$$

The *Boolean rank*, $\mathrm{r_B}(C)$, of an $n \times m$ Boolean matrix C is the least integer k such that Boolean $m \times k$ and $k \times n$ matrices with $C = A \circ B$ exist (see [1]). In [1] it is implicitly shown that:

$$\mathrm{rc}(\mathbb{K}) = \mathrm{r_B}(\mathcal{A}_I). \tag{8}$$

[1] The condition to be crossed is not necessary for Identity 5.

Lastly, we recall some aspects of dimension theory. For a concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$, its 2-*dimension*, $\dim_2(\underline{\mathfrak{B}}(\mathbb{K}))$, is the smallest number of chains of cardinality 2 in whose direct product it can be order-embedded. Since the n-fold direct product of chains of cardinality 2 is isomorphic to the powerset lattice of the n-element set $\underline{n}$, there exists $\varphi : \underline{\mathfrak{B}}(\mathbb{K}) \to \mathfrak{P}(\underline{n})$ with $(A, B) \leq (C, D) \iff \varphi(A, B) \leq \varphi(C, D)$.

A *Ferrers relation* is a relation $F \subseteq G \times M$ such that $(g, m), (h, n) \in F$ implies $(g, n) \in F$ or $(h, m) \in F$. This is equivalent to $\underline{\mathfrak{B}}(G, M, F)$ being a chain. The *length* l of F is defined as $l(F) = |\mathfrak{B}(G, M, F)| - 1$. For a context $\mathbb{K}$ its *Ferrers 2-dimension*, $\mathrm{fdim}_2(\mathbb{K})$, is the smallest number of Ferrers relations F_t, $t \in T$ with $l(F_t) < 2$, so that $I = \bigcap_{t \in T} F_t$.

The above defined dimensions are equal and are related to the rectangle cover number via the complementary context, that is:

$$\mathrm{rc}(\mathbb{K}) = \mathrm{fdim}_2(\mathbb{K}^c) = \dim_2(\underline{\mathfrak{B}}(\mathbb{K}^c)). \tag{9}$$

3 The Rectangle Cover Number of the Direct Product of Formal Contexts

In this section, we will treat the rectangle cover number of the direct product of two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$. From Identity 3, it follows that $\mathrm{rc}(\mathbb{K}_1 \check{\times} \mathbb{K}_2) \leq \mathrm{rc}(\mathbb{K}_1) + \mathrm{rc}(\mathbb{K}_2)$. We will provide a sufficient condition for equality. Therefore, we will need a proposition about the Ferrers 2-dimension of the direct sum of two contexts. This proposition and its use in Theorem 1 is inspired by [14].

Proposition 1. *For the direct sum of two contexts* $\mathbb{K}_1 = (G_1, M_1, I_1)$ *and* $\mathbb{K}_2 = (G_2, M_2, I_2)$*, it holds that* $\mathrm{fdim}_2(\mathbb{K}_1 \oplus \mathbb{K}_2) = \mathrm{fdim}_2(\mathbb{K}_1) + \mathrm{fdim}_2(\mathbb{K}_2)$.

Proof. The claim follows from Identity 9 with interchanged roles of $\mathbb{K}$ and $\mathbb{K}^c$, and the structure of the relation $(I_1 \oplus I_2)^c$ depicted in Figure 1.

Fig. 1. The relation $I_1 \oplus I_2$ and $(I_1 \oplus I_2)^c$ of the direct sum of $\mathbb{K}_1$ and $\mathbb{K}_2$.

$I_1 \oplus I_2$	M_1	M_2
G_1	I_1	$G_1 \times M_2$
G_2	$G_2 \times M_1$	I_2

$(I_1 \oplus I_2)^c$	M_1	M_2
G_1	I_1^c	$\emptyset$
G_2	$\emptyset$	I_2^c

Theorem 1. *Let* $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *be co-crossed contexts. For the rectangle cover number of their direct product it holds that:*

$$\mathrm{rc}(\mathbb{K}_1 \check{\times} \mathbb{K}_2) = \mathrm{rc}(\mathbb{K}_1) + \mathrm{rc}(\mathbb{K}_2).$$

