Binary Lattices

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Abstract. A concept lattice is said to be binary if every formal concept covers at most two other concepts and is covered by at most two. These particular lattices can be seen as a generalization of decision trees (which rely on binary yes/no decisions). A non-binary lattice is binarizable if and only if it can be embedded into a binary lattice. We show in this paper that crown-free lattices are exactly binarizable ones. We also provide an algorithm which binarizes any crown-free concept lattice by adding and modifying a minimum number of concepts.

Keywords: crown-free lattice, formal concept analysis, decision systems

1 Introduction

Binary decision systems (choosing one among two possibilities) are usually more interpretable and clearer than more complex systems (choosing one among k) and for many data structures, the binary case is the standard case. For instance, in machine learning, decision trees [3] can be defined with any number of children per nodes, but one generally uses binary decision trees.

Moreover, the use of decision trees for prediction can be seen as recursively asking whether a particular individual has or not a chosen attribute hence nodes can be seen as sets of individuals sharing some attribute(s). As formal concepts are elements of concept lattices which also represent objects with common attributes, concept lattices and decision trees are strongly linked : concept lattices can be seen as a collection of overlapping decision trees [2].

We study binary lattices associated with formal contexts. By binary, we mean a concept lattice such that each formal concept covers at most two other concepts and is covered by at most two concepts. We will focus on binarizable lattices, i.e. lattices which can be embedded into a binary lattice. This is similar to the transformation of a non-binary node in a decision tree into an equivalent sequence of binary nodes. In a concept lattice, this amounts to adding some new formal concepts and modifying some existing concepts by adding some objects and/or attributes.

We show in this paper that binarizable lattices are exactly crown-free lattices. Crown-free lattices are an interesting case of lattices as they only have a polynomial number of elements, admit strong properties and a convenient graphical representation [4], [5]. These lattices are equivalent to totally balanced hypergraphs which can be seen as a generalization of trees (they are hypergraphs with no special cycle) and can be characterized by a sequence of trees [7].

This paper is organized as follows: the next section contains basic results and definitions linked to crown-free lattices and formal concept analysis. Section 3 presents our algorithm of binarization of a crown-free set system used to prove the equivalence between crow-free and binarisable lattices. Section 4 gives an illustrative example of binarization applied to formal concept analysis. Finally, Section 5 concludes and gives some topics of future research.

2 Preliminaries

In this paper, all the sets, posets and lattices are finite.

A *poset* (partially ordered set) is a pair $(A, \leq)$ such that A is a nonempty set and $\leq$ a reflexive, antisymetric, transitive binary relation on A. $(A, \geq)$ is called the *dual* of $(A, \leq)$.

Definition 1. *A poset $(A, \leq)$ can be* embedded *into a poset $(B, \leq)$ if there exists $f : A \rightarrow B$ such that for all $A_1, A_2 \in A$, $A_1 \leq A_2$ if and only if $f(A_1) \leq f(A_2)$*

In a poset $(A, \leq)$, we note $\prec$ the *covering relation* : $\forall U, V \in A$, $U \prec V$ (V *covers* U or U *is covered* by V) if and only if $U < V$ and $\nexists X \in A$, $U < X < V$. One can then represent a poset by its Hasse diagram. On such a diagram, each element of A is represented by a node and $U, V \in A$ are linked by a segment going upward if and only if $U \prec V$.

A poset $(L, \leq)$ is a lattice if $\inf\{U, V\}$ (the largest element that is smaller than or equal to U and V, written $U \wedge V$ and also called the *meet* of U and V) and $\sup\{U, V\}$ (the smallest element that is larger than or equal to U and V, written $U \vee V$ and also called the *join* of U and V) exist for all $U, V \in L$.

In formal context analysis, a *formal context* is a triplet $K = (G, M, I)$ with G a set of objects, M a set of attributes and $I \subseteq G \times M$ a binary relation.

Definition 2. *A* formal concept *associated with a formal context $K = (G, M, I)$, is a pair (A, B) with:*

- $A \subseteq G$, $B \subseteq M$
- $\{y \in M | \forall x \in A,\ xIy\} = B$
- $\{x \in G | \forall y \in B,\ xIy\} = A$

A is called an extent *and B is called an* intent.

