On Bimodal Nonmonotonic Logics and Their Ummodal and Nonmodal Equivalents

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Abstract

We study a bimodal nonmonotonic logic MBNF suggested in [Lifschitz, 1994] as a generalization of a number of nonmonotonic formalisms We show first that it is equivalent to a certain nonmodal system involving rules of a special kind Next, it is shown that the latter admits a modal representation that uses only one modal opera tor the operator of belief Moreover, under this translation the models of MBNF correspond to "expansions of the associated modal nonmonotonic logic Finally we show that, as far as such models are concerned, MBNF is redunhle to nonmodal default consequence relations from [Bochman, 1994] These results have general consequences concerning relationship between different formalizations of nonmonotonic reasoning

Introduction

In many respects the logic of belief and negation as fail ure (MBNF) suggested in [Lifschitz 1994] can be seen as a most general formalism for nonmonotonic reasoning that provides a unifying framework both for logic programming and for such nonmonotonic systems as default and autoepistemic logic In this paper WE consider the relation of MBNF to other nonmonotonic formalisms Our main results are twofold On the one hand, MBNF is shown to be reducible to usual unimodal nonmono tonic logics On the other hand we also show that it can be reduced to nonmodal default consequence relations from [Boohman, 1993 1994]

J he paper is organized as follows We show first that MBNF is equivalent to a certain nonmodal system called *A* BNF-consequence relation that involves rules or sequents of a special kind In this framework we define a counterpart of Lifschitz models and the notion of an L-expansion that can be cor odered as their 'core' In addition we introduce the notion of _in L-extension that also turns out to correspond to well known objects from other nonmonotonic formalisms NExt we show that BNF-consequence relations admit a modal representation that uses only one modal operator the operator of belief Moreover under this translation, L-expansions correspond to S-expansions (in the sense of [Marek *et al* 1993]) of the associated modal nonmonotonic logic Finally we show that as far as Lifschitz models are concerned, MBNF is reducible to nonmodal default consequence relations from [Bochman, 1994]

These results have important implications for a general theory of nonmonotonic reasoning They show in effect, that many apparently different nonmonotonic formalisms are actually different representations of the same basic constructions and reasoning principles

1 MBNF and its reductions

The language of MBNF involves two independent modal operators, **B** for belief and **not** for negation as failure As the majority of nonmonotonic systems MBNF involves two components One is a (monotonic) host logical system the other is the notion of a model that generates the corresponding nonmonotonic inference The system is characterized semantically MBNF interpretations are triples of the form (I, S^b, S^n), where I is an ordinary interpretation (a set of atomic propositions) and S^b, S^n are sets of such interpretations Such triples are used to define the notion of truth as follows

(i) If F is an atom F is true in (I, S^b, S^n) iff $F \in I$

(ii) $\neg F$ is true in (I, S^b, S^n) iff F is not true in (I, S^b, S^n)

(iii) $F \wedge G$ is true in (I, S^b, S^n) iff F and G are both true in (I, S^b, S^n)

(iv) $\mathbf{B}F$ is true in (I, S^b, S^n) iff for all $J \in S^b$ F is true in (J, S^b, S^n)

(v) **not** F is true in (I, S^b, S^n) iff for some $J \in S^n$ F is not true in (J, S^b, S^n)

Note that there is nothing nonmonotonic in the above conditions, they can be seen as a semantic description of the host monotonic logical system for MBNF The nonmonotonic component of MBNF is embodied in the notion of a model A *model* of a theory T is defined by Lifschitz as a pair (I, S) such that T is true in (I, S, S) and there are no J, S' such that S' properly includes S and T is true in (J, S', S) This notion of a model gives rise to a corresponding notion of entailment, which is already nonmonotonic

To begin with, note that the semantic characterization of MBNF gives rise to natural *monotonic* notions of semantic entailment and equivalence. Thus, we will say that a set of propositions u *semantically entails* a proposition A if A is true in all MBNF-interpretations in which u is true. Similarly, two sets of propositions u and v will be called *semantically equivalent* if they are true in the same MBNF-interpretations. These notions of entailment and equivalence are stronger than similar notions defined in [Lifschitz, 1994], because they are defined in terms of *all* interpretations[1]. It should be clear, however, that sets of propositions that are equivalent in our sense are interchangeable in *all* contexts and constructions that are definable on the basis of MBNF-semantics, including in particular Lifschitz models.

