CONSTRAINTS FOR THE ESTIMATION OF DISPLACEMENT VECTOR FIELDS FROM IMAGE SEQUENCES

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ABSTRACT

Smoothness constraints have been used to facilitate the estimation of displacement vector fields. Differing from HORN and SCHUNCK 81 who employ a general smoothness requirement, this contribution reports an analysis of an "oriented smoothness" requirement: a change in the displacement vector field is only constrained in the direction perpendicular to the characteristic gray value variation based on which the displacement vector is estimated. An iterative solution for the resulting system of nonlinear partial differential equations is developed. It is shown how this system of equations relates to the one derived by HORN and SCHUNCK 81.

1. INTRODUCTION

The reliable estimation of displacement vector fields is of great importance for the interpretation of image sequences about scenes with temporal variations. An upsurge of interest in this topic during the last few years has led to a considerable number of contributions which attack this estimation problem. The majority among them attempt to estimate the displacement vector only at isolated image locations. Moreover, they are based on more or less ad-hoc approaches which have not been investigated sufficiently to allow a reliable judgement on their validity and their limits of applicability. A recent survey of the literature about this topic can be found in NAGEL 82a+b.

HORN and SCHUNCK 81 took an important step away from ad-hoc solutions towards a more systematic approach. They started from a well-known approximation (see NAGEL 81 for references to the earlier literature) which relates the spatial gray value gradient $\nabla g = (g_x, g_y)^T$ to the temporal change $g_t = \partial g/\partial t$ in order to estimate the interframe displacement $U = (u, v)^T$ of certain image regions

$$(\nabla g)^T \cdot U + g_t = 0 \qquad (1)$$

Since equation (1) is insufficient to determine both components of the displacement vector field U(X), HORN and SCHUNCK 81 demand in addition that the displacement vector field varies smoothly as a function of the position vector X in the image. This requirement is combined with equation (1) in the following minimization approach:

$$\iint dx\,dy \left\{ \left((\nabla g)^T \cdot U + g_t \right)^2 + \alpha^2 \left(\nabla u^T \cdot \nabla u + \nabla v^T \cdot \nabla v \right) \right\} \Rightarrow \min \qquad (2)$$

They employ the variational calculus to transform equation (2) into a system of second order partial differential equations for the two components u(X) and v(X). Using a particular approximation on a 3x3 pixel grid for the Laplacian of u and v, this system of two coupled partial differential equations has been converted into a sparse system of linear equations with two unknowns for each pixel in the image. An iterative solution of this large system of equations has been employed in order to study various examples.

Two aspects of this approach let it appear especially attractive. First, all hypotheses are introduced explicitly into a rigorous mathematical framework. Second, well-known mathematical methods are employed to derive the solution so that the consequences of the various hypotheses and approximations can be studied in detail. One can recognize three sources of difficulties in the approach of HORN and SCHUNCK 81:

i) The smoothness requirement is applied indiscriminately across all gray value edges despite the fact that such edges might separate image regions with discontinuous displacement vector fields (see SCHUNCK and HORN 81).

ii) No advantage is taken of the fact that certain nonlinear gray value variations may constrain possible displacement vectors much stronger than image areas with smoothly varying gray values or straight line gray value transitions.

iii) The solution outlined in HORN and SCHUNCK 81 does not indicate how to handle situations where a two-pixel gradient estimate or an approximation to the Laplacian based on a 3x3 pixel window appear inappropriate.

2. "ORIENTED SMOOTHNESS" REQUIREMENTS

Several approaches have been investigated in order to cope especially with difficulty (i). YACHIDA 82 assumes that a displacement vector can be estimated at prominent points. He propagates such estimates into neighboring areas with large gray value gradients, based on the method of HORN and SCHUNCK

81. His iterative improvement scheme, however, employs the inverse variance of displacement estimates from a 5x5 window as a weight in order to suppress the propagation of displacement estimates with large local variations.

CORNELIUS and KANADE 83 deactivate the smoothness requirement in the neighborhood of zero-crossing contours in order to avoid that estimates spill across potential discontinuities of the displacement vector field, for example associated with occluding contours.

