Constrained and rational default logics

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Abstract

In this paper we consider constrained and rational default logics We provide two characterizations of constrained extensions One of them is used to derive complexity results for decision problems involving constrained extensions In particular, we show that the problem of membership of a formula in at least one (in all) constrained extension(s) of a default theory is Ef-complete (Ilf-complete) We establish the relationship between constrained and rational default logics We prove that rational ex tensions determine constrained extensions and that for seminormal default theories there is *A* one-to-one correspondence between these objects We also show that the definition of a constrained extension can be extended to cover the case of default theories which may contain justification-free defaults

1 Introduction

Default logic, introduced by Reiter [1980], is one of the most extensively studied nonmonotonic systems Several recent research monographs offer a comprehensive presentation of theoretical and practical aspects of default logic [Besnard, 1989, Brewka, 1991b, Marek and Truszczynski, 1993] Default logic was designed to handle reasoning from incomplete information It allows us to draw conclusions on the basis of "the lack of evidence to the contrary" This formalism assigns to a default theory a collection of theories called *extensions* They describe possible belief sets of an agent reasoning with this theory

All its desirable properties notwithstanding, there are situations where default logic of Reiter produces countenntuitive results In particular, this logic docs not handle well incomplete information given in the form of disjunctive clauses [Poole, 1989, Brewka, 1991a, Gelfond *et al*, 1991, Mikitiuk and Truszczynski, 1993] To remedy this, several modifications of default logic

were proposed disjunctive default logic [Gelfond *et al,* 1991], cumulative default logic [Brewka 1991a], constrained default logic [Schaub, 1992] and rational default logic [Mikitiuk and Truszczynski, 1993] The first system introduces a new disjunction operator to handle "effective" disjunction The latter three take into account, in one way or another, the requirement that defaults with mutually inconsistent justifications must not be used in the construction of the same extension Not surprising]} then, they are somewhat related

Connections between cumulative default logic and constrained default logic are studied in [Schaub, 1992] It is shown there that these two systems are, in a certain sense equivalent At the same time, they are quite differ ent from default logic of Reiter Both commit to assumptions and have such properties as semi-monotonicity and orthogonality[1] In addition in each of these two logics every default theory has an extension In the logic of Reiter all these properties hold for normal default theories but fail m the general case (in fact, for normal default theories, Reiter's default logic is essentially equivalent to constrained and cumulative default logics)

In this paper, we investigate connections between ra tional and constrained (and, consequently, also cumulative) default logics Rational default logic similarly as the logic of Reiter, lacks many of the properties of constrained default logic In particular, default theories may have no ritional extensions and rational default logic does not have the properties of semi monotomcitv and orthogonality The reason is that rational default logic, unhke constrained default logic does not commit to assumptions \t the same time connections between rational and const ained default logics are quite strong We show thrt every rational extension of a default theory determines a constrained extension Moreover we show that rational and constrained default logics coincide for the class of seminormal default theories — a much wider class of theories than normal ones, for which all four versions of default logic mentioned here are equivalent

We also give a useful, proof-theoretic characteriza-

Schaub [1992] uses the term weak orthogonality

tion of the operator Υ, which was used in [Schaub, 1992] to define the notion of a constrained extension Consequently, we get an equivalent definition of constrained extensions This result allows us to design an algorithm for computing constrained extensions and to establish the complexity of reasoning with constrained extensions Since every default theory has a constrained extension, the problem of existence of a constrained extension is, clearly, in P We show that the problem to decide, given a formula φ, whether φ is in at least one constrained extension (in all constrained extensions) of a given default theory, is Σ_2^P-complete ($\pi^\wedge$-complete) In view of a recent result on the complexity of cumulative default logic [Gottlob and Mingyi, 1994] and our results on the complexity of rational default logic [Mikitiuk and Truszczynski, 1993], it follows that all these modes of reasoning have the same computational complexity

Finally, let us note that Schaub did not allow justification-free defaults in his definition of constrained default logic In this paper, we show how to extend constrained default logic to raver theories which may contain justification-free defaults

2 Preliminaries

A *default* is any expression of the form

$$\frac{\alpha \ M\beta_1, \quad ,M\beta_k}{\gamma}, \tag{1}$$

where α, β_i, $1 \le i \le k$ and γ are prepositional formulas Let d be a default of the form (1) The formula a is called the *prerequisite* of d, $p[d]$ in symbols The formulas β_i, $1 \le i \le k$, are called the justifications of d The set of justifications is denoted by $j(d)$ Finally, the formula γ is called the *consequent* of d and is denoted $c[d]$ For a collection D of defaults by $p[D]$, $j(D)$ and $c[D]$ we denote, respectively, the sets of all prerequisites, justifications and consequents of the defaults in D A default of the form $\frac{\alpha \, M\beta}{\beta}$ ($\frac{\alpha \ M\{\beta\wedge\gamma\}}{\gamma}$, resp) is called *normal* (seminormal, resp)