Proof. First, we notice that $\mathbb{K}_1^c$ and $\mathbb{K}_2^c$ are crossed contexts. From Proposition 1, and Identity 2, 4, 5 and 9, we conclude that

$$\begin{aligned}
\mathrm{rc}(\mathbb{K}_1 \check{\times} \mathbb{K}_2) &= \mathrm{fdim}_2((\mathbb{K}_1 \check{\times} \mathbb{K}_2)^c) \\
&= \mathrm{fdim}_2(\mathbb{K}_1^c \hat{\times} \mathbb{K}_2^c) \\
&= \dim_2(\underline{\mathfrak{B}}(\mathbb{K}_1^c \hat{\times} \mathbb{K}_2^c)) \\
&= \dim_2(\underline{\mathfrak{B}}(\mathbb{K}_1^c) \times \underline{\mathfrak{B}}(\mathbb{K}_2^c)) \\
&= \dim_2(\underline{\mathfrak{B}}(\mathbb{K}_1^c \oplus \mathbb{K}_2^c)) \\
&= \mathrm{fdim}_2(\mathbb{K}_1^c \oplus \mathbb{K}_2^c) \\
&= \mathrm{fdim}_2(\mathbb{K}_1^c) + \mathrm{fdim}_2(\mathbb{K}_2^c) \\
&= \mathrm{rc}(\mathbb{K}_1) + \mathrm{rc}(\mathbb{K}_2).
\end{aligned}$$

Remark 1. In case that the hypothesis for both factors to be co-crossed does not hold, the simplest example to consider would be $\mathbb{I} := (\{g\}, \{m\}, \{g\} \times \{m\})$. It is crossed and for any non-empty context $\mathbb{K}$, it holds that $\mathrm{rc}(\mathbb{K} \check{\times} \mathbb{I}) = 1 < \mathrm{rc}(\mathbb{K}) + 1 = \mathrm{rc}(\mathbb{K}) + \mathrm{rc}(\mathbb{I})$.

Without providing a formal definition, we restate Theorem 1 for Boolean matrices. Since in this case the term direct product would not be appropriate, we will use the established notion *Cartesian sum* from graph theory (see [9]). Identity 8 implies:

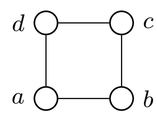
Corollary 1. *Let A_1 and A_2 be Boolean matrices with at least one empty row and one empty column. For the Boolean rank of their Cartesian sum it holds that* $\mathrm{r_B}(A_1 \check{\times} A_2) = \mathrm{r_B}(A_1) + \mathrm{r_B}(A_2)$.

4 Edge Clique Covers of Simple Graphs

In order to analyse the square cover number of tolerance spaces, we will use some results from graph theory which will be introduced in this section. A *graph* is considered as a relational structure $\mathbb{G} = (V, E)$ with *vertex set* V and an irreflexive, symmetric binary relation $E \subseteq V \times V$. Let E^{ref} denote the *reflexive closure* of E. The reflexive closure of $\mathbb{G}$ is defined as $\mathbb{G}^{\mathrm{ref}} := (V, E^{\mathrm{ref}})$. It follows that the graph $\mathbb{G}^{\mathrm{ref}}$ defines a tolerance space. On the contrary, let $\mathbb{T}$ be a tolerance space, then $\mathbb{G}_{\mathbb{T}}$ denotes the underlying graph.

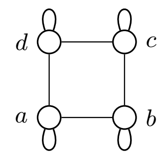
As usual, K_n denotes the *complete graph* with n vertices and $K_{m,n}$ the *complete bipartite graph* with disjoint vertex sets A and B, such that $|A| = m$ and $|B| = n$. An *n-clique* (or just *clique*) of $\mathbb{G}$ is a complete subgraph $K_n \leq \mathbb{G}$. Every clique of a graph $\mathbb{G}$ induces a clique in the reflexive closure $\mathbb{G}^{\mathrm{ref}}$ and every isolated vertex of $\mathbb{G}$ induces a 1-clique in $\mathbb{G}^{\mathrm{ref}}$. The difference between cliques and reflexive cliques is that the latter one can be identified with a formal concept and especially with a maximal square in E^{ref} in the sense of tolerance relations. Figure 2 provides an example.