WU et al. 82 propagate a displacement estimate only along a contour line between corner points. At each new contour point, they combine the estimated displacement vector from the previous contour point with new estimates of the contour direction and of the displacement component perpendicular to the contour in order to update the tangential component of the displacement vector.

HILDRETH 83 minimizes the sum of two terms, integrated along a zero-crossing contour. The first term is the squared difference between the 'measured' and the estimated displacement component perpendicular to the contour. The second term represents the squared derivative of the displacement vector field with respect to the arclength along the zero-crossing contour, expressing the smoothness requirement (see also HILDRETH 82).

The current contribution proposes a broader approach. An "oriented smoothness" Requirement will be introduced, constraining the variation of the displacement vector field only in those directions alonq which a displacement vector component can not be inferred from the spatio-temporal gray value changes. This "oriented smoothness" constraint will be tied directly to the gray value variation, th s avoiding the introduction of additional machinery such as the variance of displacement estimates or the separate, prior determination of gray value contour lines. Moreover, a single minimization task will be used both to estimate displacement vector fields from nonlinear gray value variations and to take into account the "oriented smoothness" requirement.

Two possibilities will be formulated to express an "oriented smoothness" constraint. Both can be combined into a weight matrix which constrains the variation of the displacement vector field (NAGEL 83b). A matrix with the same structure appeared in earlier investigations to estimate displacement vectors at gray value corner points (NAGEL 83a). Experiments with this earlier approach yielded encouraging results (NAGEL and ENKELMANN 82 + 83) and thus motivated the current investigation.

SMOOTHNESS PERPENDICULAR TO THE GRADIENT DIRECTION

As it is well known, equation (1) only restricts the displacement vector U along the gradient direction. It is therefore demanded that the variation of U in the direction perpendicular to the gradient direction becomes as small as possible. The first variation of U is captured by the functional matrix of first derivatives where $u_x = \partial u/\partial x$, etc.:

$$\nabla U = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} \tag{3}$$

A vector perpendicular to the gray value gradient $\nabla g = (g_x, g_y)^T$ is given by $(g_y, -g_x)^T$. The variation of U in the direction perpendicular to the gradient orientation is given by the vector

$$\begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T \nabla U \tag{4}$$

In order to minimize this variation of U we need a scalar entity which can be simply obtained by forming the scalar product of this vector (4) with itself

$$E_{gradient} = \left\{ \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T \nabla U \right\} \left\{ \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T \nabla U \right\}^T \tag{5}$$

It is a known matrix theorem that the trace of a product of two matrices which yields a square matrix remains the same if the two matrix factors are swapped - even if the two matrix factors are non-square. Since the scalar product of (4) with itself results in a square 1x1 matrix, we may write

$$E_{gradient} = trace\left\{ \left\{ \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T \nabla U \right\}^T \left\{ \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T \nabla U \right\} \right\} \tag{6a}$$

$$= trace\left\{ (\nabla U)^T \begin{pmatrix} g_y \\ -g_x \end{pmatrix} \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T \nabla U \right\} \tag{6b}$$

The outer product matrix of rank one

$$\begin{pmatrix} g_y \\ -g_x \end{pmatrix} \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T = \begin{pmatrix} g_y \cdot g_y & -g_x\, g_y \\ -g_x \cdot g_y & g_x \cdot g_x \end{pmatrix} \tag{7}$$

thus appears as a weight matrix in an expression (6b) which represents one possibility to define a norm of a matrix.

4. SMOOTHNESS PERPENDICULAR TO PRINCIPAL CURVATURE DIRECTIONS

There are situations where the gradient vanishes and nevertheless the gray values exhibit a characteristic variation, for example a relative extremum of the gray value. Since such positions are locally unique, it appears attractive to estimate their displacement because it should allow the determination of both components of $U = (u, v)^T$.

Let us assume that the image coordinate system has been aligned with the principal curvature directions of g(X) at location X. The matrix of second partial derivatives of g(X) with respect to x and y will, therefore, be diagonal

$$(\nabla\nabla g) = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} \Rightarrow \begin{pmatrix} g_{xx} & 0 \\ 0 & g_{yy} \end{pmatrix} \quad (8)$$

with $g_{xx} = \partial^2 g/\partial x^2$ etc.