Now, an important feature of MBNF is that any formula turns out to be strongly equivalent to a formula without nested occurrences of the modal operators (we will call such formulas *flat* ones), and the corresponding reduction can be accomplished using a simple and efficient procedure. Moreover, we have

Proposition 1.1 *Any set of MBNF-formulas is semantically equivalent to a set of flat MBNF clauses*

$$(1)\quad \mathbf{B}A_1 \wedge \ldots \wedge \mathbf{B}A_k \wedge \neg\mathbf{B}B_1 \wedge \ldots \wedge \neg\mathbf{B}B_l \wedge$$
$$\wedge \neg\,\mathbf{not}\,C_1 \wedge \ldots \wedge \neg\,\mathbf{not}\,C_m \wedge \mathbf{not}\,D_1 \wedge \ldots \wedge \mathbf{not}\,D_n \rightarrow A$$

where all A_i, B_i, C_i, D_i and A are objective formulas.

An objective representation of MBNF, described below, is based on an identification of the above flat clauses with rules of the form $a|b\ c|d \Vdash A$, where a, b, c, d are finite sets of objective propositions. To be more exact, a clause of the above form can be identified with a rule

$$A_1, \ldots, A_k|B_1, \ldots, B_l\ C_1, \ldots, C_m|D_1, \ldots, D_n \Vdash A$$

We will call such rules *BNF sequents*. Thus, a sequent $a|b\ c|d \Vdash A$ corresponds to a rule saying that if propositions from a are believed, propositions from b are not believed, propositions from c are not rejected by failure and propositions from d are rejected by failure, then we can infer that A is true. The following definition gives a characterization of such sequents corresponding to flat MBNF-clauses. $\vdash$ below denotes the ordinary classical consequence, $\bot$ the proposition false.

Definition 1.1 A *BNF-consequence relation* is a set of BNF-sequents satisfying the following conditions:

(Monotonicity)
$$\frac{a|b\ c|d \Vdash A}{a'|b'\ c'|d' \Vdash A}\quad \text{provided } a \subseteq a', b \subseteq b', c \subseteq c', d \subseteq d'$$

(Deductive Closure)
$$\frac{e \vdash A \qquad \{a|b\ c|d \Vdash E_i\}\ \forall E_i \in e}{a|b\ c|d \Vdash A}$$

(B-Factoring)
$$\frac{B, a|b\ c|d \Vdash A \quad a|b, B\ c|d \Vdash A}{a|b\ c|d \Vdash A}$$

(N-Factoring)
$$\frac{a|b\ B, c|d \Vdash A \quad a|b\ c|d, B \Vdash A}{a|b\ c|d \Vdash A}$$

(B-Closure)
$$\frac{a \vdash A \qquad a, A|b\ c|d \Vdash E}{a|b\ c|d \Vdash E}$$

(N-Closure)
$$\frac{c \vdash C \qquad a|b\ c, C|d \Vdash E}{a|b\ c|d \Vdash E}$$

(B-Consistency) $\qquad A|A\ \ | \Vdash \bot$

(N-Consistency) $\qquad |\ \ A|A \Vdash \bot$

Despite the number of conditions, BNF-consequence relation is a conceptually simple system. Note first that the above definition can be easily extended to infinite premise sets using the following *compactness requirement*: for any sets of propositions u, v, w and x,

$$u|v\ w|x \Vdash A \quad \text{if and only if} \quad a|b\ c|d \Vdash A,$$

for some finite a, b, c, d such that $a \subseteq u$, $b \subseteq v$, $c \subseteq w$ and $d \subseteq x$.

In what follows, we will denote by $\mathbb{C}\mathrm{n}$ the (four argument) provability operator corresponding to $\Vdash$. Let $\overline{u}$ denote the complement of a set u. Now, BNF consequence relations can be characterized in terms of the following notion of a model.