Fig. 2. The reflexive closure of a 4 cycle is depicted.



	a	b	c	d
a	0	1	0	1
b	1	0	1	0
c	0	1	0	1
d	1	0	1	0

ref $\longrightarrow$



	a	b	c	d
a	1	1	0	1
b	1	1	1	0
c	0	1	1	1
d	1	0	1	1

The *edge clique cover number* of a graph $\mathbb{G}$, $\theta_e(\mathbb{G})$, is the smallest number of cliques such that their edges cover the edges of $\mathbb{G}$. For a graph with n vertices, we have that $\theta_e(\mathbb{G}) \leq \lfloor n^2/4 \rfloor$, in which equality holds for the graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ (see [13]).

$$\theta_e(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) = \lfloor n^2/4 \rfloor. \tag{10}$$

The following proposition relates θ_e to the square covering number.

Proposition 2. *Let $\mathbb{G}$ be a graph and $\mathbb{T}$ be a tolerance space. It holds that $\theta_e(\mathbb{G}) = \mathrm{sc}(\mathbb{G}^{\mathrm{ref}})$ and $\mathrm{sc}(\mathbb{T}) = \theta_e(\mathbb{G}_{\mathbb{T}})$.*

Another graph parameter related to cliques is the *vertex clique cover number*, $\theta_v(\mathbb{G})$, that is the smallest number of cliques, such that their vertices cover all vertices of $\mathbb{G}$.

Lastly, we describe the concept lattice of a graph $\mathbb{G}$ and the relationship between graph homomorphisms (edge preserving maps) and certain maps between concept lattices of graphs. In [15], concept lattices of graphs are studied under the name *neighborhood ortholattice*. It is shown that $\underline{\mathfrak{B}}(\mathbb{G})$ is a *complete ortholattice*, that is a complete bounded lattice with an involutory antiautomorphism c, such that $x \leq c(x)$ implies $x = 0$. We define an abstract *orthogonality relation* through $x \perp y :\Longleftrightarrow x \leq c(y)$.

An *orthomap* between complete ortholattices preserves order and orthogonality, and maps only the bottom element of the domain lattice to the bottom element of the codomain lattice. The next theorem relates graph homomorphisms to orthomaps.

Theorem 2 ([15]). *A graph homomorphism from $\mathbb{G}_1$ to $\mathbb{G}_2$ exists if and only if there exists an orthomap from $\underline{\mathfrak{B}}(\mathbb{G}_1)$ to $\underline{\mathfrak{B}}(\mathbb{G}_2)$.*

Furthermore, it is shown in [15] that the concept lattice of K_n is isomorphic to the powerset lattice of an n-element set: $\underline{\mathfrak{B}}(K_n) \cong \mathfrak{P}(\underline{\mathrm{n}})$.

5 The Square Cover Number of the Direct Product of Tolerance Spaces

This section treats Conjecture 1, *i.e.*, for tolerance spaces $\mathbb{T}_1$ and $\mathbb{T}_2$:

$$\mathrm{sc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) = \mathrm{sc}(\mathbb{T}_1) + \mathrm{sc}(\mathbb{T}_2).$$

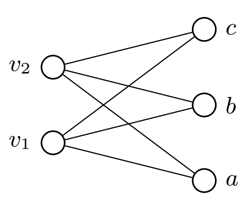
First, similar to Remark 1, we see that Conjecture 1 is false for an arbitrary tolerance space $\mathbb{T}_1$, and $\mathbb{T}_2 = (\{v\}, \{v\}, \{v\} \times \{v\})$. Second, due to Identity 3, it always holds that $\mathrm{sc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) \leq \mathrm{sc}(\mathbb{T}_1) + \mathrm{sc}(\mathbb{T}_2)$. The question for which tolerance spaces equality holds remains. An analogue to Theorem 1 can not exist, since tolerance spaces, due to their reflexivity, can not be co-crossed.

