Quantum interpretations of the four color theorem *

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Abstract

Some connections between quantum computing and quantum algebra are described, and the Four Color Theorem (4CT) is interpreted within both contexts.

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1 Introduction

Since their discovery, the implications of quantum phenomena have been considered by computer scientists, physicists, biologists and mathematicians. We are here proposing a synthesis of several of these threads, in the context of the well-known mathematical problem of four-coloring planar maps.

"Quantum computing" now refers to all those proposed theories, algorithms and techniques for exploiting the unique nature of quantum events to obtain computational advantages. While it is not yet practically feasible and may, to quote Y. Manin, be only "mathematical wishful thinking" [30], quantum computing is taken seriously by a number of academic and corporate research groups. Whether or not quantum techniques will turn out to be implementable for computing and communications, their study throws new light on the nature of computation and information.

Meanwhile, within mathematics and physics, work continues in the area known as "quantum algebra" (or sometimes "quantum geometry") in which a variety of esoteric mathematical structures are interwoven with physical models like quantum field theories, algebras of operators on Hilbert space and so on. This variety of algebraic and topological structure is a consequences of the fact that any sufficiently precise logical model, which is physically veridical, will necessarily be non-commutative. An exceedingly rich structure emerges.

It is natural to wonder whether the apparatus of quantum algebra can be utilized by quantum computing, and perhaps vice versa, whether one could test mathematical hypotheses with the aid of quantum computations. There have already been approaches suggested using quantum algebra for coding and it might also provide computational architectures for quantum computers.

In view of the common ontogeny of these disciplines of quantum computing and quantum algebra, surely, a connection must exist. The trick in quantum computing is to regard bits as

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replaced by qubits in a 2-complex-dimensional Hilbert space; these spaces and their tensor products constitute the objects of quantum algebra as well. Intuitively, the algebras of diagrams in the latter field are like the dynamic representation of a calculation, carried on in complex superposition, involving the motion of individual particles, each embodying a time-varying qubit. The diagrams correspond to the trajectories of physical objects. This is our underlying point of view.

Moreover, we believe that quantum-oriented physics, mathematics and computer science can be jointly studied in regard to a particular test-case, the Four Color Theorem (4CT). We shall give some evidence below to support this claim. Having a unified focus may also simplify the knowledge coordination issue [18] which arises when so many distinct areas of mathematics and physics all seem to be mutually interrelated.

The 4CT asserts that every planar map can be 4-colored. That is, for any "nice" division of the sphere or plane into connected pieces, there is a way to color the resulting regions with four or fewer colors (i.e., labels) so that no two adjacent regions get the same color; two regions are adjacent if they share a common boundary curve.

This famous theorem, first conjectured in 1852 and proved at last in 1976, would seem to already have a solution in the sense of a proof [4], [36]. However, for reasons outlined below, the existing results are not fully satisfactory.

The Four Color Theorem appears to be a statement about the topological properties of the plane but on deeper study it reveals aspects that are of different dimensionality. While seemingly about the vicissitudes of arbitary planar arrangements, the 4CT can be reduced to a statement involving qubits and measurement. It can also be described in terms of various elementary objects such as (i) systems of parentheses, (ii) rooted cubic oriented plane trees, (iii) cobordisms of 0-dimensional manifolds, or (iv) maximal outerplane graphs (i.e., triangulated polygons) and so it has a quantum algebra interpretation. These elementary objects constitute nonassociative algebras, knot-theoretic tangles and Lie algebras which have been used to formulate equivalent versions of the 4CT [21], [22], [5].

Thus, there is strong evidence already for the centrality of the problem in both quantum algebra and quantum computing.

Some papers on quantum computing have insisted that all computation must be physically implemented. On the other hand, it has been proposed that physics should be done on topological entitites. We believe that a successful unified theory of quantum computing and quantum algebra can be based on following both of these slogans in the case of the 4CT.

An adequate theory of the elementary objects ought to *explain* the existence of 4-colorings for planar maps and enable specific calculations for the elementary objects. The difficulty of transforming ordinary planar maps into the elementary representatives is a different issue, which will not be addressed here.

The remainder of the paper is organized as follows. In Section 2, we consider topological aspects of quantum computing and show how graphs and knots are involved. Sections 3 and 4 provide connections between quantum computing and quantum algebra. The other sections are essentially independent of these two sections. In Section 3, we represent the dynamical qubits as a spatial graph or complex superposition of such graphs as in the Temperley-Lieb algebra. Section 4 defines the quantum integers and the Jones-Wenzl idempotents and describes their relationship. We also note a connection with Bose-Einstein condensates. Section 5 reviews recent progress on the 4CT, while in Section 6, physical models of this theorem are expressed as the outcome of a quantum computer simulation. In Section 7, a conjecture is made regarding the propagation of "color force" which implies the 4CT. The conjecture is also formulated in terms of quantum algebra.

2 Topological aspects of quantum computing

Quantum computing is based on the notion of the "qubit" which is a complexification of the usual notion of the bit associated with Hamming and Shannon. A *qubit*, by contrast, is an element ξ of the two-dimensional complex Hilbert space $\mathcal{C}^2$

$$\xi = \alpha_1 e_1 + \alpha_2 e_2,$$

where e_1, e_2 is the standard orthonormal basis $(1,0),(0,1)$ (often written $|+>, |->$), and $|\alpha_1|^2 + |\alpha_2|^2 = 1$. Hence, ξ is actually an element of the unit sphere S^3 in $\mathcal{C}^2$. Bits live in the 0-sphere while qubits are in the 3-sphere, S^3, which is the one-point compactification of $\mathcal{R}^3$.