Definition 1.2 A triple (i, u, v) of sets of propositions will be called a *model* of $\Vdash$ if i is a maximal consistent set (a world) and $\mathbb{C}\mathrm{n}(u|\overline{u}\ v|\overline{v}) \subseteq i$.

Note that i is a deductively closed set. Moreover, (B-Closure) and (N-Closure) imply that if (i, u, v) is a model, then u and v must also be deductively closed sets. The following representation theorem gives a characterization of BNF-consequence relations in terms of their models.

Theorem 1.2 $a|b\ c|d \Vdash A$ *iff* $A \in i$ *for any model* (i, u, v) *such that* $a \subseteq u$, $b \subseteq \overline{u}$, $c \subseteq v$ *and* $d \subseteq \overline{v}$.

It is easy to see that *any* set of triples (i, u, v) of deductively closed sets generates a BNF-consequence relation in accordance with the above representation. Note also that models in our sense can be seen as syntactic counterparts of MBNF-interpretations: instead of sets of worlds, our models include sets of propositions that are true in them. In view of this fact, the above theorem can be used to show the correspondence between MBNF theories and BNF-consequence relations.

Recall that any MBNF theory T is semantically equivalent to some set T_1 of flat MBNF-clauses. Let $\Vdash_T$ denote the least BNF-consequence relation containing all sequents corresponding to clauses from T_1, and let F_s denote a flat MBNF-clause corresponding to a BNF-sequent s. The following theorem shows that $\Vdash_T$ is adequate for representing the 'flat content' of T.

Theorem 1.3 $s \in \Vdash_T$ *iff* T *entails* F_s

Given the correspondence between models and MBNF-interpretations, the following corollary is an immediate consequence of the theorem.

[1] Still, these stronger notions are implicit in [Lifschitz, 1994] in the form of the first-order entailment and equivalence between nonmodal reductions of MBNF-formulas obtained by replacing the two modal operators with quantifiers.

Corollary 1.4 *Models of $\Vdash_T$ exactly correspond to MBNF interpretations satisfying T.*

Thus, for any MBNF theory there is a BNF consequence relation that is in a strong sense equivalent to it. It is important to note that this correspondence between MBNF theories and BNF-consequence relations does not depend on the nonmonotonic component of MBNF, namely on Lifschitz notion of a model. In the next section we will consider such models in detail.

2 Lifschitz models

The following definition provides a characterization of Lifschitz models (or, in short, L-models) in the framework of BNF-consequence relations.

Definition 2.1 A pair $(\imath, u)$ will be called an *L-model* of $\Vdash$ if $(\imath, u, u)$ is a model and there is no model $(\jmath, u', u)$ such that $u' \subset u$.

By Corollary 1.4, this definition is adequate, since Lifschitz models of a theory T correspond to L-models of $\Vdash_T$. We consider now reformulations of the definition that would display the 'internal structure' of L-models.

A pair (u, v) of sets of propositions will be called a *bitheory* in $\Vdash$ if $u|\overline{u}\ \ v|\overline{v} \nVdash \bot$. Clearly, (u, v) is a bitheory if and only if there is a world $\imath$ such that $(\imath, u, v)$ is a model.

A bitheory (u, v) will be called *B-minimal* if there is no bitheory (u', v) such that $u' \subset u$. Now, it is easy to show that $(\imath, u)$ is an L-model if and only if

(i) $\mathbb{C}\mathrm{n}(u|\overline{u}\ \ u|\overline{u}) \subseteq \imath$,

(ii) (u, u) is a B-minimal bitheory.

For reasons that will become clear later, a set u will be called an *L-expansion* if (u, u) is a B-minimal bitheory. Thus, any L-model is a pair consisting of an L-expansion u and a maximal deductively closed set containing $\mathbb{C}\mathrm{n}(u|\overline{u}\ \ u|\overline{u})$. It is interesting to note that L-expansions correspond to what was termed models in MKNF, a preliminary version of MBNF given in [Lifschitz, 1991] (see below). In addition, L-expansions correspond also to *preferred models* of the bimodal logic GK suggested in [Lin and Shoham, 1992].