A strong curvature g_{xx} of g(X) along the x-direction represents a characteristic gray value variation which should allow to determine the x-component u of the displacement vector. We therefore demand only that the variation of the displacement vector along the other principal curvature is kept small, for example by minimizing some expression of the product $g_{xx} \cdot u_y$ as well as $g_{xx} \cdot v_y$. In analogy, a strong curvature g_{yy} of g(X) along the y-direction should result in a smoothness requirement only for the x-direction. This could be formulated by minimizing the product $g_{yy} \cdot u_x$ and $g_{yy} \cdot v_x$.

Exploiting our special coordinate conventions, these requirements can be expressed by combining equation (8) with the functional matrix ∇U from equation (3) to form

$$\begin{pmatrix} g_{yy} & 0 \\ 0 & g_{xx} \end{pmatrix} \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} g_{yy} \cdot u_x & g_{yy} \cdot v_x \\ g_{xx} \cdot u_y & g_{xx} \cdot v_y \end{pmatrix} \quad (9)$$

which contains just the products we are interested in. Now we drop the assumption that the coordinate system is aligned with the principal curvature direction and exploit our knowledge that the inverse of $\nabla\nabla$g can be expressed as

$$(\nabla\nabla g)^{-1} = \frac{1}{\det(\nabla\nabla g)} \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix} \quad (10)$$

Equation (9) can therefore be written in a general form:

$$\begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix}^T \nabla U = \begin{pmatrix} g_{yy} u_x - g_{xy} u_y & g_{yy} v_x - g_{xy} v_y \\ -g_{xy} u_x + g_{xx} u_y & -g_{xy} v_x + g_{xx} v_y \end{pmatrix} \quad (11)$$

Since the matrix on the right hand side of equation (11) can be conceived as the combination of two column vectors, we employ a matrix notation to compute the equivalent to equation (6a) for each of these column vectors:

$$E_{curvature} = \mathrm{trace}\left[\left\{ \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix}^T \nabla U \right\}^T \left\{ \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix}^T \nabla U \right\} \right] \quad (12a)$$

$$= \mathrm{trace}\left[(\nabla U)^T \left\{ \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix} \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix}^T \right\} \nabla U \right] \quad (12b)$$

5. COMBINATION OF BOTH SMOOTHNESS REQUIREMENTS

Since it cannot be said in general which requirement appears more important, both are combined in the form

$$E_{gradient} + a^2 E_{curvature} = \frac{1}{b} \mathrm{trace}\left[(\nabla U)^T F (\nabla U) \right] \quad (13a)$$

with

$$F = \left\{ \begin{pmatrix} g_y \\ -g_x \end{pmatrix} \begin{pmatrix} g_y \\ -g_x \end{pmatrix}^T + a^2 \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix} \begin{pmatrix} g_{yy} & -g_{xy} \\ -g_{xy} & g_{xx} \end{pmatrix}^T \right\} \quad (13b)$$

Here, b denotes a normalization factor whereas the factor a^2 - with the dimension of the squared unit of length - measures the relative strenghts between these two contributing weight matrices. The intent formulated so far has been to minimize the variation of U in particular directions indicated by properties of the gray value distribution at a location X. In order to obtain some normalization with respect to the size of the gradient vector or the curvature matrix, we look for a norm of the weight matrix F. Experience with a matrix of similar structure reported in NAGEL and ENKELMANN 83 suggests

$$C^{-1} = \frac{F}{\det F} \quad (14)$$

should be used as the normalized weight matrix where F is given by equation (13b). Straightforward calculations show that (14) represents the inverse of the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix} = \left((\nabla g)(\nabla g)^T + a^2 (\nabla\nabla g)(\nabla\nabla g)^T \right) \quad (15)$$

6. MINIMIZATION APPROACH

We now have all the ingredients to formulate the expression to be minimized, using the notation $g1(X) = g(X,t1)$ and $g2(X) = g(X,t2)$:

$$E = \iint dx\,dy \left\{ \left[g2(X) - g1(X-U) \right]^2 + \alpha^2 \, \mathrm{trace}\left[(\nabla U)^T C^{-1} (\nabla U) \right] \right\} \quad (16)$$

It is assumed that the gray values do not change except for position and possibly the effects of (dis-)occlusion. The first term of this expression has been introduced and discussed in NAGEL 83a. This expression specializes to the approach of HORN and SCHUNCK 81 if we make the following assumptions:

i) The influence of smoothness requirements oriented along the principal curvature directions is dropped, i.e. $a^2 = 0$.

ii) The directional sensitivity of the smoothness requirement is suppressed, i.e. the outer product matrix $(\nabla g)(\nabla g)^T$ is replaced by the unit matrix.

iii) The difference in the first term of equation (16) is approximated by a first order Taylor expansion.