In order to be able to make use of the the rectangle cover number, we will relate the square cover number, $\mathrm{sc}(\mathbb{T})$, to the rectangle cover number, $\mathrm{rc}(\mathbb{T})$. Since every square is also a rectangle, it holds that for any tolerance space, the rectangle cover number is less or equal to the square cover number.

$$\mathrm{rc}(\mathbb{T}) \leq \mathrm{sc}(\mathbb{T}). \tag{11}$$

But, the reverse inequality is wrong in general. To see this, we notice that $\mathrm{rc}(\mathbb{T}) = \mathrm{r_B}(\mathcal{A}_\tau) \leq |V|$ (see [1]). Consequently, a tolerance space with square cover number larger than $|V|$ would provide a counter example. From Identity 10 and Proposition 2, we conclude that $6 = \mathrm{sc}(K_{2,3}^{\mathrm{ref}}) > \mathrm{rc}(K_{2,3}^{\mathrm{ref}}) = 5$ (see Fig. 3).

Fig. 3. The graph $K_{2,3}$ and the adjacency matrix of $K_{2,3}^{\mathrm{ref}}$.



	v_1	v_2	a	b	c
v_1	1	0	1	1	1
v_2	0	1	1	1	1
a	1	1	1	0	0
b	1	1	0	1	0
c	1	1	0	0	1

This motivates the following definition.

Definition 1. *We will say that a tolerance space* $\mathbb{T}$ *has the* balanced covering property *(in short* BCP*) if* $\mathrm{sc}(\mathbb{T}) = \mathrm{rc}(\mathbb{T})$*.*

This definition leads immediately to:

Theorem 3. *Let* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ *be tolerance spaces with the BCP, such that* $\mathrm{rc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) = \mathrm{rc}(\mathbb{T}_1) + \mathrm{rc}(\mathbb{T}_2)$*. It follows that* $\mathrm{sc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) = \mathrm{sc}(\mathbb{T}_1) + \mathrm{sc}(\mathbb{T}_2)$*.*

Proof. It always holds that $\mathrm{sc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) \leq \mathrm{sc}(\mathbb{T}_1) + \mathrm{sc}(\mathbb{T}_2)$. From the BCP of $\mathbb{T}_1$ and $\mathbb{T}_2$, and Inequality 11, we conclude the reverse direction

$$\mathrm{sc}(\mathbb{T}_1) + \mathrm{sc}(\mathbb{T}_2) = \mathrm{rc}(\mathbb{T}_1) + \mathrm{rc}(\mathbb{T}_2) = \mathrm{rc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2) \leq \mathrm{sc}(\mathbb{T}_1 \check{\times} \mathbb{T}_2).$$

6 Tolerance Spaces with the balanced covering property

In this section, we will provide examples of tolerance spaces which have the BCP.

Example 1. *The following is inspired by [8]. A covering $\mathcal{H} \subseteq \mathfrak{P}(V)$ of V is* irredundant *if $\mathcal{H} \backslash \{X\}$ is not a covering of V for any $X \in \mathcal{H}$. An irredundant covering induces the tolerance $\tau_{\mathcal{H}} := \bigcup\{X \times X \mid X \in \mathcal{H}\}$ with underlying tolerance space $\mathbb{T}_{\mathcal{H}} := (V, \tau_{\mathcal{H}})$. It follows that $\mathrm{sc}(\mathbb{T}_{\mathcal{H}}) \leq |\mathcal{H}|$. Since, $\mathcal{H}$ is an irredundant covering, for every $X \in \mathcal{H}$ there exists $v \in X$, such that $X \times X$ is the only maximal square which is covering (v, v). Hence, the squares $X \times X$ with $X \in \mathcal{H}$ are* mandatory *(see [1] for* mandatory factors *in the sense of factor analysis) for every covering of $\tau_{\mathcal{H}}$, which implies $\mathrm{sc}(\mathbb{T}_{\mathcal{H}}) = \mathrm{rc}(\mathbb{T}_{\mathcal{H}}) = |\mathcal{H}|$.*

Furthermore, note that tolerances induced by irredundant coverings can be considered as the reflexive closure of graphs $\mathbb{G}$ with $\theta_e(\mathbb{G}) = \theta_v(\mathbb{G})$ (see [2] Theorem 1) and that equivalence relations are a special case of such tolerances.

Next, we will describe the structure of the graphs $\mathbb{G} = (V, E)$ whose underlying relation E is the complement of a tolerance induced by an irredundant covering.