A default theory is a pair *(D W)*, where D is a set of defaults and W ib a set of propositional formulas A default theory *(D,W)* is *nonnal (semtnormal,* resp) if all defaults in D are normal (sermnormal, resp) A default theor> *(D, W)* is *finite* if both D and W are finite

For a set D of defaults we define

$$Mon(D) = \left\{ \frac{p(d)}{c(d)} \quad d \in D \right\}$$

Given a set of ml ,ence rules A, b> $Cn^A(\ )$ we mean the consequence operator of the formal proof s>stem $PC+A_r$ consisting of propositional calculus and the rules in A

In [Mikutiuk and Truszczjnskj, 1993] we introduced the notions of an active set of defaults and a rational extension of a default theory

Definition 2 1 A set A of defaults is *active* with respect to sets of formulas W and S if it satisfies the following conditions

AS1 $j(A) = \emptyset$, or $j(A) \cup S$ is consistent,

AS2 $p(A) \subseteq Cn^{Mon(A)}(W)^2$

The set of all subsets of a set of defaults D which are active with respect to W and S will be denoted by $\mathcal{A}(D, W, S)$ ☐

Observe that $\emptyset$ is active with respect to every W and S Hence, $\mathcal{A}(D, W, S)$ is always non-empty An application of the Kuratowski-Zorn Lemma gives the following result

Proposition 2 1 ([Mikitiuk and Truszczynski, 1993]) Let (D, W) be a default theory and let S be a propositional theory Every $A \in \mathcal{A}(D, W, S)$ is contained in a maximal element of $\mathcal{A}(D, W, S)$ ☐

We define $\mathcal{MA}(D, W, S)$ to be the set of all maximal elements in $\mathcal{A}(D, W, S)$

Definition 2 2 A theory S is a *rational extension* for a default theory (D, W) if $S = Cn^{Mon(A)}(W)$ for some $A \in \mathcal{MA}(D, W, S)$ ☐

Schaub [1992] introduced the notion of a constrained extension by means of the following definition

Definition 2 3 ([Schaub, 1992]) Let (D, W) be a default theory and let T be a propositional theory Then $\Upsilon(T)$ is the pair of smallest sets of formulas (S', T') such that

CE1 $W \subseteq S' \subseteq T'$,

CE2 $S' = Cn(S')$, and $T' = Cn(T')$

CE3 For any default $\frac{\alpha \, M\beta}{\gamma} \in D$ if $\alpha \in S'$ and $T \cup \{\beta, \gamma\}$ is consistent then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$

A pair of sets of formulas (E, C) is a *constrained extension* of (D, W) if $\Upsilon(C) = (E, C)$ ☐

If (E, C) is a constrained extension, then we will refer to E as a *proper constrained extension*

The following example shows that the notions of an extension, a rational extension and a proper constrained extension are different

Example 2 1 Let us consider the default theory $(D, \emptyset)$ [Schaub, 1992], where

$$D = \left\{ \frac{Mb}{c}, \frac{M\neg b}{d}, \frac{M\neg c}{e}, \frac{M\neg d}{f} \right\}$$

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[2] In [Mikitiuk and Truszczynski, 1993] we used the classical reduct of A with respect to S, denoted A_S, instead of $Mon(A)$ both in **AS2** and in the definition of a rational extension If A satisfies **AS1**, then $A_S = Mon(A)$ Since in this paper we will use a different notion of a reduct (Definition 3 2), we decided to reformulate the definitions of active sets and rational extensions to avoid confusion These definitions are equivalent to the original ones

This theory has one extension $Cn(\{c,d\})$, two rational extensions $S_1 = Cn(\{c,f\})$, $S_2 = Cn(\{d,e\})$, and three constrained extensions $(Cn(\{e,f\}), Cn(\{e,\neg c,f,\neg d\}))$, $(S_1, Cn(\{c,b,f,\neg d\}))$, $(S_2, Cn(\{d,\neg b,e,\neg c\}))$ □

Schaub [1992] does not consider justification-free defaults, so in this paper we do not consider them either Hence, $j(A) = \emptyset$ only if $A = \emptyset$ Moreover, in both rational and constrained default logic one can replace all justifications of a default by their conjunction Thus we assume that every default has exactly one justification In Section 6 we will show how Schaub's definition can be extended to cover the case of justification-free defaults

3 Characterizations of constrained extensions

In this section we will give two useful characterizations of constrained extensions The first of them will be based on a proof-theoretic description of the operator $\Upsilon(T)$ We will use it in the next section to derive complexity results on reasoning with constrained default logic The characterization requires the notion of a generating default introduced in [Schaub 1992][3]