The Euler-Lagrange equations for the minimization problem of (16) yield

$$\left[g2(X) - g1(X-U) \right] \cdot \frac{\partial g1(X-U)}{\partial u} - \alpha^2 \begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{pmatrix}^T C^{-1} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = 0 \quad (17a)$$

$$\left[g2(X) - g1(X-U) \right] \cdot \frac{\partial g1(X-U)}{\partial v} - \alpha^2 \begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{pmatrix}^T C^{-1} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = 0 \quad (17b)$$

or

$$\left[g2(X) - g1(X-U) \right] \cdot \frac{\partial g1(X-U)}{\partial u} - \alpha^2 \left[- \left\{ C^{-1} \begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{pmatrix} \underline{\left.\right\}^T C} \cdot C^{-1} \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \mathrm{trace}\left\{ C^{-1} \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} \right\} \right] \quad (18a)$$

The dotted underline indicates the operand of the differentiation operators. An analogous equation(48b) results for the second component v of U.

Let *c1* denote the upper left eigenvalue of C and c2 the other eigenvalue. Assume that the coordinate system happens to be chosen in such a way that the matrix C at location X is diagonal. Then the smoothness term for the first component of the displacement vector takes the form:

$$\begin{pmatrix} \frac{d}{dx} \\ \frac{d}{dy} \end{pmatrix}^T \left\{ C^{-1} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \right\} = - \begin{pmatrix} \frac{1}{c_1} \frac{dc_{11}}{dx} + \frac{1}{c_2} \frac{dc_{12}}{dy} \\ \frac{1}{c_1} \frac{dc_{12}}{dx} + \frac{1}{c_2} \frac{dc_{22}}{dy} \end{pmatrix} \begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & \frac{1}{c_2} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} + \frac{u_{xx}}{c_1} + \frac{u_{yy}}{c_2} \quad (19)$$

The right hand side immediately shows that large eigenvalues of the matrix C tend to suppress the influence from the smoothness requirement in their direction.

"Gray value corners" can be characterized as the location of maximum curvature in the locus line of extremal gray value slope (see NAGEL 83a). At gray value corners, both eigenvalues become large and damp down the influence of the smoothness requirement. This is advantageous because a gray value corner is defined well enough to allow the determination of both displacement vector components by "virtually local" measurements.

According to our assumptions we may write for not to large U

$$g1(X-U) \cong g1(X) - (\nabla g1)^T U + \tfrac{1}{2} U^T (\nabla\nabla g) U \quad (20).$$

Since

$$\begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix} g1(X-U) = - \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} g1(X-U) + \begin{pmatrix} g1_{xx} & g1_{xy} \\ g1_{xy} & g1_{yy} \end{pmatrix} U \quad (21)$$

the coupled system (17) of partial differential equations can therefore be written as

$$0 = \left[g2(X) - g1(X-U)\right]\left[-\nabla g1 + (\nabla\nabla g1)U\right] + \alpha^2 \begin{pmatrix} \left\{C^{-1}\begin{pmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{pmatrix}\right\}^T C\cdot C^{-1}\begin{pmatrix}u_x\\ u_y\end{pmatrix} - \mathrm{trace}\left\{C^{-1}\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\right\} \\ \left\{C^{-1}\begin{pmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{pmatrix}\right\}^T C\cdot C^{-1}\begin{pmatrix}v_x\\ v_y\end{pmatrix} - \mathrm{trace}\left\{C^{-1}\begin{pmatrix}v_{xx} & v_{xy}\\ v_{xy} & v_{yy}\end{pmatrix}\right\} \end{pmatrix} \quad (22)$$