However, this description leaves out the effect of the equivalence relation $\rho\xi \equiv \xi$ for any complex number ρ of norm 1 acting on the left by scalar multiplication on the vector space $\mathcal{C}^2$. The equivalence classes are the points of complex projective space $P_1(\mathcal{C})$ which is topologically S^2 (see, e.g., [37, p. 146]).

Since bits are discrete, one naturally thinks of the dynamic version as a mere sequence. However, qubits have a continuum of states and so dynamic qubits can have emergent characteristics. If $\xi = \xi(t)$ is a dynamically varying qubit ($t \in \mathcal{R}$), then it describes a curve in the 3-sphere and its quotient spaces. The curve consists of the trajectory of a point projected from a point in $\mathcal{C}^2$ moving under a time-varying unitary transformation.

Let us consider an idealized model QC of a quantum computer with a rather analogue nature. Suppose that we imagine certain qubits which are "doing the computing" through some sort of direct transformation. While these computational registers are changing state during the program, at the end they should have returned to the original state. What are the consequences of such a model?

According to the Schrődinger equation, quantum evolution is perfectly deterministic. Hence, the qubits have no singularities. The indeterminacy of quantum computing only arises when measurements are made.

We need only assume that qubits are defined for finite intervals of time; say, $[0,T]$. Call $\xi(t)$ *periodic* if $\xi(0) = \xi(T)$. Also note that relativistic considerations are being ignored.

Our model QC is a quantum computer that can be built with internal qubits that are periodic. The input is transformed into an output so the dynamic system produced by the qubits is an open braid. That is, the input-output qubits are self-avoiding trajectories. No knowledge is necessary of the input-output qubit before time 0 (input) or after T (output).

A *spatial arc (graph)* is the homeomorphic image of the interval $[0,1]$ (a finite graph, resp.) in some space. A *knot* is the homeomorphic image of S^1. This is not quite the same as a topological C_n which is a knot with n distinguished points. Up to the usual classification of ambient isotopy type, the exact parameterization doesn't matter provided that one stays within a suitable category - e.g., piecewise-linear or smooth. A *link* is a disjoint union of knots, and one may consider oriented versions.

The following lemma says that the topology of our hypothetical QC looks something like a loom: internal dynamics are described by a link while the processed trajectories are interwoven.

Lemma 1 *Dynamic nonperiodic qubits are spatial arcs; periodic qubits are knots.*

Perhaps this is related to the model, constructed by Kitaev [24] and by Freedman and Myers [9], in which qubits are associated with the edges of a spatial graph embedded within a surface. Our results certainly seem to be similar but I do not yet understand their derivations.

A dynamic qubit can be linked with itself (i.e., knotted) and this might offer interesting possibilities for cryptography. Indeed, as the Mayans apparently knew, one can encode messages using knots and links.

A static qubit carries no more information than a static bit in the sense that, when measured, it must yield either one of the two pure states (with suitable probabilities), but a time-varying qubit is a continuous trajectory and so there are infinitely many knot types with respect to ambient isotopy.

However, if we impose various geometric conditions on the qubits, then the number of knot-types seems to be finite. For example, it is well-known that the number of knot types with a fixed crossing number is bounded (though growing quite fast in the crossing number). Milnor showed that a closed curve with total curvature less than 4π cannot be knotted. Geometric conditions could reflect the analytic nature of the defining equations as well as the physics of the quantum computer implementation.

Two obvious candidates for restriction are (i) the curvature and (ii) the minimum self-avoidance distance (i.e., the "knot thickness"). It is physically apparent that such constraints do lead to bounds on the number of feasible knot types but currently little is known; see [25], [26], [32], [28], [33]. Of course, one would like to have a complete tabulation of how the number of knot types depends on various parameters.

We assume: (1) There are geometric constraints on the knots' curvature and thickness corresponding to the physical and informational properties of QC. (2) These restrictions impose an upper bound on the number of distinct knot-topologies for the internal periodic qubits. (3) QC contains at most a finite number of qubits. (4) The internal states of QC are the link topologies.

Lemma 2 *Under these assumptions, QC has only finitely many internal states.*

The identification of state with topology amounts to the criterion of analogue computation. Finiteness is hardly surprising, but it does not seem to be implied by unrestricted theories of quantum computation.

3 From quantum computing to quantum algebra

We discuss the (sparse) existing literature on relations between quantum computing and quantum algebra at the end of this section.

Quantum algebra involves braided structures and hence knots. Knots have a connection with the "topoidal" viewpoint in which mathematical structure is represented by an elementary category (the topos) where algebra, logic and geometry coexist. Thus, it is really not at all surprising to find in knot theory so many disparate areas of mathematics (from combinatorial partitions to statistical thermodynamics to higher order category theory to operator algebras ...). Also, the topos is a fundamental aspect of microphysics.

The study of knots has led to investigation of certain algebras of diagrams. Rather than considering complex superposition of elementary bit-states (qubits), which are merely formal linear combinations of $|0>, |1>$ using complex weights as described above, one can study an algebra of equivalence classes of linear combinations of "spatial graph diagrams". In the best known case of knots and links, these are the usual diagrams which are planar drawings with over and undercrossing information.