Remark. It is easy to show that GK and MBNF are equivalent (modulo notational variants) for flat modal theories. Moreover, as is noted in [Lin and Shoham, 1992], flat formulas are sufficient for determining the notion of a preferred model of GK. Thus, as far as intended models are concerned, the two systems can be seen as equivalent.

Now we consider some additional rules and conditions that can be imposed on BNF-consequence relations. The following rules are of a special interest:

(Nontriviality)	$\bot\|\ \ \| \Vdash \bot$
(L-Coherence)	$A\|\ \ \|A \Vdash \bot$
(G-Coherence)	$\dfrac{A, a\|\ \ c\|d \Vdash \bot}{a\|\ \ A, c\|d \Vdash \bot}$

(Nontriviality) requires our beliefs to be consistent, (L-Coherence) says that believed propositions cannot be negated by failure, while (G-Coherence) is a partial inverse of (L-Coherence); it says that if assuming that some propositions are negated by failure and some are not imply that a certain set of beliefs is inconsistent, then one of the propositions from this set can be negated by failure provided the rest of these propositions is believed.

A BNF-consequence relation will be called *nontrivial* if it satisfies (Nontriviality), *coherent* if it satisfies (L-Coherence) and *strongly coherent* if it satisfies both (L-Coherence) and (G-Coherence). Let $\Vdash^c$ be the least strongly coherent consequence relation containing $\Vdash$. Clearly, it is a consequence relation obtained from $\Vdash$ by simply adding the relevant rules. The following result shows that these rules preserve L-models.

Proposition 2.1 *$\Vdash^c$ has the same L models as $\Vdash$.*

As to (Nontriviality), it can be shown to preserve L-models with consistent L-expansions.

For coherent BNF-consequence relations, the conditions for L-models can be simplified.

Proposition 2.2 *For any coherent BNF c.r.*

(i) A consistent set u is an L-expansion if and only if

$$u = \{A \mid \emptyset|A\ \ u|\overline{u} \Vdash \bot\},$$

(ii) $(\imath, u)$ is an L-model iff u is an L-expansion and $\imath$ is a maximal consistent set that includes $\mathbb{C}\mathrm{n}(\emptyset|\emptyset\ \ u|\overline{u})$.

To conclude this section, we define still another nonmonotonic object that will play an important role in what follows. A bitheory (u, v) will be called *minimal* if there is no other bitheory (u', v') such that $u' \subseteq u$ and $v' \subseteq v$. Now, a set u will be called an *L-extension* of $\Vdash$ if (u, u) is a minimal bitheory.

Clearly, any L-extension is an L-expansion. However, the former notion is stronger: it embodies both the requirement of minimality of beliefs and the principle of maximality of negation as failure assignments[2]. As we will see, such objects correspond to well known constructs in other nonmonotonic formalisms. The following proposition gives a characterization of L-extensions for coherent consequence relations.

Proposition 2.3 *A consistent set u is an L extension of a coherent BNF-c.r. $\Vdash$ iff $u = \{A \mid\ \ |A\ \ |\overline{u} \Vdash \bot\}$.*

3 A unimodal re-presentation

In this section we are going to show that coherent BNF-consequence relations admit, in turn, a modal representation that involves only one modal operator, $\mathbf{B}$. It is obtained by identifying 'not' with $\mathbf{B}\neg\mathbf{B}$[3]. To be more exact, a sequent s of the form

$$A_1, \ldots, A_k|B_1, \ldots, B_l\ \ C_1, \ldots, C_m|D_1, \ldots, D_n \Vdash A,$$

is representable by the following formula that we will denote by $\bar{s}$:

(2) $\mathbf{B}A_1 \wedge \ldots \wedge \mathbf{B}A_k \wedge \neg\mathbf{B}B_1 \wedge \ldots \wedge \neg\mathbf{B}B_l \wedge$
$\wedge\neg\mathbf{B}\neg\mathbf{B}C_1 \wedge \ldots \wedge\neg\mathbf{B}\neg\mathbf{B}C_m \wedge \mathbf{B}\neg\mathbf{B}D_1 \wedge \ldots \wedge \mathbf{B}\neg\mathbf{B}D_n \rightarrow A$

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[2] Or the requirement of minimality of assumptions in the terminology of [Lin and Shoham, 1992].

