DFG-Schwerpunktprogramm 1324

„Extraktion quantifizierbarer Information aus komplexen Systemen"

On Besov Regularity of Solutions to Nonlinear Elliptic Partial Differential Equations

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Preprint 85

Edited by

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Fachbereich 12 - Mathematik und Informatik
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35032 Marburg

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On Besov Regularity of Solutions to Nonlinear Elliptic Partial Differential Equations

Stephan Dahlke* & Winfried Sickel†

January 27, 2011

Abstract

In this paper, we study the regularity of the solutions to nonlinear elliptic equations. In particular, we are interested in smoothness estimates in the specific scale $B^\alpha_\tau(L_\tau)$, $\quad \tau = (\alpha/d + 1/2)^{-1}$, of Besov spaces which determines the approximation order of adaptive and other nonlinear numerical approximation schemes. We show that the Besov regularity is high enough to justify the use of adaptive schemes.

AMS Subject Classification: 41A25, 41A46, 41A65, 42C40, 65C99

Key Words: (Nonlinear) elliptic equation, regularity of solutions, Besov spaces, linear and nonlinear approximation methods.

1 Introduction

In this paper, we study the regularity of the solutions to nonlinear elliptic partial differential equations of the form

$$
(1) \qquad \begin{aligned} -\triangle u(x) + g(x, u(x)) &= f(x) \quad \text{in} \quad \Omega \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}
$$

*The work of this author has been supported by Deutsche Forschungsgemeinschaft (DFG), grants Da360/11–1, Da360/12–1, Da360/13–1.

†The work of this author has been supported by Deutsche Forschungsgemeinschaft (DFG), grant Si 487/14-1

on a bounded Lipschitz domain Ω contained in $\mathbb{R}^d$, $d \geq 3$. We are mainly interested in estimates of the regularity of solutions u in the specific scale $B^\alpha_\tau(L_\tau)$, $\tau = (\alpha/d + 1/2)^{-1}$, of Besov spaces. Here our aim is to obtain estimates with α as large as possible, accepting that τ becomes less than 1 (a few remarks concerning the used function spaces will be given below). This is partly motivated by some concrete applications which we will explain now in a somewhat simplified situation. For this purpose we turn for a moment to the Poisson equation

$$\begin{aligned} -\triangle u(x) &= f(x) \quad \text{in} \quad \Omega \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{2}$$

Here we would like to recall the famous $H^{3/2}$ Theorem of Jerison and Kenig [17].

Theorem 1. *(i) Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $d \geq 3$. For every $f \in L_2(\Omega)$ there exists a unique solution u of* (2) *s.t. $u \in H^{3/2}(\Omega)$. (Thm. B in* [17]*).* *(ii) There exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$, and an infinitely differentiable function f on $\overline{\Omega}$ s.t. $u \notin H^s(\Omega)$, $s > 3/2$. (Thm. 1.2 (a) in* [17]*).*

Now we comment on some of the consequences concerning approximability of u. Recall, for two quasi-Banach spaces X, Y and a linear continuous operator $T:\ X \to Y$ the n-th approximation number of T is defined as

$$a_n(T) := a_n(T, X, Y) := \inf \Big\{ \|\, T - L\, | X \to Y \| \ :\quad L \in \mathcal{L}(X,Y), \operatorname{rank} L < n \Big\}, \tag{3}$$

$n \in \mathbb{N}$, where, as usual, $\mathcal{L}(X,Y)$ denotes the collection of all linear and continuous operators mapping X into Y. Let id : $H^s(\Omega) \to L_2(\Omega)$, $s > 0$, be the identity operator. For the situation we are interested in it is known since a long time that

$$a_n(\mathrm{id}) \asymp n^{-s/d}\,, \qquad n \in \mathbb{N}\,.$$

The $H^{3/2}$ Theorem implies that the optimal rate of convergence by using linear methods of approximation is just $3/(2d)$ as long as we do not impose further properties of Ω.

On the other hand, it is well–known that the approximation order of best n-term wavelet approximation, which is a nonlinear approximation method, is determined by the smoothness of the object one wants to approximate in the specific scale $B^\alpha_\tau(L_\tau)$ of Besov spaces. More precisely, let $\mathcal{B}^* = \{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ be a suitable wavelet basis and let $\mathcal{M}_n$ denote the nonlinear manifold of all functions

$$S = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda, \qquad |\Lambda| \leq n\,, \tag{4}$$

where $|\Lambda|$ denotes the cardinality of the set Λ. Moreover, let

$$\sigma_n(u)_{L_2} := \inf_{S\in\mathcal{M}_n} \|u - S\|_{L_2} \tag{5}$$

then

$$\sum_{n=1}^{\infty}[n^{\alpha/d}\sigma_n(u)_{L_2}]^{\tau}\frac{1}{n} < \infty \quad \iff \quad u \in B^{\alpha}_{\tau}(L_\tau), \quad \tau = (\alpha/d + 1/2)^{-1}. \tag{6}$$

For details, the reader is referred to [12, 13] and the references therein. Similar results hold for nonlinear approximation in L_p and also for approximation schemes with respect to Sobolev norms, see, e.g., [10]. Therefore, to determine the approximation power of adaptive schemes and to justify their use, the Besov regularity of the unknown solution of the operator equation under consideration in the scale $\tau = (\alpha/d + 1/2)^{-1}$ has to be determined. In recent years, many results in this direction have been established, see, e.g., [6, 8, 11]. Concerning the Poisson equation (2) the following is proved in [8], see also [10] and Corollary 1 below. For $f \in L_2(\Omega)$ the unique solution u belongs to $B^{\beta}_1(L_1(\Omega))$ for any $\beta < 2$. Observe that $B^{\alpha}_1(L_1(\Omega))$ does not belong to the specific scale of Besov spaces. But combining this regularity result with the $H^{3/2}$ Theorem an interpolation argument yields that

$$u \in B^{\alpha}_{\tau}(L_\tau(\Omega))\,, \qquad \tau < 2\,, \qquad \frac{1}{\tau} < \frac{2+d}{2(d-1)}, \qquad \tau = (\alpha/d + 1/2)^{-1}\,.$$

Applying (6) we obtain

$$\sigma_n(u)_{L_2} \lesssim n^{-\alpha/d}\,, \qquad n \in \mathbb{N}\,, \qquad \alpha < \frac{3}{2}\frac{d}{d-1}\,.$$

Because of $3/(2d) < 3d/(2d-2)$ (see (3)) it makes clear that nonlinear methods of approximation can be much better than linear methods of approximation.

It will be our aim to investigate this phenomenon for the nonlinear equation (1) instead of the Poisson equation (2). We would like to emphasize that the essential novelty of this paper is twofold: in contrast to previous studies, we are concerned with regularity spaces corresponding to $\tau < 1$, i.e., we deal with quasi-Banach spaces. Moreover, we study nonlinear elliptic problems in general Lipschitz domains, whereas usually at least some smoothness of the boundary is required.

Let L_0 denote the solution operator associated to (2). Then is well-known, that

$$a_n(L_0, L_2(\Omega), L_2(\Omega)) \asymp n^{-2/d}\,, \qquad n \in \mathbb{N}\,.$$

But those linear operators of rank $\leq n$, which are known to be optimal (up to a general constant), require to much pre-calculations to be of practical importance. E.g., in [9], we discussed some examples of those operators which have used the exact solution of the Poisson problem of the first n elements of an appropriate wavelet system for Ω.

The paper is organized as follows. In Section 2, we briefly recall the Besov and Sobolev smoothness results for the solutions to (2) as far as they are needed for our purposes. Then, in Section 3, we discuss the fixed point theorems we want to exploit. In Sections 4 and 5, these fixed point theorems are used to derive the desired regularity results in Sobolev and Besov spaces, respectively. Finally, in Section 6, we discuss the consequences of our analysis for concrete numerical applications.

Remark 1. For reader's convenience, in this paper we always confine the discussion to problems in space dimensions $d \geq 3$. The case $d = 2$ can be studied analogously, but under a bit different restrictions. We refer again to [17] for details.

A few words to the function spaces

We will be very brief here. In this paper, we understand Lipschitz domain in the sense of Stein' s notion of domains with minimal smooth boundary, cf. Stein [25, VI.3]. For Besov spaces $B^s_q(L_q(\Omega))$ and Bessel potential spaces $H^s_p(\Omega)$ we refer to the monographs [29, 30]. There one can find the definitions and several equivalent characterizations (e.g., characterizations by differences). A lot of material can be found also in the papers [28] and [9].

As usual, $H^s(\Omega) := H^s_2(\Omega)$ in the sense of equivalent norms.

2 Sobolev and Besov Regularity of the Solution to the Poisson Equation

First we investigate additional regularity properties of the solution of the Poisson equation (2). The smoothness of the right-hand side is always measured in a Bessel potential space $H^{t-1}_p(\Omega)$. Besov regularity in this context means, that we are looking for membership of the solution u in Besov spaces $B^s_\tau(L_\tau(\Omega))$ with $\tau < p$.