A *spatial graph* is a graph, with a specific embedding in 3-dimensional space. Graphs are, as usual, one-dimensional cellular complexes, and we assume a finite number of cells (vertices and edges). The topology is that induced by the cells and corresponds with the subspace topology if the graph is represented within 3-dimensional space. This can always be done using straight-line segments (even in the countable case). See, e.g., [14].

If the vertices are linearly ordered, one can produce an affine embedding subject to having the vertices lie on any non-planar curve in the prescribed linear order. Similarly, one can embed any graph with its vertices lying on the unit circle and with every edge assigned to its own unique level (with a vertical segment at both vertices - up, over and down). Or, in the smooth category, one would have concentric hills (i.e., stacked surfaces, with each edge lying on its own surface). If the edges are ordered, then one can extend this to the corresponding surfaces so that the first edge appears on top and the last one on the bottom, and similarly for levels. These can be thought of, metaphorically, as "pages".

In the case of knots, or spatial graphs, one may have edges, which constitute a cycle with respect to overcrossing, so they could not be realized in this "book-like" fashion. However, the argument in Bernhart and Kainen [6] can be improved to show that by subdividing any spatial graph, it is isotopic to an embedding with three pages [6]. This fact (for links) was also proved by Dynnikov [12] who has developed a link presentation based on it.

We shall show later that by a theorem of Whitney, it suffices to prove the 4CT for two-page graphs. Little is known about the book thickness of general graphs; even for cubes there is still a factor of two between best lower and upper bounds. See Bernhart and Kainen [6] for the basic properties of this invariant and [16] for references to its independent appearance within computer science.

A *spatial graph diagram* consists of the portion of a spatial graph which can be seen through a given rectangular window. Of course, links are particular examples of spatial graphs. With an arbitrarily small deformation of space, we can assume that the spatial graph diagram meets the boundary transversally and that all crossings are regular (i.e., involving pairs of interior points of edges). Adopting the usual convention, we assume that the crossings include the "over and under"information. More generally, one can consider spatial graph diagrams with respect to any (possibly bounded) surface F, which is to say an embedding in $F \times I$; see Lickorish [27, p. 134].

When the spatial graph is a disjoint union of cycles (i.e., a link), a spatial graph diagram consists of a link together with n simple curves, $n \geq 0$, each joining a pair of distinct points on the boundary of the rectangle and with all endpoints distinct; this object is called a *tangle* (see, e.g., Rolfsen [35]). The complex vector space spanned by these tangles has a one-parameter family of equivalence relations defined by regular isotopy.

More specifically, one defines the *Temperley-Lieb algebra* TL_n as the set of equivalence classes of tangles generated by two relations defined in order to enable one to reduce the number of cyclic components to zero and also to reduce the number of crossings to zero. As a result, the resulting complex vector space is spanned by the spatial graph diagrams of the underlying simple graph nK_2.

An algebra structure is given by selecting a particular point in the boundary, together with a fixed orientation of the plane, so that the $2n$ points in the boundary of the rectangle occur with half along the top and half along the bottom. Diagrams are multiplied by vertical juxtaposition of two copies of the rectangle and then rescaling back to the original size of the rectangle with the middle $2n$ points and boundaries forgotten. There is also a tensor product structure for these algebras, which is obtained by placing the rectangles side-by-side.

Let A be a complex parameter. We form a quotient algebra using the two-sided ideal generated by all elements of the forms: (i) $D \cup O - (-A^{-2} - A^2) * D$, where O denotes a null-homotopic link component (the unknot), D is any diagram and "$*$" means scalar multiplication; and (ii) $D - A * D' + A^{-1} * D''$, where D' and D'', resp., denote the result of replacing a positive crossing in D by the two new diagrams resulting from surpressing the crossing by replacing the "X" by two horizontal (or two vertical) lines, resp. (A corresponding relation for negative crossings is implied as well.)

Using these two relations, TL_n is generated by those diagrams which correspond to a planar matching on the $2n$ points; that is, to a union of n pairwise non-intersecting topological arcs which meet the boundary of the rectangle in precisely the $2n$ given points.

As a complex vector space, TL_n has dimension given by the n-th Catalan number $c_n = (n+1)^{-1}C(2n, n)$, where $C(n, k)$ denotes the binomial coefficient n choose k. This follows since there is a suitable recursion for these diagrams. Thus, the elements of the Temperley-Lieb algebras are equinumerous with with the rooted cubic oriented plane trees and many other combinatorial structures; a list of 52 examples is given in Stanley [38].

As an algebra, TL_n has a basis of only n elements rather like elementary braids. The element **1** which consists of n vertical strands acts as the neutral element under multiplication. There are additional basis elements $\mathbf{e}_i$, $1 \leq i \leq n-1$; each consists of the (equivalence class) of $i-1$ vertical strands, then a "cup/cap"pair (joining two successive points on the top to each other and the corresponding two successive points on the bottom to each other), followed by $n-i-1$ additional vertical strands. Thus, the non-identity basis elements are precisely the elementary modifications of the identity produced by doing "surgery" on a single non-crossing pair of parallel strands, replacing them by the complementary non-crossing pair. Note that these elements, whose isotopy classes span TL_n as an algebra, are all symmetric with respect to a horizontal mirror through the middle of the square. However, their products are not mirror symmetric in this way.