Theorem 4. *Let $\mathbb{T} = (V, \tau)$ be a tolerance space and $\mathbb{G} = (V, E)$ the graph defined through $\mathbb{G} := \mathbb{T}^c$. If the tolerance τ is induced by an irredundant covering $\mathcal{H} \subseteq \mathfrak{P}(V)$ with $|\mathcal{H}| = n$, then $\mathbb{G}$ is a connected graph with K_n as a retract*[2]*.*

Proof. In [8] it is shown that $\underline{\mathfrak{B}}(\mathbb{G})$ is an atomistic boolean lattice if τ is induced by an irredundant covering. Since we only consider finite tolerance spaces, this means that $\underline{\mathfrak{B}}(\mathbb{G})$ is isomorphic to a powerset lattice. We denote the isomorphism by Φ and show that it is an orthomap.

Since it is an isomorphism it preserves order and only the bottom element of the domain lattice is mapped to the bottom element of the codomain lattice. Consequently, just the preservation of orthogonality is left to show:

$$x \perp y \Rightarrow x \leq c(y) \Rightarrow \Phi(x) \leq \Phi(c(y)) = c(\Phi(y))^3 \Rightarrow \Phi(x) \perp \Phi(y).$$

The same holds for the inverse Φ^{-1} so that we have orthomaps $\underline{\mathfrak{B}}(\mathbb{G}) \to \mathfrak{P}(\underline{\mathrm{n}})$ and $\mathfrak{P}(\underline{\mathrm{n}}) \to \underline{\mathfrak{B}}(\mathbb{G})$. Theorem 2 implies the existence of two graph homomorphisms $\varphi_1 : \mathbb{G} \to K_n$ and $\varphi_2 : K_n \to \mathbb{G}$. Since φ_2 must be an embedding, it can be defined such that $\varphi_1 \circ \varphi_2 = \mathrm{id}_{\mathbb{G}}$ holds.

Lastly, we notice that $\mathbb{G}$ must be connected. Otherwise $\underline{\mathfrak{B}}(\mathbb{G})$ would be equal to the horizontal sum of the connected components of $\mathbb{G}$ (see Identity 6). But $\underline{\mathfrak{B}}(\mathbb{G}) \cong \mathfrak{P}(\underline{\mathrm{n}})$ implies that for $n \geq 3$ and $n = 1$, the concept lattice $\underline{\mathfrak{B}}(\mathbb{G})$ can not

[2] A graph G is a *retract* of H if there exist graph homomorphisms $\varphi : G \to H$ and $\psi : H \to G$ such that the composite $\psi \circ \varphi$ is the identity on G.

[3] The last equality is a consequence of the fact that the isomorphic image of an orthocomplemented lattice is again an orthocomplemented lattice. Just define $c(\Phi(x)) := \Phi(c(x))$. Since the powerset lattice has a unique orthocomplementation, this is the only possible choise for c.

be horizontally decomposed. For $n = 2$, the graph $\mathbb{G}$ must have two connected components such that their concept lattice is a chain. This is a contradiction, since the underlying relation of a graph can not be a Ferrers relation.

Example 2. *In this example we generalize the construction of* $(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})^{\text{ref}}$ *(see Section 4 and Figure 3). For this purpose let* $\mathbb{K}$ *be a context. We consider* $\mathbb{T} := (\mathbb{K} \dot{\cup} \mathbb{K}^d)^{\text{ref}}$*, the reflexive closure of the union of* $\mathbb{K}$ *and* $\mathbb{K}^d$*. This construction yields the reflexive closure of a bipartite graph with disjoint vertex sets* G *and* M*, such that we draw a line from* $g \in G$ *to* $m \in M$ *whenever* gIm *holds. It follows that every element of* I *induces a maximal clique in this bipartite graph and hence a maximal square in* $(I \dot{\cup} I^{-1})^{\text{ref}}$ *(Fig. 4). In [6] the concepts of* $\underline{\mathfrak{B}}(\mathbb{T})$ *are*

Fig. 4. The reflexive closure of $I \dot{\cup} I^{-1}$, where E_X denotes the identity on X .