Definition 3 1 ([Schaub 1992]) Let (D,W) be a default theory and S and T sets of formulas The set of *generating defaults* for (S,T) with respect to D is defined as

$$GD_D^{(S\ T)} = \left\{ \frac{\alpha\ M\beta}{\gamma} \in D \quad \alpha \in S\ T \cup \{\beta,\gamma\} \not\vdash \bot \right\}$$ □

Schaub [1992] proved the following properties of generating defaults

Theorem 3 1 Let $(S\ T)$ be a *constrained extension of a default theory* (D,W) Then

1 $S = Cn(W \cup c(GD_D^{(S\ T)}))$

2 $T = Cn(W \cup c(GD_D^{(S\ T)}) \cup j(GD_D^{(S\ T)}))$

3 There is an enumeration $\delta_1, \delta_2,$ of the defaults in $GD_D^{(S\ T)}$ such that for every $i = 1, 2,$ $W \cup c(\{\delta_1, \quad ,\delta_{i-1}\}) \vdash p(\delta_i)$ □

This theorem implies the following useful corollary

Corollary 3 2 Let (S,T) be a *constrained extension of a default theory* (D,W) Then $S = Cn^{Mon(C D_D^{(S\ T)})}(W)$ □

To present our characterization of the operator $\Upsilon(T)$ we will need two more notions

[3] The notions of a generating default and a reduct used in this paper are different from the standard ones [Reiter, 1980 Gelfond and Lifschitz, 1988] They are tailored specifically to the needs of constrained default logic

Definition 3 2 We define the *reduct* of a set of defaults D with respect to a theory T as

$$D_T = \left\{ \frac{\alpha}{\gamma} \quad \frac{\alpha\ M\beta}{\gamma} \in D, T \cup \{\beta,\gamma\} \not\vdash \bot \right\}$$

We define the operator C by letting

$$C(D,S,T) = \left\{ \beta \wedge \gamma \quad \frac{\alpha\ M\beta}{\gamma} \in GD_D^{(S\ T)} \right\}$$ □

These concepts allow us to give a constructive description of the operator $\Upsilon(T)$

Theorem 3 3 Let $(D\ W)$ be a *default theory and let* T *be a propositional theory* Then

$$\Upsilon(T) = (Cn^{D_T}(W), Cn(W \cup C(D, Cn^{D_T}(W)\ T)))$$

Proof According to Definition 2 3, $\Upsilon(T)$ is the pair of smallest sets of formulas (S', T') satisfying **CE1–CE3** It follows from **CE3** that S' is closed under inference rules from D_T Thus by the definition of the operator $Cn^A()$, $S' = Cn^{D_T}(W)$ satisfies **CE1–CE3** We need to prove that $T' = Cn(W \cup C(D\ Cn^{D_T}(W), T))$ together with $S' = Cn^{D_T}(W)$ satisfies **CE1–CE3**

We will first prove that $Cn^{D_T}(W) \subseteq Cn(W \cup C(D\ Cn^{D_T}(W), T))$ Let $\varphi \in Cn^{D_T}(W)$ Then φ has a proof from W in $PC + D_T$, that is, there is a finite sequence $\varphi_1, \quad ,\varphi_n = \varphi$ such that for every i $1 \le i \le n$, at least one of the following conditions holds

1 $\varphi_i \in W$ or φ_i is a substitution instance of an axiom of propositional logic

2 For some $j,k < i$, φ_i follows from φ_j and φ_k by modus ponens

3 For some $j < i$, the rule $\frac{\varphi_j}{\varphi}$ belongs to D_T

We will prove by induction on the length of this proof that $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W)\ T))$ If $\varphi \in W$ or φ is a substitution instance of an axiom of propositional logic then it is clear that $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W), T))$ If φ follows from φ_j and φ_k by modus ponens then, by the inductive hypothesis, $\varphi_j, \varphi_k \in Cn(W \cup C(D, Cn^{D_T}(W), T))$ It follows that $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W), T))$ If φ was obtained by applying an inference rule $\frac{\psi}{\varphi}$ from D_T then $\psi \in Cn^{D_T}(W)$ Hence, there is a default $\frac{\psi\ M\beta}{\varphi} \in D$ such that $\psi \in Cn^{D_T}(W)$ and $T \cup \{\beta,\varphi\} \not\vdash \bot$ It follows that $\beta \wedge \varphi \in C(D, Cn^{D_T}(W), T)$ and, consequently, $\varphi \in Cn(W \cup C(D, Cn^{D_T}(W)\ T))$ Thus, $Cn^{D_T}(W) \subseteq Cn(W \cup C(D, Cn^{D_T}(W), T))$ and **CE1** holds

The theory $Cn(W \cup C(D, Cn^{D_T}(W), T))$ is closed under propositional provability, so **CE2** holds To prove **CE3**, let us consider a default $\frac{\alpha\ M\beta}{\gamma}$ such that $\alpha \in S' = Cn^{D_T}(W)$ and $T \cup \{\beta,\gamma\} \not\vdash \bot$ It follows from the definition of $C(D,S,T)$ that $\beta \wedge \gamma \in C(D, Cn^{D_T}(W), T) \subseteq$