The *concept lattice* $\mathcal{L}_K$ associated with a formal context K is the lattice of all concepts of the formal context with $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ and $B_2 \subseteq B_1$. An example of a concept lattice is represented in Figure 1 with blue semicircles representing attributes and black semicircles representing objects and Table 1 gives the concepts of this formal context. We will call *extent*

$(\emptyset, \{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8\})$
$(\{\boldsymbol{r_2}\}, \{c_6, c_7, c_8\})$
$(\{\boldsymbol{r_1}\}, \{c_3, c_4, \boldsymbol{c_5}\})$
$(\{\boldsymbol{r_3}\}, \{c_2, c_3, c_4, c_6, c_8\})$
$(\{\boldsymbol{r_5}\}, \{\boldsymbol{c_1}, c_2, c_3\})$
$(\{r_2, \boldsymbol{r_4}\}, \{\boldsymbol{c_7}, c_8\})$
$(\{r_2, r_3\}, \{\boldsymbol{c_6}, c_8\})$
$(\{r_1, r_3\}, \{c_3, \boldsymbol{c_4}\})$
$(\{r_3, r_5\}, \{\boldsymbol{c_2}, c_3\})$
$(\{r_2, r_3, r_4\}, \{\boldsymbol{c_8}\})$
$(\{r_1, r_3, r_5\}, \{\boldsymbol{c_3}\})$
$(\{r_1, r_2, r_3, r_4, r_5\}, \emptyset)$

Table 1: Example of formal concepts (associated with Table 2)

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8
r_1			×	×	×			
r_2						×	×	×
r_3		×	×	×		×		×
r_4							×	×
r_5	×	×	×					

Table 2: Example of a cross table associated with Table 1

lattice of a concept lattice the lattice $(A, \subseteq)$ with A the set of extents of the formal concepts and *intent lattice* the lattice $(B, \subseteq)$ with B the set of intents. These two lattices are the dual of each other and the concept lattice can be seen as merging them hence working on the intent lattice or on the extent lattice is equivalent to working on the concept lattice.

The intent lattice and the extent lattice of a concept lattice are lattices whose elements are subsets of the same set. They are *set systems*.

Definition 3. *S is a* set system *on a set V if :*

- *$S \subseteq 2^V$,*
- *S is closed under intersection (i.e. $A \in S, B \in S \implies A \cap B \in S$),*
- *S has a minimum and a maximum element.*

Note that all the definitions given in this paper for lattices (embdedded, binary, crown-free, ...) can be extended to set systems as for a given set system S, $(S, \subseteq)$ is a lattice.

Formal contexts can also be represented as matrices, rows being the objects and columns the attributes they can have. Table 2 gives the matrix associated with the concept lattice of Figure 1 (with 1 replaced by × symbols and 0 replaced by a blank for readability purposes).

Moreover, every finite lattice $\mathcal{L} = (L, \leq)$ can be associated with a formal context $K_{\mathcal{L}}$ ([1], [6]) such that its concept lattice $\mathcal{L}_{K_{\mathcal{L}}}$ is equivalent to the initial lattice.

We will focus on a particular type of lattices : *crown-free* lattices (i.e. lattices with no crown).

Definition 4. *A* crown *is a poset $(X_1, X'_1, \ldots, X_n, X'_n)$ such that for all $i \geq 2$, $X_i < X'_{i-1}$, $X_i < X'_i$ and $X_1 < X'_n$, $X_1 < X'_1$ and there is no other comparability relation between these elements.*

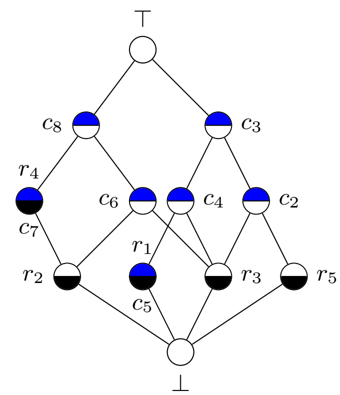


Fig. 1: Concept lattice associated with the formal context of Table 2

The Hasse diagrams of a 3-crown and of an n-crown are given in Figure 2. The concept lattice represented in Figure 1 does not contain any crown. One can remark that if $(L, \leq)$ is a 3-crown free lattice, for all $A, B, C \in L$, $A \wedge B \wedge C \in \{A \wedge B, A \wedge C, B \wedge C\}$.