The Temperley-Lieb algebra is generated by these elements and their products and complex linear combinations, subject to a family of relations: $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_j\mathbf{e}_i$ when $\|i-j\| \geq 2$, $\mathbf{e}_{i+1}\mathbf{e}_i\mathbf{e}_{i+1} = \mathbf{e}_i\mathbf{e}_{i+1}\mathbf{e}_i$ and $\mathbf{e}_i^2 = [2]_q\mathbf{e}_i$, where $q = A^2$ and $[2]_q = q + q^{-1}$. See, e.g., [35].

The first reference we could find to the idea of a linkage between the areas of quantum algebra and quantum computing appears in Fivel [8] which gives a connection between the algebraic-geometry methods of classical encryption and a physical aspect of quantum computing. Fivel relates the dynamics of the EPR states used in quantum cryptography to the theta function. Kitaev suggested the use of "anyons" (a quantum-algebraic generalization of bosons and fermions) for an analog quantum computing model; see [24],[23]. Freedman and Myers [9] describe a model in which qubits are represented by topological arcs within surfaces. The idea that quantum entanglement might have a connection with knots and links was suggested by Lomonaco [29].

For general background on knots and quantum algebra, see, e.g., Jones [13], Kassel [19], Kauffman [20], Lickorish [27] and Adams[1]. For quantum computing, many tutorials are available online.

The TL_n algebra arose within statistical physics. Kauffman and Saleur [22] describe a connection with the Potts model and then obtain an equivalent and strong forms of the 4CT utilizing this algebra. In fact, originally, Appel and Haken gave heuristic arguments (see [36]), which were used to justify what was then a massive (and expensive) computation. Their theory depended on proposed statistical behavior for large configurations with respect to coloring. The theory of chromodendra (i.e., color-trees) was constructed by Whitney and Tutte [41] to refute an earlier attempt by Yamamoto and ended up giving a quantitative analysis of the probabilities; see [36, p. 78].

4 Application to integer representation

Returning to the connection with quantum computing, we investigate an alternative way to represent integers using quantum algebra.

In quantum computing, it is common to represent an integer s as an element of a tensor product $s_{d-1} \cdots s_1 s_0$, where d is the length of the binary representation of s with s_0 as the least significant bit (i.e., the rightmost) in this string; we abuse notation by letting the bit also denote the corresponding generator of $\mathcal{C}^2$. Representing integers in this way is natural from the standpoint of computer science but certainly is not the only approach from a mathematical point-of-view. In particular, analog computation, chosen to facilitate a physical implementation, might have different preferences regarding representation.

In quantum algebra, there is a rather natural choice to represent an integer - the so-called "quantum integer". Suppose $q \neq 1$ is a complex parameter and n is a positive integer. The symbol $[n]_q$ denotes the algebraic expression

$$\frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}$$

which plainly approaches n as $q \to 1$. This quantity is also called a q-deformed integer. Similarly, quantum algebra contains other q-deformed algebraic objects.

It still seems rather mysterious to me that more complex objects should have such distortions. There is a coherence which is required in order for combinatorial theorems to have complex analytic extensions. For example, the Hardy-Ramanujan theorem shows that counting partitions can be done via the roots of the 24th roots of unity in the complex plane. The elaborate structures of quantum algebra involve phenomenal cooperation across disparate mathematical domains; that such "far-flung" (in Mac Lane's words) connections should exist is truly remarkable.

When A is not a $4k$-th root of unity (for $k \leq n$), TL_n contains a special element (the *Jones-Wenzl idempotent* $I^{(n)}$) which can be characterized by algebraic properties ([27, p. 136].) This element corresponds to a a particular weighted sum of linear generators in the algebra of diagrams. We will show that I_n will be mapped to the complex number corresponding to the quantum integer $[n+1]_q$.

Define an operator Δ on TL_n by eliminating the exterior of any tangle and then adding the interior of the identity tangle; this forms a link diagram in S^2. But the spherical link diagrams constitute a 1-dimensional space, when subjected to the corresponding relations to (i) and (ii). This gives rise to a linear functional on TL_n which we still denote by Δ.

Theorem 1 *Let A be a particular complex number and let $I^{(n)}$ be the Jones-Wenzl idempotent element in the Temperley-Lieb algebra with parameter A. Then*

$$\Delta(I^{(n)} = [n+1]_q,$$

where $q = A^2$.

Thus, there is an interesting possible choice for integer representation. For example,

$$[3]_{A^2} = \mathbf{1} + (A^2 + A^{-2})^{-1} * \mathbf{e}_1,$$

where as above "+" denotes addition in the algebra and "$*''$ is scalar multiplication.

I believe that one could use unitary evolution operators to produce the required qubit topologies. However, any implementation of quantum computing via such abstract mathematical structures will require a better understanding and, in particular, some interpretation for the extra parameter A.

A physical realization for quantum algebra has been suggested by Sun, Yu and Gao [39]; they connect q-deformed bosons with phonons and Bose-Einstein condensation. The paper [39] also proposes that a nonlinear Schrődinger equation governs the phenomena they study. This would be important since it is known that quantum computation based on a nonlinear model can achieve polynomial solutions to NP-complete problems.