$(I \dot{\cup} I^{-1})^{\text{ref}}$:	G	M
G	E_G	I
M	I^{-1}	E_M

characterized. Let $\{a\}, A \subseteq G$ *and* $\{b\}, B \subseteq M$*. The following types of concepts can occur. First, concepts which represent a row or column in* $(I \dot{\cup} I^{-1})^{\text{ref}}$,

$$(\{a\}, \{a\} \cup A^I),\ (\{b\}, B_I \cup \{b\}),\ (\{a\} \cup B, \{a\}),\ (A \cup \{b\}, \{b\}),$$

and second concepts from $\underline{\mathfrak{B}}(\mathbb{K})$*, that is* (A, A^I), (B, B_I)*, as well as the above mentioned squares* $(\{a\} \cup \{b\}, \{a\} \cup \{b\})$*. It follows that for* $|G| + |M| < |I|$*, we have that* $\text{rc}(\mathbb{T}) = |G| + |M| < \text{sc}(\mathbb{T}) = |I|$*. In the next step we remove elements from* I *until* $|G| + |M| = |I|$*, which gives us* $\text{rc}(\mathbb{T}) = \text{sc}(\mathbb{T}) \leq |G| + |M|$*. If* $|G| + |M| > |I|$*, it still holds that* $\text{rc}(\mathbb{T}) = \text{sc}(\mathbb{T})$.

Finally, we notice that a graph $\mathbb{G}$ *with* $\mathbb{G}^c = (\mathbb{K} \dot{\cup} \mathbb{K}^d)^{\text{ref}}$ *consists of complete graphs* $K_{|G|}$ *and* $K_{|M|}$*, such that their vertices are symmetrically connected through the context* $\mathbb{K}^c$*.*

Example 3. *A further example is the* symmetrization $\mathbb{K}^s$ *of a context* $\mathbb{K}$ *(see [7]). It is defined as* $\mathbb{K}^s := \mathbb{K} \oplus \mathbb{K}^d = (G \dot{\cup} M, G \dot{\cup} M, I \cup I^{-1} \cup G \times G \cup M \times M)$ *(Fig. 5). Every concept* (A, B) *of* $\mathbb{K}$ *induces a maximal square* $(A \cup B) \times (A \cup B)$*.*

More generally, every concept of $\mathbb{K}^s$ *has the form* $(A \cup D, B \cup C)$*, in which* (A, B) *and* (C, D) *are concepts of* $\mathbb{K}$*. Hence, a minimal rectangle cover of* $\mathbb{K}$ *induces a set of maximal squares which cover* I *and* I^{-1}*, but* $G \times G$ *and* $M \times M$ *may not be covered. It follows that* $\text{rc}(\mathbb{K}) \leq \text{rc}(\mathbb{K}^s) \leq \text{rc}(\mathbb{K}) + 2$ *and that* $\text{rc}(\mathbb{K}) \leq \text{sc}(\mathbb{K}^s) \leq \text{rc}(\mathbb{K}) + 2$*. If* $\text{sc}(\mathbb{K}^s) = \text{rc}(\mathbb{K})$*, the BCP* $\text{sc}(\mathbb{K}^s) = \text{rc}(\mathbb{K}^s)$ *follows from Inequality 11.*

Fig. 5. The symmetrization of $\mathbb{K}$.

$\mathbb{K}^s$	G	M
G	$G \times G$	I
M	I^{-1}	$M \times M$

A graph $\mathbb{G}$ *with* $\mathbb{G}^c = \mathbb{K} \oplus \mathbb{K}^d$ *consists of two empty graphs on* G *and* M*, such that their vertices are symmetrically connected via* $\mathbb{K}^c$

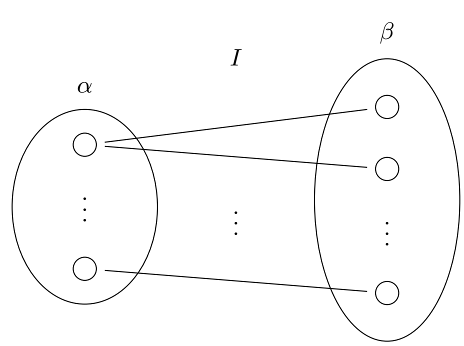
7 Construction of Tolerance Spaces

In this section, we will analyse a construction principle for tolerance spaces which is based on formal contexts. Example 2 and 3 suggest that we consider a triple $(\mathbb{A}, \mathbb{K}, \mathbb{B})$ with tolerance spaces $\mathbb{A} = (G, G, \alpha)$, $\mathbb{B} = (M, M, \beta)$ and a context $\mathbb{K} = (G, M, I)$. That triple defines the tolerance space $\mathbb{T} := (G \cup M, I \cup I^{-1} \cup \alpha \cup \beta)$ (Fig. 6).