$Cn(W \cup C(D, Cn^{D_T}(W), T))$ Hence, the pair $(Cn^{D_T}(W), Cn(W \cup C(D, Cn^{D_T}(W), T)))$ satisfies **CE1 –CE3**

Let us assume now that a pair of theories (S', T') satisfies **CE1–CE3** Then it is easy to see that $Cn^{D_T}(W) \subseteq S'$ By **CE1**, $W \subseteq T'$ Let us consider a formula $\beta \wedge \gamma \in C(D, Cn^{D_T}(W), T)$ Then there is a default $\frac{\alpha M \beta}{\gamma} \in D$ such that $\alpha \in Cn^{D_T}(W) \subseteq S'$ and $\Gamma \cup \{\beta, \gamma\} \not\vdash \perp$ It follows from **CE3** that $\beta \wedge \gamma \in T'$ Hence, $C(D, Cn^{D_T}(W), T) \subseteq T'$ Since, by **CE2**, $T' = Cn(T')$ then $Cn(W \cup C(D, Cn^{D_T}(W), T)) \subseteq T'$ and we are done $\quad\square$

As a corollary, we get a characterization of constrained extensions

Corollary 3 4 *(S T) is a constrained extension of a default theory (D, W) if and only if*

$$T = Cn(W \cup C(D, Cn^{D_T}(W), T)) \qquad (2)$$

and $S = Cn^{D_T}(W)$ $\quad\square$

This corollary makes it explicit that constrained default logic works in two stages In the first stage, possible sets of assumptions (constraints as referred to in [Schaub, 1992]) are established as solutions of the fixpoint equation (2) Then, each of them uniquely determines the corresponding proper constrained extention

The fixpoint equation (2) implies that all assumptions (theory T) needed to support a proper constrained extension S in constrained default logic must be "reprovable" This might be regarded as a weakness of constrained default logic In all versions of default logic it is required that formulas in the extensions have justifications (in terms of proofs from W by means of applicable defaults) But it is arguable whether the same should be required of assumptions "β is possible" which make defaults applicable

Corollary 3 4 allows us to strengthen a result by Schaub on pairwise maximality of constrained extensions of a default theory

Theorem 3 5 *If (S T) and (S', T') are constrained extensions of a default theory (D, W) and $T \subseteq T'$, then $T = T'$ and $S = S'$ (in particular, "T parts" of constrained extensions form an antichain)*

Proof First, observe that if (S', T') is a constrained extension of (D, W), then T' is inconsistent if and only if W is inconsistent Hence, if T' is inconsistent then T is inconsistent as well and, consequently, $T = T'$ and $S = S'$ Assume then that T' is consistent

Since (S, T) is a constrained extension of (D, W), by Corollary 3 2 we have $S = Cn^{Mon(GD_D^{(S T)})}(W)$ Moreover, for every default $\frac{\alpha M \beta}{\gamma} \in GD_D^{(S T)}$, $\beta \wedge \gamma \in T$ and, since $T \subseteq T'$, $\beta \wedge \gamma \in T'$ Since T' is consistent, it follows that $Mon(GD_D^{(S T)}) \subseteq D_{T'}$ Hence,

$$S = Cn^{Mon(GD_D^{(S T)})}(W) \subseteq Cn^{D_{T'}}(W) = S'$$

Schaub [1992] proved that if (S, T) and (S', T') are constrained extensions of a default theory and $S \subseteq S'$ and $T \subseteq T'$, then $S = S'$ and $T = T'$ $\quad\square$

Theorem 3 5 implies bounds on the number of constrained extensions of a default theory

Corollary 3 6 *Let D be a set consisting of n defaults Then for every $W \subseteq \mathcal{L}$ the default theory (D, W) has at most*

$$\binom{n}{\lceil n/2 \rceil}$$

constrained extensions

Proof Theorem 3 5 implies that the family $\mathcal{T} = \{T \ (S, T)$ is a constrained extension of $(D, W)\}$ is an antichain By Theorem 3 1, each set $T \in \mathcal{T}$ is determined by the set $GD_D^{(S T)}$ Since $\mathcal{T}$ is an antichain, the family $\{GD_D^{(S T)} \ T \in \mathcal{T}\}$ is also an antichain It is well known that the size of the largest antichain in the algebra of subsets of an n-element set has at most $\binom{n}{\lceil n/2 \rceil}$ elements Hence, the assertion follows $\quad\square$

The second characterization of constrained extensions that we present in this section is closely related to the property of semi-monotonicity of constrained default logic It exploits the fact that constrained extensions can be produced by processing defaults according to any well-ordering This is very similar to the corresponding property of normal default theories in the logic of Reiter (Theorem 4 3, [Marek and Truszczynski, 1993]) In fact, our characterization of constrained default logic provides an alternative argument that for normal default theories constrained and standard default logics coincide