5 Coloring as a test problem

The four color theorem was originally proved by Appel, Haken and Koch [2],[3], [4] using a combination of traditional mathematics and computer-implemented calculations. Though their work depended on some heuristic innovations by Heesch [11], its broad outline is basically as suggested by Birkhoff [7] 60 years earlier. A set of "unavoidable configurations" is generated so that all of its members are "reducible" (see, e.g., [36] for details and references). However, the proof is huge and cannot be checked without a computer. For this reason, it was controversial [15], [17]. In our opinion, it was not the use of computer-implemented computations to check reducibility that was the difficulty, but rather that the traditional nature of the proof's outline was masked. Furthermore, and most troubling, the argument appears almost happenstance.

With time, the 4CT itself is now accepted by the mathematical community. A new proof, using the Appel-Haken scheme but with much improved computations has been constructed by Robertson, Sanders, Seymour and Thomas [34]). They do in a day on a PC the same task done 20 years earlier using thousands of hours of mainframe computing. Moreover, the new proof needs time that grows only quadratically in the number of countries.

This progress might be analogous to improved experimental techniques for estimating the Casimir effect or predictions of general relativity. If the 4CT is regarded as the detection of the fact of planar-map 4-colorability through a computer experiment, then we could expect improved performance of the experiment - utilizing evolution of hardware, software and programming skill, including the algorithms and heuristics.

But *why* is the 4CT true and *how* do you effectively color specific maps? The necessity of the four-color property as an intrinsic aspect of planarity remains unjustified by a clear and coherent argument, and it is still not feasible to color large maps. More importantly, as pointed out by Mycielski [31] and by Kauffman and Saleur [22], there is a physicality to this mathematical problem which seems to warrant investigation.

6 Physical models of the Four-Color Theorem

Since 4-coloring the regions of a map is equivalent to 4-coloring the vertices of a planar graph, it is convenient to use graph theory; see, e.g., Harary [10] and [36].

A graph is *planar* if it can be embedded in the plane so that the vertices are represented by distinct points and the edges by disjoint curves so that the curves intersect only at common endpoints. The connected components of the complement of a plane graph are called the *regions* and resulting 2-dimensional cell complex is called a *map*.

A graph is *plane* if it is planar and one has selected one particular representation. Thus, strictly speaking, planar graphs are 1-dimensional abstract complexes while plane graphs are specific subsets of the plane (or "spatial graphs" to use our term above). We assume that graphs have no loops or parallel edges.

A graph is *maximal planar* (or a *triangulation*) if it is planar but adding any new edge would violate the planarity. Equivalently, it is maximal planar if it can be embedded in the sphere so that all regions are triangular (i.e., have three sides). Note that this, like many other features, is quite different on surfaces other than the sphere. For instance, there are graphs known which can be embedded in the torus and are maximal with respect to that property but which do not triangulate the torus.

A graph is *outerplane* if it is embedded as the vertices of a polygon, together with some of its interior diagonals; equivalently, if there is a single region of the embedding which contains all the vertices in its boundary. A graph is outerplanar if it has an outerplane embedding. It is easy to see that a graph is outerplanar if and only if it has book thickness 1. A graph is maximal outerplanar if and only if it consists of a triangulation of a polygon with no interior vertices.

The term "plane" above refers to a specific embedding within the Euclidean plane. There are, however, other possibilities for the plane such as hyperbolic, h-adic, or just defined by local relations as in Knuth's closure systems. A different but seemingly natural choice would be to complexify. However, these directions will not be explored here.

The *dual* to a plane graph is a new graph in which each region becomes a vertex and two regions meet along a common edge if and only if the two corresponding vertices are joined by a curve transverse to the common edge. If the original graph has all triangular regions, then its dual is cubic (regular of degree 3) and is called a *map*.

A graph is 4-connected if it cannot be disconnected (or made trivial) by the removal of fewer than 4 vertices. A maximal planar graph is 4-connected if and only if it contains no "separating triangles" whose removal would produce two nonempty graphs. A graph is *Hamiltonian* if and only if it contains a spanning cycle.

Theorem 2 (Whitney, 1931) *Every* 4*-connected maximal planar graph is Hamiltonian.*

See [40] and, more generally, Tutte [42].

If the 4CT were known to be true for maximal planar graphs without separating triangles, it would hold in general. For if there is a separating triangle T in a maximal planar graph G, by induction one knows that the graphs obtained by truncating without and within T, having fewer separating triangles, are 4-colorable; hence, the colorings can be renamed so as to agree on T and G would have a 4-coloring [36, p. 110].

Hence, Whitney's Theorem means that we may restrict attention to maximal planar graphs with a Hamiltonian cycle. Such graphs are precisely the same as the graphs formed by two triangulations of a polygon (inner and outer) with all vertices on the polygon; equivalently, a two page graph. It clearly further suffices to let both the outerplane graphs be maximal outerplanar. Hence, one need only show that every triangulation, with all of its vertices occuring along a polygon, has a 4-coloring of its vertices, or equivalently that there is a 4-coloring of the regions of the dual map. It is easy to show that 4-coloring the regions of a map gives a 3-coloring of the edges (one color for each of the three 1-factors of K_4), and conversely, from a 3-coloring of the edges, one can derive a 4-coloring of the regions.