[3] This identification was also used in [Lifschitz and Schwarz, 1993], though for a restricted class of theories.

In the literature on modal logics one can meet two commonly used notions of modal consequence that reflect two main notions of validity of propositions with respect to possible world models Let S be a normal modal logic A proposition *A* is called a *local S-consequence* of a set of propositions *u* (notation $u \vdash^l_S A$) if A is provable from u and theorems of § using modus ponens only *A is* called a *global* S *consequence* of u (notation $u \vdash^g_S A$), if A is provable from u and theorems of S using modus ponens and the necessitatis rule 4/LA In this study we will extensively use both these notions

Theorem 3 1 *Let* $\Vdash_S$ *be the least nontrtvial coherent conseqvence relation containing a set of sequents* S *and* S *the set of ummodal formulas corresponding to sequents from S Then a sequent s belongs to* $\Vdash_S$ *iff* $\tilde{S} \vdash^l_S \tilde{s}$ *for any normal modal logic S between* $KD4G$ *and* $KD4I$

Here KD4G denotes a modal logic corresponding to Kripke frames with transitive and directional accessibility relations (see [Bull and Segerherg, 1984]) while KDAI is a logic introduced in [Bochman, 1994] it is determined b> directional Knpke frames of depth ≤ 3

The theorem shows that the above ummodal translation is adequate for a representation of rohtrent BNF consequence relations Consequently, it is adequate for capturing th<- nonmonotonic MBNF-infcrence based on Lifschitz models Note, however that the translation itself does not depend on L-models and is adequate fnr any notion definable in terms uf (coherent) BNF consequence relations For example using the results from [Bochman, 1993], sets *u* satisfying the condition $\mathrm{Cn}_S(u|\overline{u}\ \emptyset|\emptyset) \subseteq u$ can be shown to correspond to objective subsets of stable sets containing S, while sets satisf\ing $u = \mathrm{Cn}_S(u|\overline{u}\ \emptyset|\emptyset)$ correspond to Moore's stable expansions of *S*

Remark Using the above results, we can perform a two-step translation of MBNF-formulas into ummodal ones, we first reduce an MBNF-formula to a flat one and then replace *all* occurrences *of* not in it *by B->B* Note, however, that this procedure cannot be shortened to a direct replacement of 'not in MBNF-formulas since the above ummodal translation does not account for the interaction between nested modal operators of MBNF C onsequently it is inadequate if applied to non flat MBNF-formulas

Thus, we have shown, in effect that MBNF can be returned to the family of usual modal nonmonotonic logics In the next section we will complete the picture by demonstrating that, under this reformulation, L-expansions correspond to modal S-expansions in the sense of [Marek *et* a/, 1993]

4 Biconsequence relations and L-expansions

BNF-sequents of the form $a|b\ \ c|d \Vdash \bot$ will be called $\bot$ *sequents* Such sequents correspond to *modalized* MBNF-formulas in which any propositional atom occurs in the scope of some modal operator Note that all current applications of MBNF (see e g , [inoue and Sakama, 1994, Lifschitz and Woo, 1992, Lifschitz and Schwarz, 1993]) use only formulas of this kind

Let us rewrite $\bot$-sequents as rules of a new kind as follows a sequent $a|b\ \ c|d \Vdash \bot$ will correspond to a rule

$$a\ \ d \Vdash b\ \ c$$

We will call these rules *bisequents* Such rules can be seen as representing entailment relations between *pairs* of sets of propositions In accordance with the original interpretation of BIVF-sequents, bisequents can be read as follows "If all propositions from *a* are believed and all propositions from *d* are negated by failure, then either one of the propositions from *b* should be believed or one of tht propositions from *c* should be negated b\ failure This interpretation also justifies the particular order of the four parameters chosen for representing the rule--