We assume that the set of the atoms of our language $\mathcal{L}$ is denumerable Consequently, the set of all defaults over the language $\mathcal{L}$ is denumerable

Let (D, W) be a default theory and $\prec$ a well-ordering of D We define an ordinal $\eta_\prec$ For every ordinal $\xi < \eta_\prec$ we define a set of defaults AD_ξ and a default d_ξ We also define a set of defaults $AD_\prec$ We proceed as follows If the sets $AD_\xi, \xi < \alpha$, have been defined but $\eta_\prec$ has not been defined then

1 If there is no default $d \in D \setminus \bigcup_{\xi < \alpha} AD_\xi$ such that

    (a) $W \cup c(\bigcup_{\xi < \alpha} AD_\xi) \cup j(\bigcup_{\xi < \alpha} AD_\xi) \cup j(d) \cup \{c(d)\}$ is consistent, and

    (b) $W \cup c(\bigcup_{\xi < \alpha} AD_\xi) \vdash p(d)$,

    then $\eta_\prec = \alpha$

2 Otherwise, define d_α to be the $\prec$-least default $d \in D \setminus \bigcup_{\xi < \alpha} AD_\xi$ such that the conditions (a) and (b) above hold Then set $AD_\alpha = \bigcup_{\xi < \alpha} AD_\xi \cup \{d_\alpha\}$

When the construction terminates, put $AD_\prec = \bigcup_{\xi < \eta_\prec} AD_\xi$, $S_\prec = Cn(W \cup c(AD_\prec))$ and $T_\prec = Cn(W \cup c(AD_\prec) \cup j(AD_\prec))$

This construction has the following property

Theorem 3 7 *Let (D, W) be a default theory and $\prec$ a well ordering of D Then $(S_\prec, T_\prec)$ is a constrained extension of (D, W) and $AD_\prec = GD_D^{(S_\prec T_\prec)}$*

Proof It is easy to see that if W is inconsistent then $AD_\prec = \emptyset$ and $S_\prec = T_\prec = Cn(W) = \mathcal{L}$ Thus, the asser tion follows Assume now that W is consistent Then for every $\xi < \eta_\prec$, $W \cup c(AD_\xi) \cup j(AD_\xi)$ is consistent Consequently, $W \cup c(AD_\prec) \cup j(AD_\prec)$ is consistent Thus, $T_\prec$ is consistent

Let us consider a default $d = \frac{\alpha \, M\beta}{\gamma} \in AD_\prec$ We have $\beta \in j(AD_\prec)$ and $\gamma \in c(AD_\prec)$ Thus, $\{\beta, \gamma\} \subseteq T_\prec$ and, since $T_\prec$ is consistent, $T_\prec \cup \{\beta, \gamma\}$ is consis tent Moreover, since $d \in AD_\prec$, for some $\xi < \eta_\prec$, $W \cup c(\bigcup_{\lambda < \xi} AD_\lambda) \vdash \alpha$ Consequently, $\alpha \in S_\prec$ Hence, $d \in GD_D^{(S_\prec T_\prec)}$

Consider now a default $d = \frac{\alpha \, M\beta}{\gamma} \in D \setminus AD_\prec$ Then one of the conditions (a) (b) above fails for d, that is, either $T_\prec \cup \{\beta, \gamma\}$ is inconsistent or $S_\prec \not\vdash \alpha$ Hence, $d \notin GD_D^{(S_\prec T_\prec)}$ Thus, $AD_\prec = GD_D^{(S_\prec T_\prec)}$

The last equality implies that

$$
\begin{aligned}
T_\prec &= Cn(W \cup c(GD_D^{(S_\prec T_\prec)}) \cup j(GD_D^{(S_\prec T_\prec)})) \\
&= Cn(W \cup \{\beta \wedge \gamma \; \frac{\alpha \, M\beta}{\gamma} \in GD_D^{(S_\prec T_\prec)}\}) \\
&= Cn(W \cup C(D, S_\prec, T_\prec))
\end{aligned}
$$