However, this dual map has a special property resulting from the existence of the Hamiltonian cycle (i.e., the polygon). If we take all of the edges of the map which are transverse to the polygon, they constitute a separating set of edges. Moreover, if they are removed, what remains is acyclic. In fact, we see that there are two cubic trees, dual to the inner and outer triangulations of the polygon, which contain in common this separating set as their outermost edges.

The 4CT may thus be stated: any two rooted cubic plane trees with equal size have edge 3-colorings so that roots and pairs of corresponding outermost edges (leaves) get the same color. Equivalently, given any two such trees, there is an edge 3-coloring of the first tree which when interpreted algebraically in the second tree as vector cross-product leads to the same value at the root of the second tree as at the first (Mycielski [36, p. 129]), Kauffman [21]). As Kauffman remarks, a non-zero value at the root of the second tree must actually be equal to the value of the color at the root of the first tree since, in the case of all non-zero entries, the cross-product multiplication of standard unit vectors in 3-space is simply multiplication of pure quaternions, which is associative.

Through this trick of representing three abstract colors as the three standard unit vectors $\{i, j, k\}$ in $\mathcal{R}^3$, the effect of the second tree is just that of a particular bracketing (i.e., parenthesization) of the cross-product acting on the sequence of color-values along the leaves of the first tree.

While the color values sequences at the leaves are in $\{i, j, k\}^{n+1}$ there are actually only 2^n achievable sequences since coloring the edges of a cubic plane tree is entirely determined (up to obvious color symmetries) by the cyclic order in which the colors appear at each of the n internal vertices. This transforms the 4CT into a statement about the existence of a particular bit (or qubit) string.

Since the tree is a rooted acyclic plane graph, we can use the planar orientation to uniquely enumerate the vertices. Just start at the root and traverse a small regular neighborhood of the tree in the plane in the counterclockwise sense. For algebraic purposes, it is convenient to thus identify each interior vertex of a tree with a unique index depending on the root and choice of orientation, assuming also given the choice of tree-order - say, depth-first. This can be done for all the vertices, or merely for the cubic vertices.

Given an edge coloring of the tree, each cubic vertex receives the sign "+" if the 3 colors appearing on its edges are in counterclockwise order and "-" if the order is clockwise. This requires that we first specify some (arbitrary but henceforth fixed) cyclic order on the colors i, j, k - say, the alphabetic one and also that we specify an orientation of the surface - say, the counterclockwise sense.

Beginning with an assignment of $+/-$ signs to the internal vertices, a coloring of the root extends to a unique edge-coloring of the tree. The state of the cyclic order at each internal vertex is also called its *Heawood character*. We shall use the word *state* to refer to either the $+/-$ at a particular vertex, or a string of such binary values associated with a graph.

The 4CT is therefore equivalent to the following:

Proposition 1 *For any two cubic plane trees with the same number of cubic vertices, there is a state for the first tree such that the resulting edge-coloring provides a non-zero value at the root of the second tree.*

We shall also say that the two trees are *compatably colored.*

Later, it will be convenient to think of these signs as the two possible standard basis elements in the two-dimensional complex Hilbert space, so that an assignment of Heawood characters is just a standard basis element of the n-fold tensor product.

Mycielski and Kauffman require that for every pair of rooted, cubic plane trees (equivalently, systems of parentheses), there exists a sequence of algebraic variables at the leaves which produces a non-zero effect at both of the roots. This treats the two trees symmetrically as an unordered pair.

Instead, we consider a *communication-oriented* model where ordered pairs of trees are the objects. The first tree sends a signal which is the sequence of algebraic values (in i, j, k) on its leaves, and the signal depends on the Heawood characters at its internal vertices. The second tree computes a value at its root depending on the signal it receives. Our version of the 4CT is that for any pair of trees, there must be a way to choose the Heawood characters of the first tree to produce a non-zero value at the root of the second. Thus, the tree-communication model is asymmetric.

The model being proposed here is *wavelike* in the sense that the first tree causes a branching (which we may view as a complex superposition of all its possible collective states) that propagates twisting force across space from the root of the first tree to the root of the second. The first tree gives a topologically constrained unfolding of the signal at the root into the sequence of colors on the leaves, a colored 1-dimensional wave, and the second tree collapses the colored wave - using the bracketing determined by the topology of the second tree and vector cross product. Hence, the 4CT is the assertion that inertial force can be propagated through such a wavelike process.

Now we establish a connection between the 4CT and quantum computing.

Imagine a gedanken device which consists of two rooted cubic plane trees, each with n internal vertices, with each internal vertex of the first tree given an equal superposition of the two +/- states. For simplicity, let us assume that $n = 2m$ is even. We can write $0, i, j, k$, respectively, as $|00>, |01>, |10>, |11>$. An internal vertex transforms the input $|01>$ into $|10>, |11>$ on its output edges (reading in counterclockwise order) if the sign of the vertex is $+$, with output reversed if the sign is reversed, and similarly for the other inputs. M. Biskup has remarked that such a vertex resembles a quantum gate in that it has the same number of inputs and outputs. (In fact, if, for a coloring of the first tree which does extend to the second, the internal vertices of the second tree are also labeled using the counterclockwise convention, each of them determines a sign and running everything in the opposite direction, with the two trees reversing roles, these signs give a coloring of the second tree that extends to the first.)