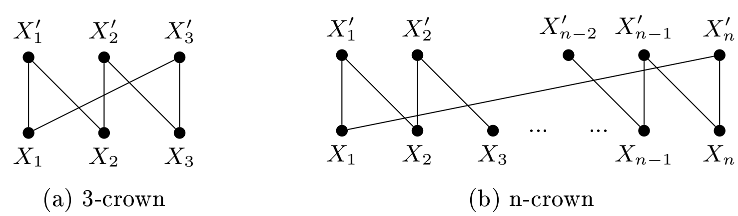


Fig. 2: Hasse diagram of a crown

3 Binary lattices and crown-free lattices

In binary decision trees, each internal node has exactly two children. More generally, binary trees are trees such that each node has at most two children. Moreover the structure of tree implies that every node has only one parent. The use of a decision tree is based on the simple idea to ask successive yes/no questions about an individual in order to classify it according to the known individuals. We extend the definition of these binary structures to lattices, taking into account that elements of a lattice can be covered by more than one element. In these structures, yes/no questions about a single attribute are replaced by a question about two different attributes. An object can then have only one of the two attributes or both of them.

Definition 5. *Let $\mathcal{L} = (L, \leq)$ be a finite lattice. $\mathcal{L}$ is said to be* binary *if*

$$\forall v \in L, v \neq \bot \implies \begin{cases} |\{u \in L \mid u \prec v\}| \leq 2 \\ |\{w \in L \mid v \prec w\}| \leq 2 \end{cases}$$

If only the first condition is respected, $\mathcal{L}$ is said to be lower-binary.

We will characterize binary lattices and those which can be embedded into a binary lattice.

Definition 6. *Let $\mathcal{L} = (L, \leq)$ be a finite lattice. $\mathcal{L}$ is* binarizable *if and only if there exists a lattice $\mathcal{L}' = (L', \leq)$ such that $\mathcal{L}$ can be embedded in $\mathcal{L}'$ and $\mathcal{L}'$ is binary.*

We will first prove that binarizable lattices are crown-free (Proposition 8). The intuition of this result can be summarized as follows : considering an element Y covering more than three elements, in an attempt to binarize the lattice, Y would cover two elements Y' and Y'' which must be the joins of the incomparable elements they cover. We show that in a crown, $Y' = Y$ or $Y'' = Y$ which is not possible. The intuition is easy to understand on a 3-crown. For example in Figure 2, it is impossible to create the union of X_1' and X_2' without X_3'.

Property 7. Let $\mathcal{L} = (L, \leq)$ be a finite lattice and $X_1, X_1', \ldots, X_n, X_n' \in L$. If $(X_1, X_1', \ldots, X_n, X_n')$ is a crown of $\mathcal{L}$ then $(X_1, X_1 \vee X_2, \ldots, X_n, X_n \vee X_1)$ is a crown of $\mathcal{L}$.

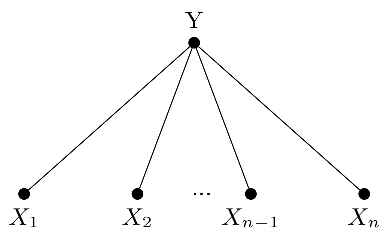
Proof. We show that $X_i \vee X_{i+1} \parallel X_j$ for all $j \neq i, i+1 \mod n$. Indeed, $X_i \parallel X_j$ and $X_{i+1} \parallel X_j$ by definition of a crown hence $X_i \vee X_{i+1} \nleq X_j$. Moreover $X_j \parallel X_i'$ by definition of a crown. Yet for all $i \leq n$, $X_i \leq X_i'$ and $X_{i+1} \leq X_i'$ hence $X_i \vee X_{i+1} \leq X_i'$. Hence $X_j \nleq X_i \vee X_{i+1}$. Hence $X_j \parallel X_i \vee X_{i+1}$. □

We will therefore only consider cycles of the form $(X_1, X_1 \vee X_2, \ldots, X_n, X_n \vee X_1)$ in the proofs.