Thus to end the proof we need to show that $S_\prec = Cn^{D_{T_\prec}}(W)$ Let us notice that the condition (b) above implies that $Cn^{Mon(AD_\prec)}(W) = Cn(W \cup c(AD_\prec)) = S_\prec$ Thus, $S_\prec = Cn^{Mon(GD_D^{(S_\prec T_\prec)})}(W)$ Let $A = \{\frac{\alpha \, M\beta}{\gamma} \in D \quad T_\prec \cup \{\beta \, \gamma\} \not\vdash \bot\}$ Then $D_{T_\prec} = Mon(A)$ and $GD_D^{(S_\prec T_\prec)} = \{d \in A \quad p(d) \in S_\prec\}$ Hence $GD_D^{(S_\prec T_\prec)} \subseteq A$ and

$$
S_\prec = Cn^{Mon(GD_D^{(S_\prec T_\prec)})}(W)
$$

$$
\subseteq Cn^{Mon(A)}(W) = Cn^{D_{T_\prec}}(W)
$$

Thus, to show that $S_\prec = Cn^{D_{T_\prec}}(W)$, we need to prove that $Cn^{D_{T_\prec}}(W) \subseteq Cn^{Mon(GD_D^{(S_\prec T_\prec)})}(W)$ To this end, let us consider a formula $\varphi \in Cn^{D_{T_\prec}}(W)$ The for mula φ has a proof from W in $PC + D_{T_\prec}$ One can prove by induction on the length of this proof that $\varphi \in Cn^{Mon(GD_D^{(S_\prec T_\prec)})}(W)$ (due to space restrictions, we omit the details of the argument) □

The converse result is also true

Theorem 3 8 *Let (S, T) be a constrained extension of a default theory (D, W) Then for any well ordering $\prec$ of D such that defaults from $GD_D^{(S T)}$ precede all other defaults, $AD_\prec = GD_D^{(S T)}$, $S_\prec = S$ and $T_\prec = T$* □

This result will be proved in a full version of the paper

4 Computational aspects of constrained default logic

Corollary 3 4 allows us to design an algorithm for com puting all constrained extensions for a finite default the ory (D, W) Let us consider a set of defaults $A \subseteq D$ Let $S = Cn^{Mon(A)}(W)$, $B = \{d \in A \quad p(d) \in S\}$, $C = \{\beta \wedge \gamma \; \frac{\alpha \, M\beta}{\gamma} \in B\}$, $T = Cn(W \cup C)$ and $E = \{\frac{\alpha \, M\beta}{\gamma} \in D \quad T \cup \{\beta, \gamma\} \not\vdash \bot\}$ It is easy to see that if $E = A$ then (S, T) is a constrained extension for (D, W) Indeed, $D_T = Mon(E)$ Thus, if $E = A$ then $Mon(A) = D_T$ and $S = Cn^{D_T}(W)$ Moreover, $B = GD_D^{(S T)}$ Hence, $C = C(D S, T) = C(D, Cn^{D_T}(W), T)$ and $T = Cn(W \cup C) = Cn(W \cup C(D, Cn^{D_T}(W), T))$ Thus, (S, T) is a constrained extension for (D, W) Con sidering all sets $A \subseteq D$, we will get all constrained exten sions of (D, W) Indeed, if (S, T) is a constrained exten sion then it will be found by considering $A = \{\frac{\alpha \, M\beta}{\gamma} \in D \quad T \cup \{\beta, \gamma\} \not\vdash \bot\}$ Thus we have the following algo rithm

For every $A \subseteq D$

1 compute U such that $Cn(U) = Cn^{Mon(A)}(W)$ (the theory $Cn^{Mon(A)}(W)$ is infinite however U is finite see [Marek and Truszczynski 1993]),

2 compute $B = \{d \in A \quad U \vdash p(d)\}$ and let $C = \{\beta \wedge \gamma \; \frac{\alpha \, M\beta}{\gamma} \in B\}$,

3 compute $E = \{\frac{\alpha \, M\beta}{\gamma} \in D \quad W \cup C \cup \{\beta, \gamma\} \not\vdash \bot\}$,

4 if $E = A$ then output $(Cn(U), Cn(W \cup C))$ as a constrained extension of (D, W)

The complexity of the above algorithm is determined by the number of calls to a propositional consistency checking procedure Assume that the number of defaults in D is n Given $A \subseteq D$, we need at most n^2 calls to such a procedure to compute U at most n calls to compute B and n calls to compute E Hence, for every $A \subseteq D$, we need $O(n^2)$ calls to a propositional consistency checking procedure, and $O(n^2 2^n)$ calls to such a procedure for the whole algorithm

We have the following complexity result (see [Garey and Johnson, 1979] for a discussion of complexity classes)

Theorem 4 1 *The following problems*

IN-SOME *Given a finite default theory (D, W) and a formula φ, decide if φ is in some constrained exten sion for (D, W)*