2.1 Sobolev Regularity

Here we recall the fundamental result of Jerison and Kenig [17]. First we need to introduce some notation. To each Ω we associate a real number $\mu := \mu(\Omega) \in (0, 1]$ and an open hexagon H_μ, see Fig. 1 below, given by

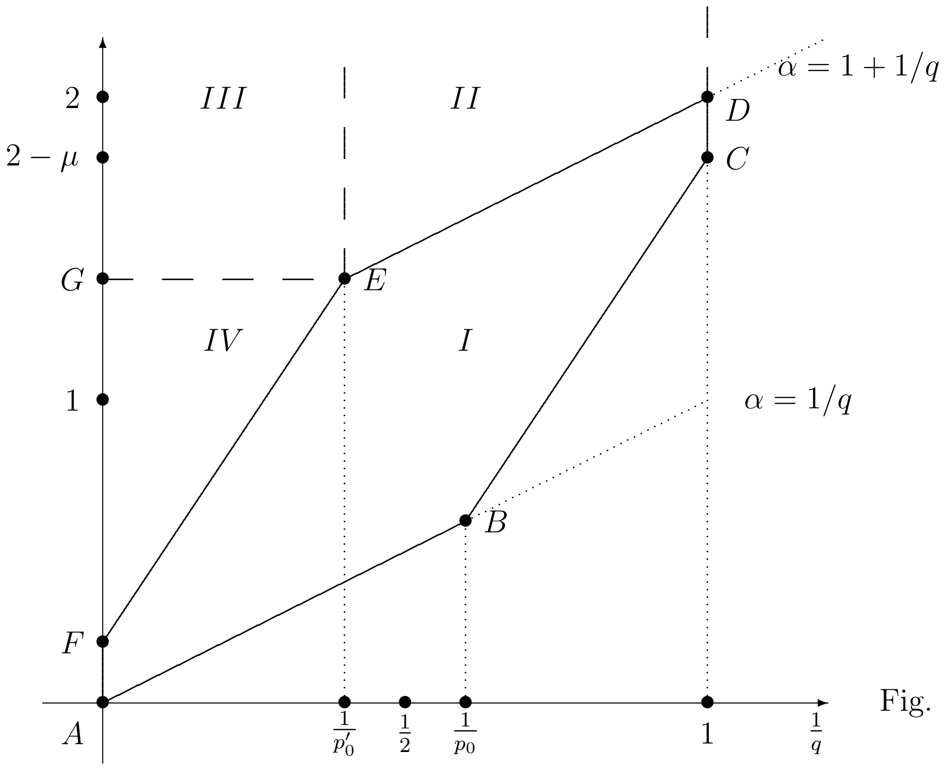


Fig. 1

the following collection of points $ABCDEF$:

$$A := (0,0), \qquad B := (1/p_0, 1/p_0), \qquad C := (1, 2-\mu),$$
$$D := (1,2), \qquad E := (1/p_0', 1 + 1/p_0'), \qquad F := (0, \mu)\,.$$

The value of $p_0 := p_0(\Omega)$ is fixed by

$$\frac{1}{p_0} := \frac{1}{2} + \frac{\mu}{2}\,. \tag{7}$$

Alternatively H_μ can be defined by the following set of inequalities: $(1/p, t+1) \in H_\mu$ if one of the following holds:

(a) $p_0 < p < p_0'$ and $\frac{1}{p} < t + 1 < 1 + 1/p$;

(b) $\quad 1 < p \le p_0$ and $\frac{3}{p} - 1 - \mu < t+1 < 1 + \frac{1}{p}$;

(c) $\quad p_0' \le p < \infty$ and $\frac{1}{p} < t+1 < \frac{3}{p} + \mu$.

Jerison and Kenig discussed the regularity of the solution of the Poisson problem for the region $I = H_\mu$. They concentrated on investigations of the Sobolev regularity of the solution, i.e., if $f \in H^{t-1}_p(\Omega)$, then they asked for $u \in H^s_p(\Omega)$ with s as large as possible. By tr we denote the trace operator with respect to the boundary of Ω. In such a generality the definition of the trace needs some care. Here we follow [18] and [27], see also [17]. First we associate to $u \in H^{t+1}_p(\Omega)$ an extension $\mathcal{E}u \in H^{t+1}_p(\mathbb{R}^d)$ and afterwards we take the restriction of $\mathcal{E}u$ to the boundary $\partial\Omega$. The technical details of this procedure, even in a more general context, are explained, e.g., in [18, pp. 205-209], [27, 9.1] or [30, 5.1.1]. Observe in this context that for Lipschitz domains Ω the boundary $\partial\Omega$ is a so-called d^*-set with $d^* = d-1$. Then Thm. 1.1 from [17] reads as follows.

Proposition 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \ge 3$. There exists μ, $0 < \mu \le 1$, depending only on the Lipschitz character of Ω such that for every $f \in H^{t-1}_p(\Omega)$ there is a unique solution*

$$u \in \mathcal{H}^{t+1}_p(\Omega) := \left\{ v \in H^{t+1}_p(\Omega) : \quad \operatorname{tr} v = 0 \right\}, \tag{8}$$

to the Poisson problem (2) provided the pair $(t+1, 1/p)$ belongs to the open hexagon H_μ. Moreover, the estimate

$$\| \, u \, | H^{t+1}_p(\Omega) \| \le c \, \| \, f \, | H^{t-1}_p(\Omega) \|$$

holds with c independent of f.

Remark 2. (i) If Ω is a C^1 domain, then p_0 may be chosen to be 1, see [17, Thm. 1.1]. We refer also to [17] for a discussion of the optimality of the restrictions in Prop. 1.
(ii) It is easily seen that the hexagon H_μ is a subset of the strip

$$\{(1/p, s) : \quad 1 < p < \infty, \; 1/p < s < 1 + 1/p\} \, .$$

Under these restrictions the set $\mathcal{H}^{t+1}_p(\Omega)$ is nothing but the closure of the test functions in the norm of the space $H^{t+1}_p(\Omega)$, see, e.g., [17, Prop. 3.3] or [27, Prop. 19.5]. For smooth domains we refer to [15] and [30, Thm. 5.21].
(iii) Detailed regularity investigations for solutions of elliptic equations in polyhedral domains have been undertaken in the recent monograph [19] by Maz'ya and

Rossmann. However, in general they measure the regularity in weighted Sobolev spaces. A first attempt to understand the consequences of such a type of regularity for nonlinear approximation has been undertaken in [11]. There we have delat with the situation on cones. An extension to polyhedral domains would be strongly desirable.

2.2 Besov Regularity

In [8] the authors investigated the Besov regularity of the solutions of the Poisson equation (2) motivated by the connection of this type of regularity to the power of nonlinear approximation schemes. We recall these results in a form used previously in [10]. We shall only consider the region II, i.e., $(1/p, t+1) \in II$, see Figure 1. This situation has been investigated in [8, Thm. 4.1].

Lemma 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let H_μ denote the associated hexagon and p_0 the specific number occurring in the definition of H_μ. Let $\varepsilon > 0$. Let $1 < p < p_0'$ and let $t \geq 1/p$. Then the solution u of the Poisson problem* (2) *with right-hand side $f \in H_p^{t-1}(\Omega)$ belongs to all spaces $B_\tau^{\alpha-\varepsilon}(L_\tau(\Omega))$, where*

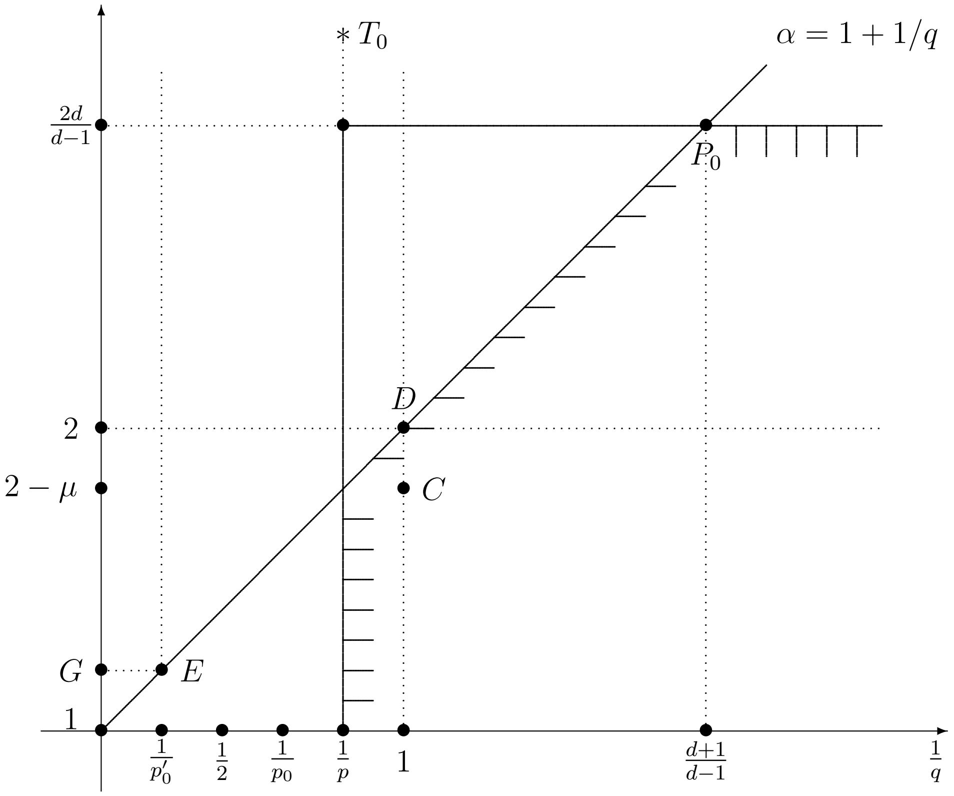
$$\begin{aligned}(1/\tau, \alpha) \quad &\in \quad \Big(\Big\{ (1/q, \beta) : \ \beta \leq \min(t+1, 1+1/q) \, , \quad \frac{d-1}{d+1} < q \leq p \Big\} \\ &\cup \quad \Big\{ (1/q, \beta) : \ \beta \leq \min\Big(t+1, \frac{2d}{d-1}\Big) \, , \quad 0 < q \leq \frac{d-1}{d+1} \Big\} \Big) \, .\end{aligned}$$