Each of the 2^n +/- states appears with weight 2^{-m} so the sum of the squares of the weights is 1. By distributivity of cross product over addition, the resulting value at the root of the second tree is $c2^{-m}|01> + (2^n - c)2^{-m}|00>$, where c is the number of ways to assign Heawood characters that produce a coloring of both trees. Of course, one can easily renormalize by dividing both complex coefficients by the sum of the squares of their amplitudes. Then the probability of obtaining either of the two pure states is the square of the absolute value of the normalized coefficients.

The 4CT says that the probability of measuring $|01>$ at the root of the second tree is non-zero. How does this gedanken device "know" about the four-color theorem? A proof which is intrinsic to the quantum world of the machine would provide a new way to prove the 4CT and a practical implementation of this device would give actual colorings. Perhaps a version of Grover's algorithm that utilizes the structure can find the coloring efficiently - e.g., by an amplification technique.

We know that the quantum event must happen but still not why. The conjecture below explains four coloring in terms of quantum algebra.

7 The color force

We shall enrich the model by considering an adjacency relation on the trees. This graph-theoretic structure which relates all of the rooted cubic plane trees (and other avatars of Catalan) will be denoted by $\mathcal{T}_n$ and is known to be connected. Instead of using trees, it will sometimes be more convenient to use maximal outerplane graphs. The following graph, to the author's knowledge, was first studied by MacLane and by Stasheff in connection with abstract associativity.

Two maximal outerplane graphs in $\mathcal{T}_n$ are *adjacent* if they differ by "flipping" one diagonal of a quadrilateral to the other. For example $\mathcal{T}_3$ has five vertices, corresponding to the five distinct ways to triangulate a pentagon and is itself regular of degree 2; $\mathcal{T}_4$ has 14 vertices and is regular of degree 3. The equivalent adjacency relation between trees corresponds to replacing $(ab)c$ by $a(bc)$ in the parenthesis system notation.

The four color property merely asserts the existence of a coloring but says nothing about its dynamical nature. We cannot rule out the following conjecture: For any two trees T,T' in $\mathcal{T}_n$, there is a path between them in the graph and a particular coloring of T so that the path induces a sequence of colorings in all of the intermediate trees in the path. The existence of such a colored path amounts to the assertion of a color "force".

Every path in $\mathcal{T}_n$ determines homomorphisms from an n-fold tensor product of $\mathcal{C}^2$ to itself. It turns out that a simple way to guarantee the existence of paths with the above property is to select those paths with particular algebraic conditions. The conjecture is that the algebraic conditions do hold.

Conjecture 1 *For any positive integer n and for any τ_1, τ_2 in $\mathcal{T}_n$, there is a path in $\mathcal{T}_n$ (that is, a sequence of edge flipping operations) and also a state for τ_1 such that*

(i) the sequence of flips begins with τ_1 and ends with τ_2;

(ii) labels of the endvertices of each flipped edge are flipped in sign, while all other labels remain constant;

(ii) the product of the labels on each flipped edge is positive.

One could also phrase this conjecture as the assertion of connectivity for the graph whose vertices consist of rooted cubic plane trees, where two trees are adjacent if their exists a Heawood character for the first which induces a color path as above.

If the two trees are identical, any state will do. If the two trees are adjacent, then it is not difficult to check that a state that gives the same sign to the two vertices of the flipped edge, and only such an assignment, produces a coloring of the first tree that extends to the second. Thus, the conjecture also holds when the trees are adjacent in $\mathcal{T}_n$.

It can also be shown that the conjecture holds for the pair (L, R) of extremal trees in $\mathcal{T}_n$, which are indeed the two vertices of maximum eccentricity. Of course, here the order doesn't matter by symmetry.

On each step, one obtains a state for the next tree by reversing the value at each vertex of the flipped edge. All other vertices retain their original value. Call the new state the *adjustment* of the original state with respect to the chosen edge. In terms of edge coloring, the two trees receive the same coloring in the sense that the flipped edge and its four consecutive neighboring edges intersect one another in a different way but receive the same labels, and all other edges are unchanged in color. In particular, outermost edges never change color.

We remark that, in a perhaps more physical view, the flipped edge is first annihilated and then a new edge is created.

Beginning with a first tree and assuming that it has Heawood characters which permit one to successively adjust it to produce the second tree, one can edge color the second tree. Each step changes the colors only in the first-neighborhood of the flipped edge and the vertex-parity condition exactly guarantees that the rearrangement remains a coloring. Externally, nothing changes and, in particular, the outermost leaves of the tree are never affected. Therefore, this process actually finds a coloring of the second tree which agrees with that of the first along the outermost leaves.

Thus, we have proved the following:

Theorem 3 *The conjecture implies the 4CT.*

Our theorem gives a formulation of the notion of color-propagation which describes a minimal means for adjusting a coloring with respect to a path in $\mathcal{T}_n$ using a local relabeling; only five edges are rearranged per step and all edges keep their original colorings. The conjecture above asserts the existence of such a color path. In certain special cases, such as when the trees differ only by a sequence of disjoint transformations, as might occur under some low amplitude noise process, the conjecture does hold.

An interesting question is to find a means to determine for two trees (or other standard objects, e.g., polygonal triangulations) whether they are relatively close in the graph $\mathcal{T}_n$ and R. Kotecký has asked about the statistics.