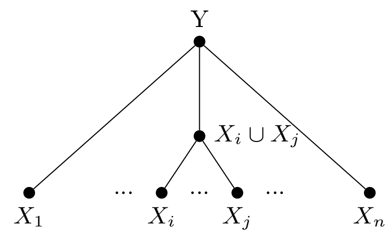
Proposition 8. *Let $\mathcal{L} = (L, \leq)$ be a finite lattice. If $\mathcal{L}$ is binarizable then $\mathcal{L}$ is crown-free.*

Proof. We first show that any binary lattice is crown-free by induction on the size of the crown. Suppose $(L, \leq)$ is a binary lattice containing a 3-crown. Let $(X_1, X_1 \vee X_2, X_2, X_2 \vee X_3, X_3, X_3 \vee X_1)$ a 3-crown and $Y = \sup(X_1, X_2, X_3) = \sup(X_1 \vee X_2, X_2 \vee X_3, X_3 \vee X_1)$. $(L, \leq)$ is binary so Y covers at most two elements Y' and Y'' and $\{X_i \leq Y' \cup Y''\} = \{X_1, X_2, X_3\}$. Y is the supremum of $X_1 \vee X_2, X_2 \vee X_3, X_3 \vee X_1$ and $Y' \prec Y, Y'' \prec Y$ hence, by the pigeonhole property, we can suppose without loss of generality that $X_2 \vee X_3 \leq Y'$ and $X_3 \vee X_1 \leq Y'$. Hence $X_1 < Y'$ and $X_2 < Y'$ so $X_1 \vee X_2 \leq Y'$ hence $Y' \geq Y$ which is a contradiction.

Suppose that any $(L, \leq)$ containing a crown of size inferior or equal to $n-1$ (with $n > 3$) is not binary. Let $(L, \leq)$ a binary lattice containing an n-crown. Let $(X_1, X_1 \vee X_2, \ldots X_n, X_n \vee X_1)$ a crown and $Y = \sup(X_1, \ldots X_n)$. $(L, \leq)$ is



(a) Initial configuration



(b) Example of first step

Fig. 3: Binarization

binary hence Y covers at most two elements Y' and Y''. $Y = \sup(X_1, \ldots, X_n)$ hence $1 \leq |\{X_i \leq Y' | i \leq n\}| < n$ and $1 \leq |\{X_i \leq Y'' \mid i \leq n\}| < n$. Y is binary hence $\{X_i \leq Y' \mid i \leq n\} \cup \{X_i \leq Y'' \mid i \leq n\} = \{X_i \mid 1 \leq i \leq n\}$ so we can suppose without loss of generality that $|\{X_i \leq Y' \mid i \leq n\}| \geq 2$. Suppose moreover that $X_1 \nleq Y'$. Let $j = \min\{i \leq n \mid X_i \leq Y'\}$ and $j' = \max\{i \leq n \mid X_i \leq Y'\}$. $(X_{j'}, X_{j'} \vee X_{j'+1 \mod n}, X_{j'+1 \mod n} \ldots, X_n, X_n \vee X_1, X_1, X_1 \vee X_2, X_2 \ldots, X_j, Y')$ is a crown of size inferior or equal to $n-1$ and superior to 3. Indeed for all $i < j$ and for all $i > j'$, $X_i \nleq Y'$. Hence the lattice has a crown of size inferior or equal to $n-1$ hence by induction hypothesis the lattice is not binary.

By definition of embedding and of a crown, for all lattice $(L, \leq)$ containing a crown, if $(L, \leq)$ can be embedded in $(L', \leq)$ by a function f, then $(X_1, X_1', \ldots X_n, X_n')$ is a crown in $(L, \leq)$ if and only if $(f(X_1), f(X_1'), \ldots, f(X_n), f(X_n'))$ is a crown in $(L', \leq)$. By the previous result, if $(L', \leq)$ has a crown then $(L', \leq)$ is not binary hence if $(L, \leq)$ has a crown, $(L, \leq)$ is not binarizable.