We shall try to make this more transparent by using two further figures. Let $T_0 := (1/p, t+1)$, where $1 < p < p_0'$ and $t > (d+1)/(d-1)$. Then our solution u belongs to all spaces $B_\tau^\alpha(L_\tau(\Omega))$ such that $(1/\tau, \alpha)$ is contained in the intersection of the (infinite open) rectangle with corners in $(1/p, 0)$ and $(1/p, (2d)/(d-1))$ with the area $\alpha < 1 + 1/q$.

If the point T_0 would be shifted along the line $(1/p, t+1)$ to higher values of t, the regularity assertions for u would not change. Also a shift to the left, i.e., an increasing of p, does not help. Most important for us will be the fact that $f \in H_p^{t-1}(\Omega)$ implies

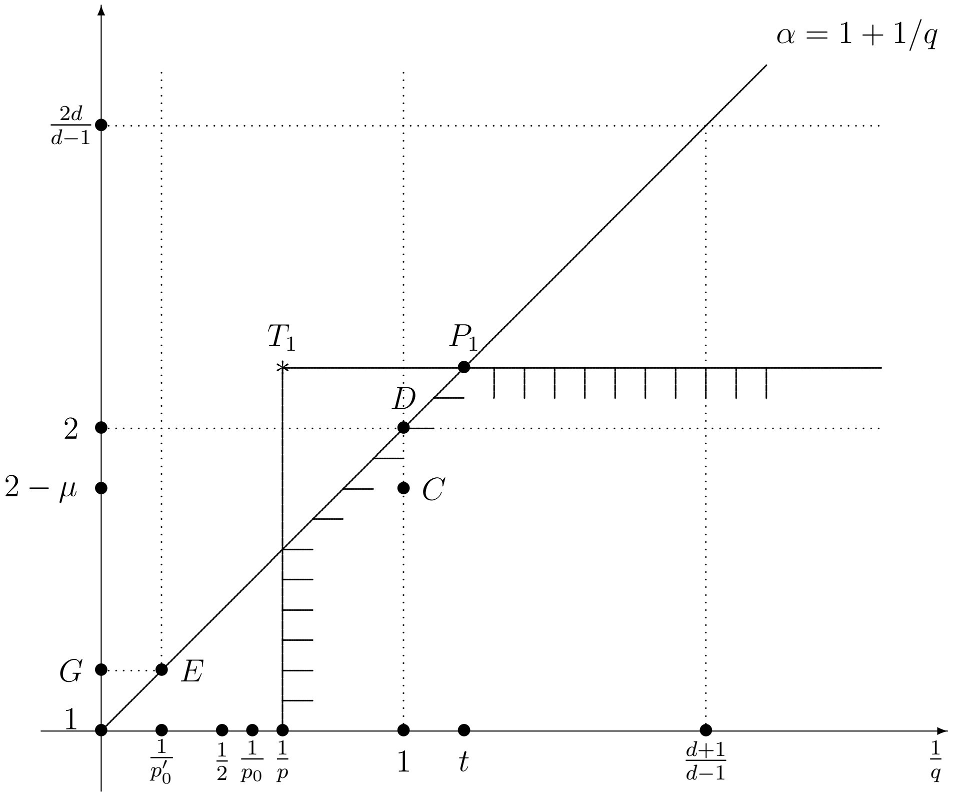
$$u \in B_\tau^\alpha(L_\tau(\Omega)) \, , \qquad \tau := \frac{d-1}{d+1} \, , \quad \alpha < 1 + 1/\tau = \frac{2d}{d-1} \, , \tag{9}$$

see the point P_0 in our Fig. 2.



$*\,T_0$
$\alpha = 1 + 1/q$
$\frac{2d}{d-1}$
P_0
D
2
$2-\mu$
C
G
E
1
$\frac{1}{p_0'}$
$\frac{1}{2}$
$\frac{1}{p_0}$
$\frac{1}{p}$
1
$\frac{d+1}{d-1}$
$\frac{1}{q}$

Fig. 2



$\alpha = 1 + 1/q$
$\frac{2d}{d-1}$
T_1
P_1
D
2
$2-\mu$
C
G
E
1
$\frac{1}{p_0'}$
$\frac{1}{2}$
$\frac{1}{p_0}$
$\frac{1}{p}$
1
t
$\frac{d+1}{d-1}$
$\frac{1}{q}$

Fig. 3

Now we turn to a second situation. Let $T_1 := (1/p, t+1)$, where $1 < p < p_0'$ and $1/p < t < (d+1)/(d-1)$. Then our solution u belongs to all spaces $B^\alpha_\tau(L_\tau(\Omega))$ such that $(1/\tau, \alpha)$ belongs to the intersection of the (infinite open) rectangle with corners in $(1/p, 0)$ and $(1/p, t+1)$ with the area $\alpha < 1 + 1/q$. If the point T_1 would be shifted along the line $(1/p, t+1)$ to larger values of p, the regularity assertions for u would not change. But this time a shift parallel to the t-axis helps, of course. Most important for us will be the fact that $f \in H^{t-1}_p(\Omega)$ implies

$$u \in B^\alpha_\tau(L_\tau(\Omega)), \qquad \tau := \frac{1}{t}, \quad \alpha < 1 + 1/\tau = 1 + t, \tag{10}$$

see the point P_1 in our Fig. 3.

We summarize our findings by concentrating on the limiting situation (which corresponds to the points P_0 and P_1, respectively). We put

$$\mathcal{B}^\alpha_\tau(L_\tau(\Omega)) := \Big\{ u \in B^\alpha_\tau(L_\tau(\Omega)) : \quad \operatorname{tr} u = 0 \Big\}, \tag{11}$$

compare with (8).

Corollary 1. *Let $1 < p < \infty$ and $\varepsilon > 0$. Let L_0 be the solution operator of the Poisson problem, i.e., $L_0 f = u$.*
(i) *Let $t = 1$. Then we have*

$$L_0 \in \mathcal{L}\Big(L_p(\Omega), \mathcal{B}^{2-\varepsilon}_1(L_1(\Omega))\Big).$$

(ii) *Let $1 < t < (d+1)/(d-1)$. Then we have*

$$L_0 \in \mathcal{L}\Big(H^{t-1}_p(\Omega), \mathcal{B}^{1+t-\varepsilon}_{1/t}(L_{1/t}(\Omega))\Big).$$

(iii) *Let $t \geq (d+1)/(d-1)$. Then we have*

$$L_0 \in \mathcal{L}\Big(H^{t-1}_p(\Omega), \mathcal{B}^{\frac{2d}{d-1}-\varepsilon}_\tau(L_\tau(\Omega))\Big), \qquad \tau = \frac{d-1}{d+1}.$$

3 Nonlinear Equations in Quasi-Banach Spaces

3.1 Fixed Points of Nonlinear Operators and Quasi-Banach Spaces

As announced we want to work with quasi-Banach spaces. Those spaces are not locally convex in general. Hence, many methods, well-known in the framework of

Banach spaces, can not be applied. Here we recall an abstract result, valid in the framework of admissible quasi-Banach spaces, see [20, Thm. 6.3.1].

Definition 1. *A quasi-normed space A is said to be* admissible*, if for every compact subset $K \subset A$ and for every $\varepsilon > 0$ there exists a continuous map $T : K \to A$ such that $T(K)$ is contained in a finite-dimensional subset of A and $x \in K$ implies*

$$\| Tx - x|A \| \leq \varepsilon .$$

Concerning the function spaces under consideration here we have the following.