Given two trees, the problem of choosing a specific path and coloring reveals an additional connection between the 4CT and the subject matter of quantum algebra and quantum computing.

The elementary edge-flip operations can be interpreted algebraically. Let $V = \mathcal{C}^2$ with two generators $|+>, |->$. An *R-matrix* is a linear automorphism c of $V \otimes V$ which satisfies the Yang-Baxter equation [19, p. 167],

$$1 \otimes c \circ c \otimes 1 \circ 1 \otimes c = c \otimes 1 \circ 1 \otimes c \circ c \otimes 1$$

between endomorphisms of $V \otimes V \otimes V$.

Let τ be the endomorphism of V which reverses the generators. We define $c : V \otimes V \to V \otimes V$ by $c|b1, b2 >= |\tau(b_2), \tau(b_1) >$. It is simple to check that this c does satisfy the Yang-Baxter equation. For example, applying the left-hand-side to $| + +- >$, we get $(1 \otimes c)(c \otimes 1)| + +- >= (1 \otimes c)| - -- >= | - ++ >$, while applying the right-hand-side to the same element gives $(c \otimes 1)(1 \otimes c)| - -- >= (c \otimes 1)| - ++ >= | - ++ >$.

Similarly, one can define $c' : V \otimes V \to V \otimes V$ by $c'|b1, b2 >= |\tau(b_1), \tau(b_2) >$, but c' is not an R-matrix.

Now we use these two maps c and c' to determine two different actions of the space of oriented paths in $\mathcal{T}_n$ on $\mathcal{V} = \bigotimes_{j=1}^{n} V$. If $a = (\tau_1, \tau_2)$ is an arc of the directed graph $\mathcal{D}_n$ obtained by replacing each edge of $\mathcal{T}_n$ by both of its possible ordered arcs, define an *arc operator* which is an endomorphism of the n-fold tensor product $\mathcal{V}$. In each tree, τ_1, τ_2, the vertices are indexed according to the depth-first, clockwise order so a defines a permutation σ_a on the n indices which reflects the topological reindexing which results from the edge annihilation/creation event.

We define a linear automorphism $\mathcal{F}_a$ of $\mathcal{V}$ which corresponds to the edge-flip. Suppose the two vertices at either end of the removed edge e are indexed by, say, $r < s$ depending on the particular plane embedding. Let the ends of the added edge be indexed by $t < u$. (See our remark about annihilation and creation above.) Now we define $\mathcal{F}_a$ on $(b_1 \otimes \cdots \otimes b_n)$

where each b_i is $+$ or $-$, which is the canonical tensor-product basis, by $(b_1 \otimes \cdots \otimes b_n) \mapsto (b'_1 \otimes \cdots \otimes b'_n)$, where $b'_t = \tau b_s$, $b'_u = \tau b_r$ and for $j \neq t, u$, $b'_j = b_k$ for $k = \sigma_a^{-1}(j)$.

The effect of $\mathcal{F}_a$ is to interweave c with the identity according to the topological reindexing, and one can plainly do the same thing with c'. Define $\mathcal{F}'_a$ for any a on the path analogously with $\mathcal{F}_a$ by replacing c with c' which also reverses the sign of the generator but does not interchange the factors of $V \otimes V$. So $\mathcal{F}'_a(b_1 \otimes \cdots \otimes b_n) = (b'_1 \otimes \cdots \otimes b'_n)$, where $b'_t = \tau b_r$, $b'_u = \tau b_s$ and for $j \neq t, u$, $b'_j = b_k$ for $k = \sigma_a^{-1}(j)$. Let ϕ_a denote the difference of these two arc operators $\phi_a := \mathcal{F}_a - \mathcal{F}'_a$ for any edge a in $\mathcal{T}_n$.

Lemma 3 *If e is an edge in some tree T in $\mathcal{T}_n$ with $r < s$ the indices of its two endpoints and a is the arc in the graph in $\mathcal{D}_n$ corresponding to flipping the edge, then $\phi_a(b_1 \otimes \cdots \otimes b_n) = 0$ if and only if $b_r = b_s$.*

Now we extend to paths. Let $\mathcal{P} = \mathcal{P}(\tau_1, \tau_2) = \{P : P \text{ is a path between } \tau_1 \text{ and } \tau_2 \text{ in } \mathcal{T}_n\}$ and let $\mathcal{P}_k$ denote the paths of length k. For any P in $\mathcal{P}$, define $\mathcal{F}(P)$ as the composition (in $\mathcal{V}$) of the morphisms $\mathcal{F}_a$ in the order in which they appear in the path. Similarly, one defines $\mathcal{F}'(P)$ another automorphism of $\mathcal{V}$.

For any P in $\mathcal{P}_k$ and any integer m, $1 \leq m \leq k$, define $P(m)$ to be the initial subpath of P with length m. It is convenient to define $\Phi_P = \bigcap_{m=1}^{k} ker(\mathcal{F}(P(m)) - \mathcal{F}'(P(m)))$.

One can then prove the following:

Theorem 4 *A path P in $\mathcal{P}_k(\tau_1, \tau_2)$ is a color path if and only if $\Phi_P \neq 0$.*

The color force conjecture above may now be restated as follows:

For any pair τ_1, τ_2 in $\mathcal{T}_n$, the direct sum of Φ_P, $P \in \mathcal{P}(\tau_1, \tau_2)$, is nonzero.

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