NOT-IN-ALL *Given a finite default theory (D, W) and a formula φ, decide if there is a constrained extension for (D, W) not containing φ,*

are Σ_2^P-complete The problem

IN-ALL *Given a finite default theory (D, W) and a formula φ decide if φ is in all constrained extensions of (D, W),*

is Π_2^P-complete

Proof To verify that a formula φ belongs to some (does not belong to all, resp) constrained extensions of (D, W), we can nondeterministically guess a set of defaults $A \subseteq D$, verify that $(Cn(U), Cn(W \cup C))$ is a constrained extension for (D, W) (U and C are as defined in the above algorithm) and verify that $U \vdash \varphi$ ($U \not\vdash \varphi$, resp), what requires one more call to a propositional consistency checking procedure It follows that the problems IN-SOME and NOT IN ALL are in Σ_2^P Since the problem NOT-IN-ALL is in Σ_2^P, the problem IN-ALL is in Π_2^P Observe next that S is an extension for a normal default theory if and only if (S, S) is a constrained extension for this theory (see [Schaub, 1992]) Hence, the hardness of all three problems in their respective complexity classes follows from the fact that the problems IN-SOME, NOT-IN-ALL and IN-ALL for extensions of normal default theories are Σ_2^P-hard and Π_2^P-hard, respectively (see [Gottlob, 1992]) □

5 Connections between constrained and rational default logics

First, we will show that rational extensions determine constrained extensions That is, we will show that for every rational extension S of a default theory, there is T such that (S, T) is a constrained extension of this theory

Theorem 5 1 *Let E be a rational extension for a default theory (D, W) and let $A \in \mathcal{MA}(D, W, E)$ be such that $E = Cn^{Mon(A)}(W)$ $(= Cn(W \cup c(A)))$ Let $C = Cn(E \cup j(A))$ $(= Cn(W \cup c(A) \cup j(A)))$ Then (E, C) is a constrained extension of (D, W)*

Proof We need to show that $\Upsilon(C) = (E, C)$ We will prove first that pair (E, C) satisfies **CE1**-**CE3** with respect to C The conditions **CE1** and **CE2** are obviously satisfied Now let a default $d = \frac{\alpha M \beta}{\gamma}$ be such that $\alpha \in E$ and $C \cup \{\beta, \gamma\}$ is consistent Since $C \cup \{\beta, \gamma\}$ is consistent, then $E \cup j(A \cup \{d\}) = E \cup j(A) \cup \{\beta\}$ is consistent Moreover, since $p(A) \subseteq Cn^{Mon(A)}(W) = E$ and $\alpha \in E$, we have $p(A \cup \{d\}) = p(A) \cup \{\alpha\} \subseteq E = Cn^{Mon(A)}(W) \subseteq Cn^{Mon(A \cup \{d\})}(W)$ Thus, $(A \cup \{d\}) \in A(D, W, E)$ By the maximality of A, $d \in A$ Hence, $\gamma \in c(A) \subseteq E$ and $\{\beta, \gamma\} \subseteq c(A) \cup j(A) \subseteq C$ It follows that $\gamma \in E$ and $\beta \wedge \gamma \in Cn(C) = C$ and **CE3** holds

Now, we need to prove that if a pair (S, T) satisfies **CE1**-**CE3** with respect to C then $E \subseteq S$ and $C \subseteq T$ Let us notice that, since $c(A) \cup j(A) \subseteq C$, for $d = \frac{\alpha M \beta}{\gamma} \in A$, $C \cup \{\beta, \gamma\} = C$ By **AS1** $j(A) = \emptyset$ or $j(A) \cup E$ is consistent Thus, we have either $C = Cn(E \cup j(A))$ is consistent or $A = \emptyset$ (let us recall that we do not consider justification-free defaults) In both cases for $d = \frac{\alpha M \beta}{\gamma} \in A$, $C \cup \{\beta, \gamma\}$ is consistent

Thus, **CE3** implies that S is closed under inference rules from $Mon(A)$ Moreover, by **CE1**, $W \subseteq S$ and, by **CE2**, S is closed under propositional provability It follows that $E = Cn^{Mon(A)}(W) \subseteq S$ We have $p(A) \subseteq E \subseteq S$ So, it follows from **CE3** and **CE2** that $j(A) \cup c(A) \subseteq T$ Since, by **CE1**, $W \subseteq T$ and, by **CE2**, $T = Cn(T)$, then $C \subseteq T$ and we are done □

The converse statement is not true Every default theory has at least one constrained extension [Schaub, 1992] and there are default theories that do not have rational extensions [Mikituk and Truszczynski, 1993] We will show that for seminormal default theories an exact one-to-one correspondence between rational and constrained extensions can be established But first we will prove an auxiliary result

Theorem 5 2 *Let (S, T) be a constrained extension of a default theory (D, W) Then $GD_D^{(S, T)} \in A(D, W, S)$*

Proof It follows from Theorem 3 1 that $T = Cn(S \cup j(GD_D^{(S, T)}))$ If T is consistent then $j(GD_D^{(S, T)}) \cup S$ is also consistent If T is inconsistent then it follows from the definition of $GD_D^{(S, T)}$ that $GD_D^{(S, T)} = \emptyset$, so $j(GD_D^{(S, T)}) = \emptyset$ Thus, in both cases **AS1** holds

According to the definition of $GD_D^{(S, T)}$ $p(GD_D^{(S, T)}) \subseteq S$ By Corollary 3 2, $S = Cn^{Mon(GD_D^{(S, T)})}(W)$ Hence, **AS2** also holds □