In this proof we only used the fact that for all Y in a binary lattice, Y covers at most two elements. If Y is covered by more than 3 elements, the same proof can be applied to the dual of the lattice in which Y covers more than 3 elements. □

We will now show that any crown-free lattice can be embedded in a binary lattice (Proposition 14). We will work on set systems associated with lattices and use Algorithm 1 in order to transform any crown-free set system into a lower-binary set system. Each non-binary element is transformed into a binary one by creating unions of some of the elements it covers. In order to keep the closure under intersection of the set system, the elements used to created the new elements have to be chosen wisely.

Definition 9. *Let $\{X_1, \ldots, X_n\}$ be a set of incomparable subsets of the same set. X_i and X_j are said to be of* maximal intersection *among $\{X_1, X_2, \ldots X_n\}$ if and only if there does not exist $k \neq i, j$ such that $X_i \cap X_j \subsetneq X_i \cap X_k$ or $X_i \cap X_j \subsetneq X_j \cap X_k$.*

One step of the process is illustrated in Figure 3.

Algorithm 1: Lower-binarization of a crown-free set system

```
Data: S a set system
Result: B(S) a set system such that every element covers at most two other
        elements and S can be embedded into B(S)
B(S) = S
for Y ∈ S do
    C = {X ∈ B(S) | X ≺ Y}
    while |C| > 2 do
        Find X_i, X_j of maximal intersection among C
        B(S) = B(S) ∪ {X_i ∪ X_j}
        C = C ∪ {X_i ∪ X_j}\{X_i, X_j}
    return B(S)
```

The process adds as few elements as possible to the set system to make it binary. Indeed, if an element Y covers k elements $X_1, \ldots, X_k$, our construction adds exactly $k-2$ elements.

The following technical lemmas will be used to prove Proposition 13. Note that all these lemmas only require the set system to have no 3-crowns and not to be crown-free.

Lemma 10. *Let L be a set system with no 3-crown and $\{X_1, \ldots, X_n\} \subset L$ a set of incomparable elements of L.*

Let X_i and X_j of maximal intersection among $\{X_1, \ldots X_n\}$.

Then,

$$\forall l \in \{1, \ldots, n\}, X_l \cap (X_i \cup X_j) = \begin{cases} X_i \cap X_l \\ or \\ X_j \cap X_l \end{cases}$$

Proof. Let X_i, X_j of maximal intersection among $\{X_1, \ldots, X_n\}$.

L has no 3-crowns, so $X_i \cap X_j \cap X_l \in \{X_i \cap X_j, X_i \cap X_l, X_j \cap X_l\}$.

X_i and X_j are of maximal intersection so $X_i \cap X_j \nsubseteq X_l \cap X_i$ and $X_i \cap X_j \nsubseteq X_l \cap X_j$. Hence $X_i \cap X_j \cap X_l \in \{X_i \cap X_l, X_j \cap X_l\}$ i.e. $X_j \cap X_l \subseteq X_i \cap X_l$ or $X_i \cap X_l \subseteq X_j \cap X_l$. Yet

$$\forall l, X_l \cap (X_i \cup X_j) = (X_i \cap X_l) \cup (X_j \cap X_l)$$

Hence

$$\forall l, X_l \cap (X_i \cup X_j) = \begin{cases} X_i \cap X_l \\ or \\ X_j \cap X_l \end{cases} \iff \begin{cases} X_j \cap X_l \subseteq X_i \cap X_l \\ or \\ X_i \cap X_l \subseteq X_j \cap X_l \end{cases}$$

□

Lemma 11. *Let L be a set system with no 3-crown and $\{X_1, \ldots, X_n\} \subset L$ the set of elements covered by $Y \in L$. Taking X_i, X_j, two elements of maximal intersection among $\{X_1, \ldots X_n\}$:*

$$\forall Z \in L \backslash \{X_1, \ldots, X_n, Y\}, Z \cap (X_i \cup X_j) \in L \cup \{X \cup Y\}.$$