Proposition 2. *Let Ω be a bounded Lipschitz domain.*
(i) *Let $1 < p < \infty$ and $t \in \mathbb{R}$. Then the spaces $H^t_p(\Omega)$ are admissible.*
(ii) *Let $1 < p, q \leq \infty$ and $t \in \mathbb{R}$. Then the spaces $B^t_q(L_p(\Omega))$ are admissible.*
(iii) *Let $1 < p < \infty$, $1 \leq q \leq \infty$ and $t > 1/p$. Then the spaces $\mathcal{B}^t_q(L_p(\Omega))$ are admissible.*
(iv) *Let $1 < p < \infty$ and $t > 1/p$. Then the spaces $\mathcal{H}^t_p(\Omega)$ are admissible.*

Proof. Step 1. The proof of (i) and (ii), given in [20, Thm. 6.2.3./4] for smooth domains, carries over to the present situation by taking into account the existence of a linear and continuous extension operator $\mathcal{E}$ such that

$$\mathcal{E} \in \mathcal{L}\Big(H^t_p(\Omega), H^t_p(\mathbb{R}^d)\Big) \cap \mathcal{L}\Big(B^t_q(L_p(\Omega)), B^t_q(L_p(\mathbb{R}^d))\Big) .$$

For this we refer to [26] and [21].
Step 2. We shall use the results of [18, Thm. 8.1.2]. It follows that the trace is a well-defined bounded linear operator s.t.

$$\mathrm{tr} \, : \, B^t_q(L_p(\Omega)) \to B^{t-1/p}_q(L_p(\partial\Omega))$$

and there exists a linear and bounded extension operator

$$\mathcal{E}_{\partial\Omega} : \, B^{t-1/p}_q(L_p(\partial\Omega)) \to B^t_q(L_p(\Omega))$$

satisfying $\mathrm{tr} \circ \mathcal{E}_{\partial\Omega} = \mathrm{id}$. Now we make use of Step 1. For a given compact set $K \subset \mathcal{B}^t_q(L_p(\Omega)) \subset B^t_q(L_p(\Omega))$ and a given $\varepsilon > 0$ let T denote the continuous map from Def. 1. We define

$$\mathcal{T} := T - \mathcal{E}_{\partial\Omega} \circ \mathrm{tr} \circ T - \mathcal{E}_{\partial\Omega} \circ \mathrm{tr} \circ \mathrm{id}$$

Then, for $f \in \mathcal{B}^t_q(L_p(\Omega))$, we find

$$\mathcal{T} f = Tf - \mathcal{E}_{\partial\Omega} \circ \mathrm{tr}\,(Tf) - \mathcal{E}_{\partial\Omega} \circ \mathrm{tr}\, f$$

and

$$\mathrm{tr}\, \mathcal{T} f = \mathrm{tr}\,(Tf) - \mathrm{tr}\,(Tf) - \mathrm{tr}\, f = 0\,,$$

i.e., $\mathcal{T}$ maps $\mathcal{B}^t_q(L_p(\Omega))$ into $\mathcal{B}^t_q(L_p(\Omega))$. In addition we obtain

$$\begin{aligned} \| \mathcal{T} f - f \|_{B^t_q(L_p(\Omega))} &\leq \| Tf - f \|_{B^t_q(L_p(\Omega))} + \| \mathcal{E}_{\partial\Omega} \circ \mathrm{tr}\,(Tf - f) \| \\ &\leq \varepsilon\,(1 + \|\mathcal{E}_{\partial\Omega}\|\, \|\mathrm{tr}\,\|)\,. \end{aligned}$$

This proves the admissibility of $\mathcal{B}^t_q(L_p(\Omega))$.

Step 3. By using [18, Thm. 7.1]. the arguments are the same as in Step 2. □

Remark 3. For t not a natural number part (iii) of Prop. 2 extends to $p = 1$. This time we have to use Thm. 6.1, Thm. 6.1.2, Thm. 6.2.3 in [18].

Let X and Y are admissible quasi-Banach spaces. Furthermore we assume that $L : Y \to X$ is a linear and continuous operator, and $N : X \to Y$ is (in general) a nonlinear map. We are looking for a fix point of the problem

$$u = (L \circ N)u. \tag{12}$$

Proposition 3. *Let X, Y, L, and N be as above. Suppose, that there exist $\eta \geq 0$, $\vartheta \geq 0$ and $\delta \geq 0$ such that*

$$\| Nu|Y \| \leq \eta + \vartheta\, \| u|X \|^{\delta} \tag{13}$$

holds for all $u \in X$. Furthermore we assume that the mapping $L \circ N : X \to X$ is completely continuous. Then there exists at least one solution $u \in X$ of (12) *provided one of the following conditions is satisfied:*

$$\begin{array}{lll} \text{(a)} & \delta \in [0,1), \\ \text{(b)} & \delta = 1,\ \vartheta < \| L \|^{-1}, \\ \text{(c)} & \delta > 1 \ \ \textit{and}\ \ \eta\, \| L \| < \Big[\dfrac{1}{\vartheta\, \| L \|}\Big]^{\frac{1}{\delta-1}} \Big[(\dfrac{1}{\delta})^{\frac{1}{\delta-1}} - (\dfrac{1}{\delta})^{\frac{\delta}{\delta-1}} \Big]. \end{array} \tag{14}$$

Remark 4. (i) A proof of this proposition, based on the topological degree of a mapping (Leray–Schauder principle), is given in [20, Thm. 6.3.1]. It generalizes an earlier result of Fučík [16, Theorem 7.3], who used Schauder's fixed point theorem (which is enough for Banach spaces).

(ii) As already stated in the introduction, in this paper we confine the discussion to the case $\delta \leq 1$. The case $\delta > 1$ will be studied in a forthcoming paper.

3.2 Estimates of Compositions of Functions - I

Concerning g we suppose that it satisfies the usual Caratheodory condition with respect Ω:
(i) For all $\xi \in \mathbb{R}$ the function $x \mapsto g(x,\xi)$ is Lebesgue measurable on Ω.
(ii) For almost all $x \in \Omega$ the function $\xi \mapsto g(x,\xi)$ is continuous on $\mathbb{R}$.
Very often this set of conditions is abbreviated as $g \in \mathrm{Car}\,(\Omega \times \mathbb{R})$.
For given f the nonlinear mapping N is defined to be

$$Nu(x) := f(x) - g(x, u(x)), \qquad x \in \Omega\,. \tag{15}$$

The investigation of composition operators and associated estimates in the framework of Bessel potential and Besov spaces is an active field of research. We refer to the monograph [20, Chapt. 5] and the recent survey [2] for the state of the art. The more specific problem $X := B^\alpha_\tau(L_\tau(\Omega))$ and $Y := W^m_p(\Omega)$ with $0 < \tau$, $p < \infty$, $m \in \mathbb{N}_0$ and $\alpha > 0$, has been investigated in [22, 23, 24], but for the more simple operator $Nu(x) := f(x) - g(u(x))$, $x \in \Omega$. Hence, we have to adapt some arguments. The letter I will be reserved for identity operators (embedding operators). With $\|\, I\,|\mathcal{L}(U,V)\|$ we denote the operator norm if I is considered as a mapping of the quasi-Banach space U into the quasi-Banach space V. Let $g \in \mathrm{Car}(\Omega \times \mathbb{R})$ satisfy the growth condition

$$|g(x,\xi)| \le a + b\,|\xi|^\delta\,, \qquad a,b,\delta \ge 0, \quad x \in \Omega, \quad \xi \in \mathbb{R}\,. \tag{16}$$

Then

$$\begin{aligned} \|\, Nu\,\|_p &\le \|\, f\,\|_p + a\,|\Omega|^{1/p} + b\,\Big(\int_\Omega |u(x)|^{\delta p}\,dx\Big)^{1/p} \\ &\le \eta + b\,\|\, u\,\|^\delta_{\delta p} \\ &\le \eta + b\,\|\, I\,|\mathcal{L}(X, L_{\delta p}(\Omega))\|^\delta\,\|\, u\,\|^\delta_X\,, \end{aligned} \tag{17}$$

as long as we have the continuous embedding

$$X = B^\alpha_\tau(L_\tau(\Omega)) \hookrightarrow L_{\delta p}(\Omega)\,.$$

To guarantee this embedding our parameters have to satisfy the inequality

$$\alpha > d\,\max\,\Big(0, \frac{1}{\tau} - \frac{1}{\max(1,\delta\,p)}\Big)\,, \tag{18}$$

see, e.g., [29, 1.11.1].