Theorem 5 3 *Let (S, T) be a constrained extension of a seminormal default theory (D, W) Then S is a rational extension of (D, W) and $GD_D^{(S, T)} \in \mathcal{MA}(D, W, S)$*

Proof By Theorem 5 2, $GD_D^{(S, T)} \in A(D, W, S)$ and, by Corollary 3 2 $S = Cn^{Mon(GD_D^{(S, T)})}(W)$ Thus we need to prove the maximality of $GD_D^{(S, T)}$ in $A(D, W, S)$ only Since every default in D is seminormal, we can rewrite the definitions of $GD_D^{(S, T)}$, D_T and $C(D, S, T)$ as

$$GD_D^{(S, T)} = \left\{ \frac{\alpha M \beta}{\gamma} \in D \quad \alpha \in S, T \cup \{\beta\} \not\vdash \bot \right\},$$

$$D_T = \left\{ \frac{\alpha}{\gamma} \quad \frac{\alpha M \beta}{\gamma} \in D, T \cup \{\beta\} \not\vdash \bot \right\},$$

$$C(D, S, T) = \left\{ \beta \quad \frac{\alpha M \beta}{\gamma} \in GD_D^{(S, T)} \right\} = j(GD_D^{(S, T)})$$

Let us denote $GD_D^{(S, T)}$ by A Since, by Corollary 3 4 $S = Cn^{D_T}(W)$ and $T = Cn(W \cup C(D, Cn^{D_T}(W), T))$ we get $T = Cn(W \cup j(A))$

Let us consider now a default $d = \frac{\alpha M \beta}{\gamma} \in D \setminus A$ We have either $\alpha \notin S = Cn^{Mon(A)}(W)$ or $T \cup \{\beta\}$ is inconsistent If $T \cup \{\beta\}$ is inconsistent then $W \cup j(A) \cup \{\beta\} = W \cup j(A \cup \{d\})$ is inconsistent and the set $A \cup \{d\}$ does not satisfy **AS1** (since d is seminormal, $j(\{d\}) \neq \emptyset$) If $\alpha \notin Cn^{Mon(A)}(W)$ then $Cn^{Mon(A \cup \{d\})}(W) = Cn^{Mon(A)}(W)$ and $p(A \cup \{d\}) = p(A) \cup \{\alpha\} \not\subseteq Cn^{Mon(A \cup \{d\})}(W)$, so $A \cup \{d\}$ does not satisfy **AS2** By Proposition 4 2 from an

extended version of [Mikitiuk and Truszczynski, 1993], $A \in \mathcal{MA}(D, W, S)$ and we are done □

Schaub ([1992]) proved that every default theory has a constrained extension, so we have the following corollary (proved first by other methods in an extended version of [Mikitiuk and Truszczynski, 1993])

Corollary 5 4 *Every seminormal default theory has a rational extension* □

6 The case of justification-free defaults

We close this paper with a remark that constrained default logic can be extended to cover the case of default theories that may contain justification-free defaults To this end, one has to replace **CE3** by the following two conditions

CE3$_1$ For any default $\frac{\alpha M \beta}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\beta, \gamma\}$ is consistent then $\gamma \in S'$ and $\beta \wedge \gamma \in T'$

CE3$_2$ For any default $\frac{\alpha}{\gamma} \in D$, if $\alpha \in S'$ and $T \cup \{\gamma\}$ is consistent then $\gamma \in S'$

The definitions of $GD_D^{(S,T)}$, D_T and $C(D, S, T)$ must be modified in the same way Under such modifications all results presented in this paper, except for Theorem 5 1 remain true Moreover, the following, slightly modified version of Theorem 5 1 holds (the assumption of consistency of a rational extension is added)

Theorem 6 1 *Let E be a consistent rational extension for a default theory (D, W) and let $A \in \mathcal{MA}(D, W, E)$ be such that $E = Cn^{Mon(A)}(W) \ (= Cn(W \cup c(A)))$ Let $C = Cn(E \cup j(A)) \ (= Cn(W \cup c(A) \cup j(A)))$ Then (E, C) is a constrained extension of (D, W)*

Proofs of these results will be included in a full version of the paper

7 Conclusions

In this paper we showed that constrained and rational default logic* are closely related While Reiter s default logic and constrained default logic coincide on the class of normal default theories, rational and constrained default logics coincide on a much wider class of seminonnal default theories

We showed that basic problems of reasoning with constrained extensions are complete for the second level of the polynomial hierarchy (with the exception of the existence of an extension problem, which is trivially in P) We also proposed algorithms to compute constrained extensions

Constrained default logic was originally introduced onlj for default theories without justification-free defaults In the paper, we proposed a modification of the original definition of Schaub, which allows for defaultb to be justification-free Under our definition, all major properties of constrained default logic remain true

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