Proof. If $Z \cap X_i \subseteq X_j$ then $Z \cap (X_i \cup X_j) = Z \cap X_j$. As L is closed under intersection, $Z \cap X_j \in L$. The same goes for X_j.
Suppose now $Z \cap X_i \backslash \{X_j\} \neq \emptyset$ and $Z \cap X_j \backslash \{X_i\} \neq \emptyset$. Hence $X_i \cap X_j \subseteq Z$, as L has no 3-crowns. So $X_i \cap X_j \subseteq Z \cap Y$.
If $Y \subseteq Z$, the result is obvious. If $Z \parallel Y$ or $Z \subsetneq Y$ then $Z \cap Y \subsetneq Y$ so there exists k such that $Z \cap Y \subseteq X_k$ as Y exactly covers the elements $(X_1, \ldots, X_n)$. $X_i \cap X_j \subseteq Z \cap Y \subseteq X_k$ and X_i, X_j are of maximal intersection so $k = i$ or $k = j$. Hence $Z \cap Y \subseteq X_i$ (or symmetrically $Z \cap Y \subseteq X_j$) which leads to $Z \cap Y \subseteq Z \cap X_i$ so $Z \cap Y \subseteq Z \cap (X_i \cup X_j)$.
Moreover as $X_i \cup X_j \subseteq Y$, $Z \cap (X_i \cup X_j) \subseteq Z \cap Y$, by double inclusion $Z \cap (X_i \cup X_j) = Z \cap Y$. Yet $Z \cap Y \in L$ as L is closed under intersection, which completes the proof. $\square$

Lemma 12. *Let L be a set system with no 3-crown and $\{X_1, \ldots, X_n\} \in L$ the set of elements covered by $Y \in L$. For all X_i, X_j of maximal intersection among $\{X_1, \ldots, X_n\}$, all elements of $\{X_1, \ldots X_n\} \cup \{X_i \cup X_j\} \setminus \{X_i, X_j\}$ are incomparable.*

Proof. We prove that for all $k \neq i, j$, $X_i \cup X_j \parallel X_k$.
As, for all $k \neq i, j$, X_i, X_j and X_k are incomparable, $X_i \nsubseteq X_k$ and $X_j \nsubseteq X_k$ hence $X_i \cup X_j \nsubseteq X_k$.
As L has no 3-crown, $X_i \cap X_j \cap X_k \in \{X_i \cap X_j, X_i \cap X_k, X_j \cap X_k\}$ so as X_i and X_j are of maximal intersection, $X_i \cap X_j \not\subset X_i \cap X_k$ and $X_i \cap X_j \not\subset X_j \cap X_k$. So $X_i \cap X_j \cap X_k \in \{X_i \cap X_k, X_j \cap X_k\}$. Suppose $X_i \cap X_j \cap X_k = X_i \cap X_k$. We then have $X_i \cap X_k \subseteq X_j$. Moreover, as $(X_1, \ldots X_n)$ are incomparable, there exists $x \in X_k$ such that $x \notin X_j$. $X_i \cap X_k \subseteq X_j$ hence $x \notin X_i$. So there exists $x \in X_k, x \notin X_i \cup X_j$ so $X_k \nsubseteq X_i \cup X_j$. $\square$

Proposition 13. *Algorithm 1 applied on a set system S returns a set system $\mathcal{B}(S)$ such that:*

- *for all $X \in \mathcal{B}(S)$, X covers at most two elements,*
- *$S \subseteq \mathcal{B}(S)$,*
- *for all $X \in S$, X is covered by the same number of elements in S and in $\mathcal{B}(S)$.*

Proof. By Lemma 10 and 11, creating new elements as described in the construction preserves the closure under intersection of the system. Lemma 10 shows that the intersection of any element of the considered part of the system and the new element is already in the system and Lemma 11 shows the same for other elements. Moreover, the minimum element and the maximum element are unchanged as the algorithm only adds unions of two elements hence the system is a set system. Lemma 12 shows that if X_i and X_j are chosen from the set of incomparable elements $\{X_1, \ldots, X_n\}$ covered by an element Y to create a new element, the only changes in the covering relation of the system are $X_i \cup X_j \prec Y$, $X_i \prec X_i \cup X_j$ and $X_j \prec X_i \cup X_j$. Hence, no new crown is added and the set system is still crown-free. Moreover, for $k = i, j$, X_k is not anymore covered by

Y but by $X_i \cup X_j$ which keeps unchanged the number of covering elements of X_k. The other elements are unchanged. Lemma 12 also proves that the process can be iterated if the set system still has no 3-crowns.