7. APPROXIMATIVE SOLUTION

In order to get some insight into the structure of solutions of the system of two equations (22) we assume that an approximative solution Ug (X) is known. The true solution U(X) should differ from U_0 (X) by a small correction vector field AU(X):

$$U(X) = U_0(X) + \Delta U(X) \quad (23)$$

AU(X) = (Au(X).Av(X)) is supposed to be small so that we may neglect terms with higher than first powers of components of AU:

$$g1(X-U) = g1(X-U_0-\Delta U) \cong g1(X-U_0) - \left(\nabla g1(X-U_0)\right)^T$$

The second bracket in equation (22) contains a term which corresponds to the slope of g1 at location X-U. This term is approximated by the slope of g1 at location X-U0 :

$$\left[-\nabla g1 + (\nabla\nabla g1)U\right] = -\left[\nabla g1 - (\nabla\nabla g1)(U_0 + \Delta U)\right] \quad (25a)$$

$$\cong -\left[\nabla g1(X) - (\nabla\nabla g1(X))U_0\right]$$

$$\cong -\nabla g1(X-U_0) \quad (25b)$$

Introduction of these approximations into equation (22) yields a system of equations which are linear in the components of the correction vector field U(X):

It appears natural to evaluate all partial derivatives of g1(X) at the location X-U_0, including those which enter through the "smoothness" term multiplied *by alpha*^{2}. We obtain

$$(\nabla g1)(\nabla g1)^T \Delta U = -\left[g2(X) - g1(X-U_0)\right]\nabla g1 + \alpha^2\left[\text{as in equation (26)}\right] \quad (27)$$

This system of linear equations for AU cannot be solved without further consideration because the matrix (Vg1)(Vg1) has been derived from an outer product and, therefore, has only rank one rather than two.

A way to cope with this problem is suggested by earlier investigations. For digitized images, the partial derivatives of g(X) *are* computed using the operators of BEAUDET 78 which are based on a bivariate polynomial approximation of g(X) within a square window centered at X. The same operators should be used to compute the partial derivatives of U(X) which appear, for example, in equation (22). In this manner, a more general approach than that described by HORN and SCHUNCK 81 is employed here. The use of these operators implies that the entities appearing in equation (22) are the result of a kind of averaging across the operator window.

These considerations are applied in the following manner. The start values U_0(X) are set identically

$$0 = \left[g2(X) - g1(X-U_0) + \left(\nabla g1(X-U_0)\right)^T\Delta U\right]\left[-\nabla g1(X-U_0)\right] + \alpha^2 \begin{pmatrix} \left\{C^{-1}\begin{pmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{pmatrix}\right\}^T C\cdot C^{-1}\begin{pmatrix}u_x\\ u_y\end{pmatrix} - \mathrm{trace}\left\{C^{-1}\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\right\} \\ \left\{C^{-1}\begin{pmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{pmatrix}\right\}^T C\cdot C^{-1}\begin{pmatrix}v_x\\ v_y\end{pmatrix} - \mathrm{trace}\left\{C^{-1}\begin{pmatrix}v_{xx} & v_{xy}\\ v_{xy} & v_{yy}\end{pmatrix}\right\} \end{pmatrix} \quad (26)$$

zero. Moreover, the strength α^2 of the smoothness term in equation (27) is taken to be a function of the iteration count. It is set to zero initially. If the window is reasonably small, we may represent the first derivative of g as a linear function of the deviation (ξ, η) from the window center:

$$\nabla g\,(x+\xi\,,\,y+\eta) = \nabla g(x) + \left(\nabla\nabla g(x)\right)\begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{28}$$

Introduction of these approximations into equation (27), forming the average over the window and using the fact that the window is squared and centered at the origin of the local coordinate system (ξ, η) with the consequences

$$\sum_{\xi,\eta} \xi = \sum_{\xi,\eta} \eta = \sum_{\xi,\eta} \xi\cdot\eta = 0 \tag{29}$$

as well as $\overline{\xi^2} = \overline{\eta^2}$ leads to the result:

$$\overline{(\nabla g)(\nabla g)^T} = \begin{pmatrix} g_x^2 + \overline{\xi^2}\left(g_{xx}^2 + g_{xy}^2\right) & g_x g_y + \overline{\xi^2} g_{xy}\left(g_{xx} + g_{yy}\right) \\ g_x g_y + \overline{\xi^2} g_{xy}\left(g_{xx} + g_{yy}\right) & g_y^2 + \overline{\xi^2}\left(g_{xy}^2 + g_{yy}^2\right) \end{pmatrix} \tag{30}$$