Lemma 2. *Let $0 < \tau \le \infty$, $1 < p < \infty$ and $f \in L_p(\Omega)$. Let $g \in Car(\Omega \times \mathbb{R})$ satisfy the growth condition* (16) *for some $a, b \ge 0$ and some $0 < \delta \le 1$. If* (18) *is satisfied, then N, as defined in* (15)*, is a continuous and bounded mapping s.t.*

$$\| Nu \|_{L_p(\Omega)} \le \eta + \vartheta \| u \|^{\delta}_{B^{\alpha}_{\tau}(L_\tau(\Omega))} \tag{19}$$

where

$$\eta := \| f \|_p + a\, |\Omega|^{1/p} \tag{20}$$

and

$$\vartheta := b \, \| I \, | \mathcal{L}(B^{\alpha}_{\tau}(L_\tau(\Omega)), L_{\delta p}(\Omega)) \|^{\delta} \, . \tag{21}$$

Proof. Boundedness of N under the given restrictions in (18) has been shown above. It suffices to comment on the continuity. The operator $u \mapsto g(\,\cdot\,, u(\,\cdot\,))$ is continuous considered as mapping of $L_{\delta p}(\Omega)$ into $L_p(\Omega)$ in case $\delta p \ge 1$, see [1, 3.4, 3.7]. By the same reference, in case $\delta p < 1$ this operator is continuous considered as mapping of $L_1(\Omega)$ into $L_{1/\delta}(\Omega)$. Since $L_{1/\delta}(\Omega) \hookrightarrow L_p(\Omega)$ we have continuity also as a mapping of $L_1(\Omega)$ into $L_p(\Omega)$. The restriction (18) implies the continuity of the embedding $B^{\alpha}_{\tau}(L_\tau(\Omega))$ into $L_1(\Omega)$. The proof is complete. □

3.3 Estimates of Compositions of Functions - II

Now we consider estimates of compositions with respect to norms in Bessel potential spaces $H^t_p(\Omega)$, $1 < p < \infty$ and $0 < t < 1$. We consider uniformly continuous functions $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$|g(x,\xi)| \le a + b|\xi|^{\delta} \tag{22}$$

$$|g(x,\xi) - g(x,\eta)| \le c_1 \, |\xi - \eta|^{\delta} \tag{23}$$

$$|g(x,\xi) - g(y,\xi)| \le c_2 \, |x - y|^{\delta} \tag{24}$$

with some $0 < \delta \le 1$. In the proof of the next lemma we shall use also Lizorkin-Triebel spaces $F^s_{p,q}(\Omega)$ on domains, we refer to [29] for a discussion of these spaces. By ω_d we denote the volume of the unit ball in $\mathbb{R}$.

Lemma 3. *Let g satisfy the conditions* (22)-(24) *for some $a, b \ge 0$ and some $0 < \delta \le 1$. Let*

$$d \max\Big(0, \frac{1}{p} - \delta, \frac{1}{2} - \delta\Big) < t < \delta \tag{25}$$

and

$$\alpha > \frac{t}{\delta} + d\,\max\Big(0, \frac{1}{\tau} - \frac{1}{p\,\delta}\Big)\,. \tag{26}$$

Then, for any $f \in H^t_p(\Omega)$*, the operator* N *is a bounded and continuous mapping, defined on* $B^\alpha_\tau(L_\tau(\Omega))$ *and with values in* $H^t_p(\Omega)$*. In particular we have*

$$\| \, Nu \, \|_{H^t_p(\Omega)} \le \eta^* + \vartheta^* \, \| \, u \, \|^\delta_{B^\alpha_\tau(L_\tau(\Omega))}\,, \tag{27}$$

where

$$\eta^* \quad := \quad \eta + c_2\, \omega_d\, |\Omega|^{1/p}\, \frac{1}{\sqrt{2(\delta - t)}} \tag{28}$$

and

$$\vartheta^* \quad := \quad \vartheta + c_1\, \omega_d^{1-\delta}\, \| \, I\, | \mathcal{L}(B^\alpha_\tau(L_\tau(\Omega)), F^{t/\delta}_{p\delta,q}(\Omega)) \|^\delta\,. \tag{29}$$

Proof. Step 1. Boundedness of N. We shall use the following characterization of $H^t_p(\Omega)$ by means of differences. Let $x \in \Omega$ and $0 < s < \infty$. We put

$$V_\Omega(x,s) := \Big\{ h \in \mathbb{R}^d :\ |h| < s,\ x + \tau h \in \Omega \text{ for all } 0 \le \tau \le 1 \Big\}$$

and

$$\| \, u \, \|_{H^t_p(\Omega)} := \| \, u \, \|_{L_p(\Omega)} + \Big(\int_\Omega \Big(\int_0^1 s^{-2t} \Big[s^{-d} \int_{V_\Omega(x,s)} |\Delta^1_h u(x)|\, dh \Big]^2 \frac{ds}{s} \Big)^{p/2} dx \Big)^{1/p}\,.$$

If $0 < t < 1$, the finiteness of $\| \, u \, \|_{H^t_p(\Omega)}$ characterizes $H^t_p(\Omega)$, see [29, Thm. 4.10]. Of course, $\| \, g(x, u(x)) \, \|_{L_p(\Omega)}$ can be estimated as in Lemma 2. To estimate the second expression in $\| \cdot \|_{H^t_p(\Omega)}$ we first observe

$$\begin{aligned}
&|g(x+h, u(x+h)) - g(x, u(x))| \\
\le\ & |g(x+h, u(x+h)) - g(x+h, u(x))| + |g(x+h, u(x)) - g(x, u(x))| \\
\le\ & c_1\, |u(x+h) - u(x)|^\delta + c_2\, |h|^\delta\,.
\end{aligned}$$

It is elementary to derive

$$\Big(\int_\Omega \Big(\int_0^1 s^{-2t} \Big[s^{-d} \int_{V_\Omega(x,s)} |h|^\delta\, dh \Big]^2 \frac{ds}{s} \Big)^{p/2} dx \Big)^{1/p} \le \omega_d\, |\Omega|^{1/p}\, \frac{1}{\sqrt{2(\delta - t)}}\,,$$

if $\delta > t$. Let $q := 2\,\delta$. Furthermore, by means of Hölder's inequality applied with $1 = \delta + (1-\delta)$, we have

$$\begin{aligned}
\Big(\int_\Omega & \Big(\int_0^1 s^{-2t}\,\Big[s^{-d}\int_{V_\Omega(x,s)} |\Delta_h^1 u(x)|^\delta\, dh\Big]^2 \frac{ds}{s}\Big)^{p/2} dx\Big)^{1/p} \\
& \le\ \omega_d^{1-\delta}\Big(\int_\Omega\Big(\int_0^1 s^{-2t}\,\Big[s^{-d}\int_{V_\Omega(x,s)} |\Delta_h^1 u(x)|\, dh\Big]^{2\delta} \frac{ds}{s}\Big)^{p/2} dx\Big)^{1/p} \\
& =\ \omega_d^{1-\delta}\Big(\int_\Omega\Big(\int_0^1 s^{-tq/\delta}\,\Big[s^{-d}\int_{V_\Omega(x,s)} |\Delta_h^1 u(x)|\, dh\Big]^{q} \frac{ds}{s}\Big)^{(p\,\delta)/q} dx\Big)^{\delta/(p\,\delta)} \\
& \le\ \omega_d^{1-\delta} \|\, u\,\|^\delta_{F^{t/\delta}_{p\delta,q}(\Omega)}\,,
\end{aligned}$$

where $F^{t/\delta}_{p\delta,q}(\Omega)$ denotes a Lizorkin-Triebel space, at least under some additional restrictions, see [29, Thm. 4.10]. These restrictions are summarized in (25). Observe further, that (25) and (26) imply the inequality $\alpha > d(\frac{1}{\tau}-1)$. Finally, the continuous embedding

$$B^\alpha_\tau(L_\tau(\Omega)) \hookrightarrow F^{t/\delta}_{p\delta,q}(\Omega)$$

is guaranteed by (26). Hence, with η and ϑ as in (19), we obtain

$$\begin{aligned}
\|\, g(x,u(x))\,\|_{H^t_p(\Omega)} & \le\ \eta + c_2\,\omega_d\,|\Omega|^{1/p}\,\frac{1}{\sqrt{2(\delta-t)}} \\
& \quad +\ \vartheta\,\|\, u\,\|^\delta_{B^\alpha_\tau(L_\tau(\Omega))} + c_1\,\omega_d^{1-\delta}\|\, u\,\|^\delta_{F^{t/\delta}_{p\delta,q}(\Omega)} \\
& \le\ \eta^* + \vartheta^*\,\|\, u\,\|^\delta_{B^\alpha_\tau(L_\tau(\Omega))}\,,
\end{aligned} \tag{30}$$

where η^* and ϑ^* are as in (28) and (29).

Step 2. Continuity of N. Since (25) and (26) imply (18) our mapping $N:\ B^\alpha_\tau(L_\tau(\Omega)) \to L_p(\Omega)$ is continuous, see Lemma 2. Let $0 < \theta < 1$. Then

$$[H^t_p(\Omega), L_p(\Omega)]_\theta = H^{t(1-\theta)}_p(\Omega)$$

see [29, 1.11.8], where $[\,\cdot\,,\,\cdot\,]_\theta$ denotes the complex method of interpolation. We are going to employ the associated interpolation inequality

$$\|\, u\,|H^{t(1-\theta)}_p(\Omega)\| \le \|\, u\,|H^t_p(\Omega)\|^{1-\theta}\,\|\, u\,|L_p(\Omega)\|^\theta\,, \qquad u \in H^t_p(\Omega)\,.$$

Replacing u by $Nu_1 - Nu_2$ and using (30), we find

$$\begin{aligned}
\|\, Nu_1 \ - \ Nu_2\,\|_{H^{t(1-\theta)}_p(\Omega)} & \le \|\, Nu_1 - Nu_2\,|H^t_p(\Omega)\|^{1-\theta}\,\|\, Nu_1 - Nu_2\,|L_p(\Omega)\|^\theta \\
& \le\ \Big(2\,\eta^* + \vartheta^*(\|\, u_1\,\|^\delta_{B^\alpha_\tau(L_\tau(\Omega))} + \|\, u_2\,\|^\delta_{B^\alpha_\tau(L_\tau(\Omega))})\Big)^{1-\theta}\,\|\, Nu_1 - Nu_2\,|L_p(\Omega)\|^\theta\,.
\end{aligned}$$

If u_1 is approaching u_2, then the second factor on the right-hand side tends to zero whereas the first one remains bounded.