The algorithm ends when all elements of the initial system cover at most two elements. Moreover, the elements added by the construction cover exactly two elements by construction. □

Proposition 14. *Let $\mathcal{L} = (L, \leq)$ be a finite lattice. If $\mathcal{L}$ is crown-free then $\mathcal{L}$ is binarizable.*

Proof. Let $(A, \subseteq)$ be the extent lattice of $K_{\mathcal{L}}$ (the formal context associated with $\mathcal{L}$). Algorithm 1 can be applied on the set system A. By Proposition 13, the obtained set system $\mathcal{B}(A)$ is such that $A \subseteq \mathcal{B}(A)$ and $\mathcal{B}(A)$ is lower-binary. Moreover $\mathcal{L}$ can trivially be embedded in the lattice $\mathcal{L}' = (\mathcal{B}(A), \subseteq)$.

Let $(B, \subseteq)$ be the intent lattice of $\mathcal{L}'$. By Proposition 13, $\mathcal{B}(B)$, the result of Algorithm 1 applied on B, is binary. Finally, $\mathcal{L}'$ can be embedded in the binary lattice $\mathcal{L}'' = (\mathcal{B}(B), \subseteq)$ so $\mathcal{L}$ can also be embedded into $\mathcal{L}''$ by transitivity of the embedding. □

Propositions 8 and 14 prove the main result of this paper.

Theorem 15. *Let $(L, \leq)$ be a finite lattice. $(L, \leq)$ is binarizable if and only if $(L, \leq)$ is crown-free.*

4 Example

We will now apply our binarization algorithm on a small concept lattice. Table 3a gives a small formal context describing some animals to apply our algorithm. The formal concepts associated with this context are given in Table 3b. The representation of the Hasse diagram of the concept lattice associated with this formal context (Figure 4) gives an easy way to see non-binary formal concepts.

Here, the concept representing the *duck* is covered by three concepts and the concept representing the attribute *swim* covers three concepts. This representation also shows that the lattice is crown-free so it it binarizable and our algorithm can be applied.

A binarization of the set system associated with the extent lattice of this concept lattice gives the Hasse diagram given in Figure 5a with the new latent node (a new attribute) represented as a rectangle. We deliberately ignore the fact that the element $\perp$ is not binary as binarizing it would not increase interpretability or help using the model in machine learning. The Hasse diagram associated with a binarization of the extents and of the intents is represented in Figure 5b and is associated with the formal context of Table 4. The process creates two new formal concepts : one associated with a new object *obj* (which could be interpreted as the existence of a bird eating seeds, for example a canary) and one associated with a new attribute *att* (which suggests the existence of an attribute allowing to distinguish salmon, shark, barracuda and crocodile from frog and duck) and modifies

	scale	teeth	swim	fly	seed	feather	air
salmon	×		×				
shark		×	×				
barracuda	×	×	×				
frog			×				×
crocodile		×	×				×
eagle				×		×	×
ostrich					×	×	×
duck			×	×	×	×	×

(a) Cross-table of animals data

({∅}, {scale, teeth, swim, fly, seed, feather, air})
({*crocodile*}, {teeth, swim, air})
({*duck*}, {swim, fly, seed, feather, air})
({*barracuda*}, {scale, teeth, swim})
({*ostrich*, duck}, {*seed*, feather, air})
({*salmon*, barracuda}, {*scale*, swim})
({*eagle*, duck}, {*fly*, feather, air})
({*shark*, barracuda, crocodile}, {*teeth*, swim})
({*frog*, crocodile, duck}, {air, swim})
({eagle, ostrich, duck}, {*feather*, air})
({frog, crocodile, eagle, ostrich, duck}, {*air*})
({salmon, shark, barracuda, frog, crocodile}, {*swim*})

(b) Formal concepts of animals formal context

Table 3: Animal example formal context

	scale	teeth	**att**	swim	fly	seed	feather	air
salmon	×		×	×				
shark		×	×	×				
barracuda	×	×	×	×				
frog				×				×
crocodile		×	×	×				×
eagle					×		×	×
ostrich						×	×	×
obj					×	×	×	×
duck				×	×	×	×	×