If we equate $\overline{\xi^2} = \overline{\eta^2}$ with the constant a^2 which has been introduced to denote the relative strength between the gradient term and the curvature term in the smoothness requirement - see equation (13) - then equation (30) becomes identical to the definition of the matrix C in equation (15).

This matrix has the full rank of two if both principal curvatures of the gray value distribution at location X differ significantly from *zero.* In this case we may invert the modified equation (27) and obtain

$$\Delta U = -C^{-1}\,\overline{\left[g2(x) - g1(x-U_0)\right]\nabla g1(x-U_0)} + \alpha^2 C^{-1}\begin{pmatrix} \left\{C^{-1}\begin{pmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{pmatrix}\right\}^T C\,C^{-1}\begin{pmatrix}u_x\\ u_y\end{pmatrix} - \mathrm{trace}\left\{C^{-1}\begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\right\} \\ \left\{C^{-1}\begin{pmatrix}\frac{d}{dx}\\ \frac{d}{dy}\end{pmatrix}\right\}^T C\,C^{-1}\begin{pmatrix}v_x\\ v_y\end{pmatrix} - \mathrm{trace}\left\{C^{-1}\begin{pmatrix}v_{xx} & v_{xy}\\ v_{xy} & v_{yy}\end{pmatrix}\right\} \end{pmatrix} \tag{31a}$$

This result has to be handled in the following manner: initially, alpha^2 = o and $U_0(X) = 0$. It has been shown (NAGEL 83a, NAGEL and ENKELMANN 83) that

$$\Delta U = -C^{-1}\,\overline{\left[g2(x) - g1(x-U_0)\right]\nabla g1(x-U_0)} \tag{31b}$$

yields a very reasonable estimate of $U = U_n + \Delta U = U$ at image locations where the conditions for a "gray value corner" are satisfied. It is, therefore, suggested to estimate U at such locations using equation (31b). Once these estimates have stabilized after a certain number of iterations, the start value U(X) at all other locations is set to zero and the smoothness constraint is switched on by setting alpha^2 to the desired value. During the subsequent iterations, equations (31a) are employed, using the BEAUDET operators to compute the required derivatives of $g1(X-U_c)$ as well as U(X). The smoothness term will spread the non-zero estimates of the displacement vectors which have been obtained during the first few iterations at the "gray value corners". The matrix C is prevented from becoming singular by introduction of minimum values for the partial derivatives of g derived from the errors in the gray value measurements (see appendix of NAGEL 83a).

8. DISCUSSION

In order to overcome some problems with the approach of HORN and SCHUNCK 81 , the concept of "oriented smoothness constraint" has been introduced. As a result, a coupled system of nonlinear partial differential equations (22) for the displacement vector field U(X) has been derived. In order to study solutions of this system, an iterative method has been employed which is based on earlier experience. This approach comprises three steps:

i) Estimate both components of the displacement vector at "gray value corners" - see equation (31 b);

ii) Improve these estimates through iteration at such locations, still with suppressed smoothness constraint;

iii) Spread the resulting estimates by further iterations, but now taking into account the "oriented smoothness constraint" as represented in equation (31a).

The first two steps have already been implemented during investigations of the approach presented in NAGEL 83a. The very encouraging results have been presented in NAGEL and ENKELMANN 82 for step i) and in NAGEL and ENKELMANN 83 for step ii). The experience gained through these studies led to the - heuristic - proposal to use the inverse of the matrix given in equation (30) as a weight for some kind of oriented smoothness requirement (NAGEL 83b) The current contribution is an attempt to derive this heuristic suggestion in a rigorous manner from clearly stated assumptions. Moreover, all approximations are outlined in a derivation of a closed expression for an interative solution to the discretized version of the resulting system of coupled nonlinear partial differential equations. This approach is currently being studied experimentally.

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