This proves the continuity of $N:\ B^\alpha_\tau(L_\tau(\Omega)) \to H^{t(1-\theta)}_p(\Omega)$. Since we may choose θ close to 0 we have derived continuity of $N:\ B^\alpha_\tau(L_\tau(\Omega)) \to H^{t-\varepsilon}_p(\Omega)$, where $\varepsilon > 0$. Finally we notice that, for given α, the inequalities (25) and (26) are strict with respect to t. Hence, we may replace $t-\varepsilon$ by t. $\square$

4 Sobolev Regularity of the Solution of (1)

For later use we need an extension of the regularity results of Jerison and Kenig to the nonlinear situation. Of course, these investigations depend on the growth and the regularity of g.

This paper represents a first approach to those regularity investigations in Besov spaces with a small integrability parameter. It is expected that, by using more sophisticated methods in combination with specific examples, see, e.g. [14], [16] and [31], our estimates can be improved.

4.1 Problem (1) in Lebesgue Spaces

In this first subsection we study right-hand sides f belonging to $L_p(\Omega)$.

According to Prop. 3 we split our considerations into the cases $0 < \delta < 1$ and $\delta = 1$. For given p, t and $\varepsilon > 0$ we define q by

$$\frac{1}{q} := \min\Big(1, \frac{1}{p} + \frac{1}{pd} - \frac{t}{d}\Big) - \varepsilon\,. \tag{31}$$

Theorem 2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with associated hexagon H_μ. Let $d \ge 3$ and $1 < p < p_0'$. Further we suppose $(1/p, t+1) \in H_\mu$. Let $g \in Car(\Omega \times \mathbb{R})$ satisfy the growth condition (16) for some $a, b \ge 0$ and some $0 < \delta < 1$. Then for any $f \in H^{t-1}_p(\Omega)$ the nonlinear problem (1) has at least one solution $u \in \mathcal{H}^\beta_p(\Omega)$, $\beta < t+1$, if*

$$t+1 > d \lim_{\varepsilon \downarrow 0} \Big(\frac{1}{p} - \frac{1}{\max(1, \delta q)}\Big)\,. \tag{32}$$

Proof. As above, L_0 denotes the solution operator of the (linear) Poisson equation. ¿From Proposition 1 we know

$$L_0 \in \mathcal{L}\Big(H^{t-1}_p(\Omega), \mathcal{H}^{t+1}_p(\Omega)\Big)\,.$$

Now we use the compactness of the embedding

$$I : \mathcal{H}_p^{t+1}(\Omega) \to \mathcal{H}_p^{\beta}(\Omega)\,, \qquad \beta < t+1\,,$$

see Proposition 4.6 in [29]. Hence $L := I \circ L_0$ is a compact operator mapping $H_p^{t-1}(\Omega)$ into $\mathcal{H}_p^{\beta}(\Omega)$, $\beta < t+1$. Observe, that for the specific value of q defined above and ε sufficiently small we know $L_q(\Omega) \hookrightarrow H_p^{t-1}(\Omega)$. This implies

$$\| Nu \|_{H_p^{t-1}(\Omega)} \le \| I \,|\mathcal{L}(L_q(\Omega), H_p^{t-1}(\Omega))\| \, \| Nu \|_{L_q(\Omega)} \le \eta + \vartheta \, \| u \|^{\delta}_{H_p^{\beta}(\Omega)} \tag{33}$$

if

$$\beta > d \, \max\Big(0, \frac{1}{p} - \frac{1}{\max(1,\delta q)}\Big)\,,$$

see Lemma 2. Notice, that this condition guarantees

$$\Big(B_p^{\beta}(L_p(\Omega)) \cup H_p^{\beta}(\Omega)\Big) \hookrightarrow L_q(\Omega)\,.$$

The operator $N : H_p^{\beta}(\Omega) \to L_q(\Omega)$ is also continuous, see Lemma 2. Hence,

$$L \circ N : \, \mathcal{H}_p^{\beta}(\Omega) \to \mathcal{H}_p^{\beta}(\Omega)$$

is completely continuous for all β. Since $\delta < 1$ we can apply Proposition 3. Furthermore, the spaces $\mathcal{H}_p^{\beta}(\Omega)$ are admissible if $\beta > 1/p$. This yields the existence of a fix point $u \in \mathcal{H}_p^{\beta}(\Omega)$ of the mapping $L \circ N$ if β satisfies the inequalities

$$\max\Big[\Big(\frac{1}{p}, d\,\Big(\frac{1}{p} - \frac{1}{\max(1,\delta q)}\Big)\Big] < \beta < t+1\,.$$

We already know, that $t+1 > 1/p$, see the definition of H_μ. Now the existence of some β, satisfying these inequalities, are guaranteed by (32). Now the claim follows by the monotonicity of the scale $\mathcal{H}_p^{\beta}(\Omega)$ with respect to β. □

Theorem 3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ with associated hexagon H_μ. Let $d \ge 3$ and $1 < p < p_0'$. Further we suppose $(1/p, t+1) \in H_\mu$. Let $g \in Car(\Omega \times \mathbb{R})$ satisfy the growth condition (16) for some $a, b \ge 0$ and $\delta = 1$. If $b = b(\Omega)$ is sufficiently small, then for any $f \in H_p^{t-1}(\Omega)$ the nonlinear problem (1) has at least one solution $u \in \mathcal{H}_p^{\beta}(\Omega)$, $\beta < t+1$.*

Proof. We argue as in the previous theorem. First, observe that the additional restriction (32) disappears in case $\delta = 1$. Further, one has to notice that ϑ becomes small if b becomes small, see (33) and the proof of Theorem 5. □

5 Besov Regularity of the Solution of (1)

We shall discuss two situations: (a) $t = 1$ and (b) $1 < t < 1 + \delta$.

5.1 The Problem (1) in Lebesgue Spaces

In this first subsection we study right-hand sides f belonging to $L_p(\Omega)$. According to Prop. 3 we split our considerations into the cases $0 < \delta < 1$ and $\delta = 1$.

Theorem 4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $d \geq 3$ and $1 < p < \infty$. Let $g \in Car(\Omega \times \mathbb{R})$ satisfy the growth condition* (16) *for some $a, b \geq 0$ and some $0 < \delta < 1$. Then for any $f \in L_p(\Omega)$ the nonlinear problem* (1) *has at least one solution $u \in \mathcal{B}_1^\alpha(L_1(\Omega))$, $\alpha < 2$.*

Proof. By L_0 we denote the solution operator of the Poisson equation. ¿From Corollary 1 we know

$$L_0 \in \mathcal{L}\Big(L_p(\Omega), \mathcal{B}_1^\alpha(L_1(\Omega))\Big)\,, \qquad \alpha < 2\,.$$

Now we use the compactness of the embedding

$$I : \mathcal{B}_1^\alpha(L_1(\Omega)) \to \mathcal{B}_1^\beta(L_1(\Omega))\,, \qquad \beta < \alpha\,,$$

see Prop. 4.6 in [29]. Hence $L := I \circ L_0$ is a compact operator mapping $L_p(\Omega)$ into $\mathcal{B}_1^\beta(L_1(\Omega))$, $\beta < 2$. The estimate (19) yields

$$\| \, Nu \, \|_p \leq \eta + \vartheta \, \| \, u \, \|^\delta_{B_1^\beta(L_1(\Omega))}\,.$$

and $N : B_1^\beta(L_1(\Omega)) \to L_p(\Omega)$ is also continuous, see Lemma 2. Consequently

$$L \circ N : \; \mathcal{B}_1^\beta(L_1(\Omega)) \to \mathcal{B}_1^\beta(L_1(\Omega))$$

is completely continuous for all β, $0 < \beta < 2$. The spaces $\mathcal{B}_1^\beta(L_1(\Omega))$ are admissible in case $\beta \in (1, 2)$. Since $\delta < 1$ we can apply Proposition 3. This yields the existence of a fixed point $u \in \mathcal{B}_1^\beta(L_1(\Omega))$ of the mapping $L \circ N$. □

Theorem 5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $d \geq 3$ and $1 < p < \infty$. Let $g \in Car(\Omega \times \mathbb{R})$ satisfy the growth condition* (16) *for some $a, b \geq 0$ and $\delta = 1$. If $b = b(\Omega)$ is sufficiently small, then we have that for any $f \in L_p(\Omega)$ the nonlinear problem* (1) *has at least one solution $u \in \mathcal{B}_1^\alpha(L_1(\Omega))$, $\alpha < 2$.*

Proof. We argue as in the previous theorem. It is sufficient to observe that

$$\vartheta = b \, \| \, I \, | \mathcal{L}(B_\tau^\alpha(L_\tau(\Omega)), L_{\delta p}(\Omega)) \|^\delta \xrightarrow[b \to 0]{} 0\,. \qquad □$$

5.2 The Problem (1) in Bessel-Potential Spaces

This is the more interesting situation.