Fig. 6. The triple $(\mathbb{A}, \mathbb{K}, \mathbb{B})$ defines the tolerance space $\mathbb{T}$.

$\tau :$	G	M
G	α	I
M	I^{-1}	β

Fig. 7. The bipartite structure of the triple $(\mathbb{A}, \mathbb{K}, \mathbb{B})$.



The interaction of α, I and β determines the structure of the tolerance space. This fact can be interpreted as a bipartite graph defined through I, such that α and β are tolerance relations on the disjoint vertex sets (Fig. 7). In the context of clique partitions of graphs, this was already observed in [4].

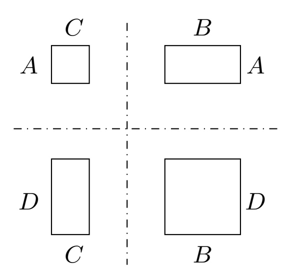
If I is the empty relation, then $\mathrm{rc}(\mathbb{T}) = \mathrm{rc}(\mathbb{A})+\mathrm{rc}(\mathbb{B})$ and $\mathrm{sc}(\mathbb{T}) = \mathrm{sc}(\mathbb{A})+\mathrm{sc}(\mathbb{B})$. For $||G| - |M|| \leq 1$ and $I = G \times M$, as well as α, β equal to the identity relation, the square cover number $\mathrm{sc}(\mathbb{T})$ is maximal. In this case, an increase of the elements of α and β can only reduce the square cover number. Generally, if I has many edges, more edges in α and β are necessary for a small square cover number, because one has to connect the edges of I into just a few maximal squares.

For arbitrary I, α and β, we state the following theorem.

Theorem 5. *Let $(\mathbb{A}, \mathbb{K}, \mathbb{B})$ be defined as above. For $A, C \subseteq G$ and $B, D \subseteq M$, let $A \times B$ and $C \times D$ be subsets of I. If $A \times C \subseteq \alpha$ and $D \times B \subseteq \beta$, then $(A \cup D) \times (B \cup C)$ is a rectangle of the underlying tolerance space. This rectangle is maximal if A, B, C and D are maximal with respect to the above stated inclusions. For $A = C$ and $B = D$ a (maximal) square is induced.*

Proof. *If $C \times D \subseteq I$, then $D \times C \subseteq I^{-1}$. The rest of the proof is graphical (Fig. 8).*

Fig. 8. The structure of maximal rectangles.



Note that in order to induce a maximal rectangle, neither $A \times B$ and $C \times D$ have to be maximal in I, nor $A \times C$ and $D \times B$ have to be maximal in α and β.

Corollary 2. *Let $\mathbb{T} = (\mathbb{A}, \mathbb{K}, \mathbb{B})$ be defined as above. A concept $(A, B) \in \mathfrak{B}(\mathbb{K})$ induces a maximal rectangle in $\mathbb{T}$ if there exists no rectangle $C \times D \subseteq I$ such that $A \times C \subseteq \alpha$ and $D \times B \subseteq \beta$. Furthermore, $(A, C) \in \mathfrak{B}(\mathbb{A})$ and $(D, B) \in \mathfrak{B}(\mathbb{B})$ induce a maximal rectangle in $\mathbb{T}$ if there exist no rectangles $A \times B \subseteq I$ and $C \times D \subseteq I$.*

8 Conclusion

This paper analysed rectangle covers of the direct product of formal contexts. If the contexts are co-crossed, then the rectangle cover number of the direct product is equal the sum of each factors' rectangle cover number.

In the next step, we treated the square cover number of the direct product of tolerance spaces. If each factor has the balanced covering property (BCP), which means that its square cover number is equal to its rectangle cover number, then additivity of the rectangle cover number with respect to the direct product transfers to the square cover number.

Lastly, we provided a variety of examples for tolerance spaces which have the BCP, analysed the corresponding graphs and introduced a construction principle for tolerance spaces based on formal contexts.

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