Table 4: Binarized formal context

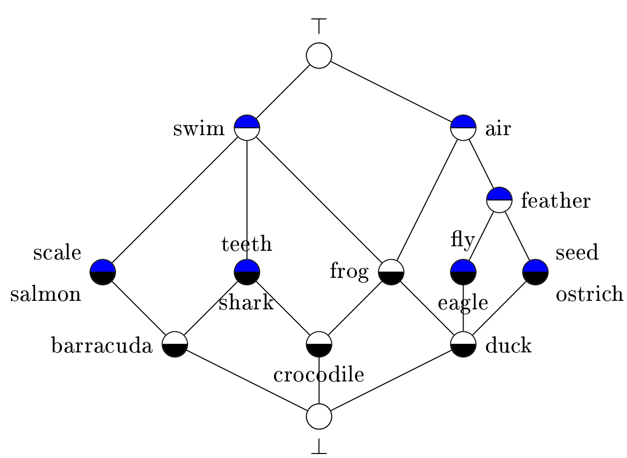
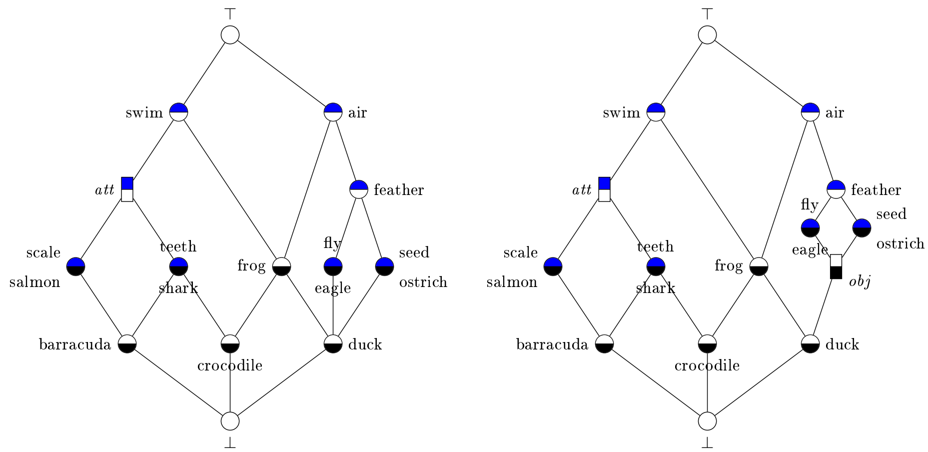


Fig. 4: Hasse diagram of the lattice associated with Table 3b



(a) Binarization of the extents

(b) Binarization of the extents and the intents

Fig. 5: Hasse diagram of binarization of the lattice associated with Table 3b

other formal concepts using this new attribute and object. The binarization process does not give a unique solution. Indeed, *att* could have been associated with a new concept $\{(shark, frog, barracuda, crocodile, duck), (teeth, att, swim)\}$ instead of $\{salmon, shark, barracuda, crocodile), (scale, att, swim)\}$ as the two intersections concerned are incomparable. This allows to make interactive systems giving the user different choices to binarize each concept. The model can then be used in the same way a decision tree is used : beginning from the top of the lattice, a new element is propagated in the nodes asking at each node whether it has the attribute of the right child or of the left child of the current node beforce classifying it as close to some known concept.

5 Conclusion

We presented a simple and efficient algorithm to transform a crown-free set system into a binary one. This construction allows us to prove the equivalence between binary lattices and crown-free ones. Our algorithm can be easily used in formal context analysis in order to modify a concept lattice to obtain a binary one, adding some objects and attributes. Moreover, our algorithm can be independently applied on the intents only or on the extents only or on both. The system being binary, it is easy to interpret and understand. The equivalence between binary lattices and crown-free ones makes of crown-free lattices a perfect candidate to extend machine learning ideas developped in decision trees to more complex systems. Indeed, it can then be used in machine learning to predict the class of a given object by propagating it in the concept lattice recursively asking whether the object has or not the attribute represented by the predecessor of the concept in the lattice. Moreover, the binarization we proposed is not unique but always adds the same number of elements to the lattice, which allows the development of interactive systems to build the model. Topics for future work include :

- a top-down process to build binary lattices inspired from decision trees,
- machine learning applications of lattice structures,
- study of the interest of intersecting classes in the machine learning perspective and the classification one.

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