Theorem 6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $d \geq 3$, $1 < p < \infty$ and $t > 1$. Let g be as in Lemma 3 with some $\delta < 1$. Further we assume*

$$\frac{d}{d+1} < \delta \,. \tag{34}$$

Then for any $f \in H_p^{t-1}(\Omega)$ the nonlinear problem (1) has at least one solution $u \in \mathcal{B}_v^\alpha(L_v(\Omega))$, where

$$\begin{cases} \alpha \;<\; \delta + 2\,, \qquad v := 1/(\delta+1) \qquad \textit{if} \qquad \delta + 1 \leq (d+1)/(d-1)\,; \\[2ex] \alpha \;<\; \frac{2d}{d-1}\,, \qquad v := \frac{d-1}{d+1} \qquad \textit{if} \qquad \delta + 1 > (d+1)/(d-1)\,. \end{cases} \tag{35}$$

Proof. Thanks to Thm. 4 we know there is a solution $u \in B_1^\beta(L_1(\Omega))$, $\beta < 2$. Hence we have the identity $u = (L_0 \circ N)u$ for this particular u. Using Lemma 3 we find in case $1 < q < \infty$

$$\| \, Nu \, \|_{H_q^\gamma(\Omega)} \leq \eta^* + \vartheta^* \, \| \, u \, \|^\delta_{B_1^\beta(L_1(\Omega))} \,,$$

if

$$d \, \max\Big(0, \frac{1}{q} - \delta, \frac{1}{2} - \delta\Big) < \gamma < \delta \qquad \text{and} \qquad d \, \max\Big(0, 1 - \frac{1}{q\delta}\Big) + \frac{\gamma}{\delta} < \beta \,.$$

For $q \downarrow 1$ these conditions reduce to $d(1-\delta) < \gamma < \delta$ and $\gamma/\delta < \beta$. Since (34) is equivalent to $d(1-\delta) < \delta$ there exist always such γ. We choose γ sufficiently close to δ and q sufficiently close to 1 (both is always possible by simple monotonicity arguments). Hence $N(u) \in H_q^\gamma(\Omega)$, $\gamma < \delta$. Now we apply Cor. 1 with respect to the new right-hand side $N(u)$. Then $\gamma + 1$ takes over the role of t in Cor. 1. If $\delta + 1 > (d+1)/(d-1)$, then we may have $\gamma + 1 > (d + 1/(d-1))$ and obtain $L_0(N(u)) \in B_\tau^{\frac{2d}{d-1} - \varepsilon}(L_\tau(\Omega))$, $\tau = (d-1)/(d+1)$, using Cor. 1(iii).
If $\delta + 1 \leq (d+1)/(d-1)$, then, as a consequence of Cor. 1(ii), we find

$$L_0(N(u)) \in B^{\gamma+2}_{1/(\gamma+1)}(L_{1/(\gamma+1)}(\Omega)) \hookrightarrow B_v^\alpha(L_v(\Omega))\,, \qquad \alpha < \delta + 2\,, \quad v := 1/(\delta+1)\,.$$

This proves the claim. $\square$

Remark 5. (i) Observe, that $t > 1$ implies that the point $(1/p, t+1)$ belongs to the regions $II \cup III$, see Fig. 1.

(ii) Elementary calculations yield: if $d = 3$ then, because of $\delta \le 1$, we are always in case $\delta + 1 \le (d+1)/(d-1)$; if $d \ge 4$, then (34) implies $\delta + 1 > (d+1)/(d-1)$. We prefered the formulation given above since one can relax the restriction (34). Therefore one has to choose the parameter q in the proof in dependence of δ. We omit details.
(iii) Thm. 6 could be partly improved in case we could apply the fixed point argument in spaces with $p < 1$. Therefore we would need an extension of Prop. 2(iii).

Theorem 7. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let $1 < p < \infty$ and $t > 1$. Let g be as in Lemma 3 with $\delta = 1$. If c_1, c_2 and b are sufficiently small, see (22) - (24), then for any $f \in H^{t-1}_p(\Omega)$ the nonlinear problem (1) has at least one solution $u \in \mathcal{B}^\alpha_v(L_v(\Omega))$, where*

$$\begin{cases} \alpha < 3\,, & v := 1/2 \quad \textit{if} \quad d = 3\,; \\ \alpha < \frac{2d}{d-1}\,, & v := \frac{d-1}{d+1} \quad \textit{if} \quad d \ge 4\,. \end{cases} \tag{36}$$

Proof. It is enough to give some comments. Employing Lemma 2 (with $\delta = 1$) and Lemma 3 we find that ϑ, see Proposition 3, becomes small, if b and c_1, c_2 are small. □

6 Numerical Applications

As we mentioned in the introduction our analysis was partially motivated by some concrete practical applications. In recent years, the numerical treatment of elliptic boundary value problems has become a field of increasing importance with many applications in science and engineering. Usually, even in lower dimensions, the discretization of these equations leads to systems involving up to millions of unknowns. Therefore, to increase efficiency, very often *adaptive* schemes are the method of choice. Generally speaking, an adaptive numerical scheme is an updating strategy. Based on a local *a posteriori error estimator*, the underlying grid or the underlying function space is refined only in regions where the numerical approximation is still far away from the exact solution. A lot of convincing numerical applications indicate the usefulness of this approach. Nevertheless, still the following question concerning the *theoretical foundation* of adaptive schemes arises: is it possible to determine the order of convergence of adaptive numerical schemes, and do we gain efficiency when compared with nonadaptive schemes? To answer this question, in particular recent results from the realm of adaptive *wavelet* methods have turned out to be useful. Indeed, for linear elliptic problems, it has been possible to design adaptive wavelet schemes that are guaranteed to converge with optimal order, in the sense that they asymptotically realize the approximation order of *best n-term wavelet approximation*, see [3, 4]. It has turned out that the Besov smoothness of the unknown solution u to (2) is definitely high enough to justify the use of adaptive schemes.

In recent studies, much effort has been spent to generalize the adaptive wavelet algorithms also to certain *nonlinear* equations. In [5], again an adaptive wavelet algorithms that is guaranteed to converge with optimal order has been derived. As a particular case, the scope of the analysis in [5] covers specific *semilinear* equations of the form (1). Then, to justify the use of adaptive schemes, one is again faced with the task to determine the Besov smoothness of the solution u to (1). In this paper, we provide a first answer.

It is important to note that in our setting the smoothness bounds are *not only* caused by the nonlinear terms, but just by the non-regularity of the domain Ω.

We are particularly interested in adaptive algorithms based on wavelets. The benchmark of adaptive wavelet schemes is the best n-term wavelet approximation. Therefore, to estimate the power of adaptive wavelet algorithms and to justify their use, the performance of best n-term approximation has to be studied.

Let $\mathcal{B}^* = \{\psi_\lambda\}_{\lambda\in\mathcal{J}}$ be a wavelet system on Ω that characterizes Sobolev spaces $H^r(\Omega)$ as well as Besov spaces $B^\beta_q(L_p(\Omega))$ at the same time and for a sufficiently large range of parameters β, r, p. We refer to [9] for a detailed description of these conditions and for references where appropriate wavelet systems are constructed. Let $\sigma_n(u, \mathcal{B}^*)_{H^r(\Omega)}$ denote the best n-term approximation in $H^r(\Omega)$ with respect to the wavelet system $\mathcal{B}^*$. Then following basic characterization has been shown in [7]:

$$(37)\quad \sum_{n=1}^{\infty}[n^{(\alpha-r)/d}\sigma_n(g,\mathcal{B}^*)_{H^r(\Omega)}]^\tau \frac{1}{n} < \infty \iff g \in B^\alpha_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\alpha - r}{d} + \frac{1}{2},$$

compare also with (6) for the specific case $r = 0$, i.e., best n-term approximation in $L_2(\Omega)$. Eq. (37) obviously implies

$$(38)\qquad g \in B^\alpha_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\alpha - r}{d} + \frac{1}{2} \quad \Longrightarrow \sigma_n(g,\mathcal{B}^*)_{H^r(\Omega)} = \mathcal{O}(n^{-(\alpha-r)/d}).$$

Therefore, the approximation order of best n-term approximation depends on the Besov regularity of the object one want to approximate. Nonlinear approximation schemes perform better when compared with linear approximation schemes if the Besov smoothness of the object under consideration is higher than its Sobolev regularity. In Sections 4 and 5 we have discussed both, the Sobolev and the Besov smoothness of the solutions to (1), and we have seen that, just as for the linear case, the Besov smoothness is generically higher. Therefore, to numerically approximate the solutions to (1), the use of nonlinear approximation methods, and in particular the use of adaptive numerical schemes, is completely justified. To explain these relationships in more detail, we shall discuss two special cases. The first one is concerned with right-hand sides f contained in $L_2(\Omega)$ and best n-term approximation in $L_2(\Omega)$, whereas the second one deals with smoother right–hand sides and approximation in $H^1(\Omega)$.

Theorem 8. *Let $d = 3$ and let $g \in Car(\Omega \times \mathbb{R})$ satisfy the growth condition* (16) *for some a, b and $\delta < 1$. If $f \in L_2(\Omega)$, then there is a solution u of problem* (1) *s.t.*

$$\sigma_n(u, \mathcal{B}^*)_{L_2(\Omega)} \lesssim n^{-(2/3-\epsilon)}$$

where ϵ may be chosen arbitrary small.

Proof. We have two different regularity results, namely Thm. 2 and Thm. 4. First we deal with the consequences of Thm. 2. The point $(1/p, 2)$ does not belong to H_μ. A simple monotonicity argument yields that we can apply Thm. 2 with respect to the point $(1/2, 1+\alpha)$, $\alpha < 1/2$. Hence $u \in H^\beta(\Omega)$, $\beta < 3/2$. Here we have used that (32) is satisfied for all $0 < \delta < 1$. Now we discuss the consequences of Thm. 4. We obtain $u \in B^\alpha_1(L_1(\Omega))$, $\alpha < 2$. We shall use $H^s(\Omega) = B^s_2(L_2(\Omega))$ in the sense of equivalent norms and real interpolation, see [28] or [29, Thm. 1.110]. It holds

$$\Big(H^\beta(\Omega), B^\alpha_1(L_1(\Omega))\Big)_{\Theta,\tau} = B^s_\tau(L_\tau(\Omega))\,,$$

where

$$s = (1-\Theta)\beta + \Theta\alpha \qquad \text{and} \qquad \frac{1}{\tau} = \frac{1-\Theta}{2} + \Theta\,.$$

Now we look for the intersection of the straight lines

$$g(x) := 3(x - 1/2) \qquad \text{and} \qquad h(x) := x + 1\,, \qquad x \in \mathbb{R}\,.$$

Here x has taken over the role of $1/\tau$. The function h describes the regularity of u and the function g describes the specific scale $B^s_\tau(L_\tau(\Omega))$, $\tau = (s/d + 1/2)^{-1}$ of Besov spaces we are interested in (line of Sobolev embeddings into $L_2(\Omega)$).

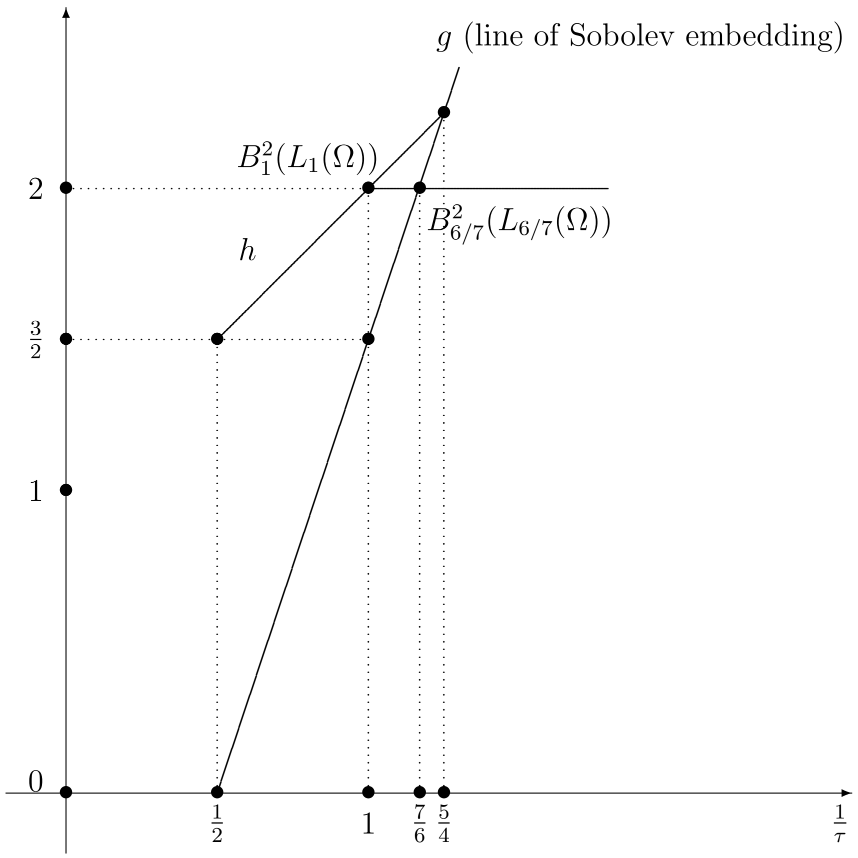


Fig. 4

The intersection of g and h takes place in the point $(5/4,\ 9/4)$. However, we do not know that u belongs to the spaces on the line h for $p < 1$. But we can use the elementary embedding

$$B^\alpha_1(L_1(\Omega)) \hookrightarrow B^\alpha_\tau(L_\tau(\Omega))\,, \qquad \tau < 1\,.$$

The line $(x, 2)$ meets g in point $(7/6, 2)$, see Fig. 4. This implies

$$u \in B^s_\tau(L_\tau(\Omega)), \qquad \frac{6}{7} < \tau < 2\,, \qquad s := 3\left(\frac{1}{\tau} - \frac{1}{2}\right).$$

An application of (38) proves our claim. □

Remark 6. (i) By Theorem 8, we see that best n-term approximation provides approximation order $\mathcal{O}(n^{-(2/3-\epsilon)})$. In contrast to this, a linear approximation scheme would only yield approximation order $\mathcal{O}(n^{-1/2})$, see (3) and Thm. 2. Therefore, best n-term approximation is indeed superior when compared with linear methods.

(ii) A typical example where the conditions of Theorem 8 are satisfied would be

$$\begin{aligned} -\triangle u(x) + |u|^\delta &= f(x) \quad \text{in} \quad \Omega, \qquad 0 < \delta < 1\,, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

As a second example, we study the case of smoother right–hand sides and $\delta = 1$. We shall in particular discuss the case of nonlinear approximation in H^1. The reason is that the adaptive wavelet algorithms usually work with the energy norm which is equivalent to the H^1-norm, see again [3] for details. For simplicity we concentrate on a special case, namely $d = 3$ and $p = 2$, the other cases can be studied analogously.

Theorem 9. *Let $p = 2, d = 3, \delta = 1, t > 1$ and $\varepsilon > 0$. Let g be as in Lemma 3 and suppose that the conditions* (22), (23) *and* (24) *are satisfied with sufficiently small constants c_1, c_2 and b. Then, for any f in $H^{t-1}(\Omega)$ there is a solution u of* (1) *s.t.*

$$\sigma_n(u, \mathcal{B}^*)_{H^1(\Omega)} \lesssim n^{-(1/4-\varepsilon)}, \qquad n \in \mathbb{N}.$$

Proof. We argue as in the previous proof. We have two different regularity results, namely Thm. 3 and Thm. 7. First we deal with the consequences of Thm. 3. It follows $u \in H^\beta(\Omega)$, $\beta < 3/2$. From Thm. 7 we derive $u \in B^\alpha_{1/2}(L_{1/2}(\Omega))$, $\alpha < 3$. Again we make use of the interpolation argument and obtain, that

$$u \in B^\alpha_\tau(L_\tau(\Omega)), \qquad \frac{1}{2} < \tau < 2, \qquad \alpha < \frac{1}{\tau} + 1.$$

Now we look for the intersection of the straight lines

$$g(x) := 3(x - 1/2) + 1 \qquad \text{and} \qquad h(x) := x + 1, \qquad x \in \mathbb{R}.$$

Again x replaces $1/\tau$. As above the function h describes the regularity of u and the function h describes the specific scale $B^s_\tau(L_\tau(\Omega))$, $\tau = ((s-1)/d + 1/2)^{-1}$ of Besov spaces which are embedded into $H^1(\Omega)$. The intersection takes place in the point $(3/4,\ 7/4)$. This yields

$$u \in B^s_\tau(L_\tau(\Omega)), \qquad \frac{4}{3} < \tau < 2, \qquad s := 1 + 3\left(\frac{1}{\tau} - \frac{1}{2}\right).$$

Now an application of (38) yields the result. □

Remark 7. Since the Sobolev regularity of u is limited by $3/2$, for nonadaptive discretization methods and $d = 3$ only the approximation order $\mathcal{O}(n^{-(1/6-\epsilon)})$ in H^1 can be expected, see (3). Therefore, compared with nonadaptive schemes, the approximation order that can be achieved by adaptive schemes is significantly higher, and therefore the use of adaptivity is again completely justified.

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