Definition 4 1 A set of bisequents will be called a *b1 consequence relation* if it satisifies the following condi tions and ʳules

$$\frac{a\ \ b \Vdash c\ \ d}{a'\ \ b' \Vdash c'\ \ d'} \quad \text{provided } a \subseteq a', b \subseteq b'\ c \subseteq c'\ d \subseteq d'$$

$$A\ \Vdash A \qquad\qquad A \Vdash\ A$$

$$\frac{a\ \ b \Vdash A, c\ \ d \quad A\ a\ \ b \Vdash c\ \ d}{a\ \ b \Vdash c\ \ d}$$

$$\frac{a\ \ b \Vdash c\ \ A, d \quad a\ \ A\ b \Vdash c\ \ d}{a\ \ b \Vdash c\ \ d}$$

$$\frac{a \vdash A \quad a, A\ \ b \Vdash c\ \ d}{a\ \ b \Vdash c\ \ d}$$

$$\frac{d \vdash D \quad a\ \ b \Vdash c\ \ d, D}{a\ \ b \Vdash c\ \ d}$$

It is eas> to see that the abovt conditions are simply reformulations of the corresponding conditions for BNF-consequence relations Note that by Theorem 1 2 *12*-btquents are determined by bitheories only Accords ingly, *models of a.* biconsequence relation can be defined as pairs *(u i)* such that $u\ \ \overline{v} \nVdash \overline{u}\ \ v$ Then we have

Theorem 4 1 *IfW-isa biconsequence relation then* $a\ \ b \Vdash c\ \ d$ *iff for any model (u, v) if* $a \subseteq u$ *and* b C T *then either* $c \cap u \neq \emptyset$ *or* $d \cap \overline{v} \neq \emptyset$

Clearly, an\ BNF-consequence relation contains a biconsequence sub-relation that is determined by its bitheones Moreover, these biconsequence subrt lations can be ihown to be *conservative* in the sense that thev prove all and only $\bot$-sequents that are provable in tht correspond ing BNF-consequence relation

Instead of L-models we now have only *It-expansions* defined as sets u such that (u u) is a B-minmial model while L-extensions are defined as minimal modi ls Then it is easy to see that L-expansions and I extensions of a BNF-consequence relation coincide with the corresponding objects of the associated biconsequence relation

The conditions of coherence strong coherence and nontnviality can be immediately transferred to bicon sequence relations (since they are formulated in terms of $\bot$-sequents), as before, they will preserve I-expansions Moreover, in this case the latter can he characterized as sets satisfying

$$u = \{A \mid\ \ \overline{u} \Vdash A\ \ u\},$$

while L-extensions will be characterized by

$$u = \{A \mid \ \overline{u} \Vdash A\ \}$$

The correspondence between BNF- and biconsequence relations implies that bisequents admit the same unimodal representation as $\bot$-sequents. Moreover, it can be shown that in this case a local modal consequence determined by any modal logic in an extended range KD4G–S4F would be appropriate for nontrivial coherent biconsequence relations.

The next theorem shows that a *global* modal consequence is adequate for representing strongly coherent biconsequence relations.

Theorem 4.2 *A sequent s belongs to the least nontrivial strongly coherent biconsequence relation containing* $\mathcal{S}$ *if and only if* $\tilde{S} \vdash^g_{\mathbb{S}} \tilde{s}$ *for any normal modal logic* $\mathbb{S}$ *between KD4G and S4F.*

The main result of this section is that under the unimodal translation L-expansions correspond to modal $\mathbb{S}$-expansions in the sense of [Marek *et al.*, 1993]. $St(u)$ below denotes the unique modal stable set having u as its objective subset.

Theorem 4.3 *If* $\Vdash^2_S$ *is the least strongly coherent biconsequence relation containing a set of bisequents S, then for any normal modal logic* $\mathbb{S}$ *in the range S4.2–S4F:*

- *(i) A consistent set u is an L-expansion in* $\Vdash^2_S$ *iff* $St(u)$ *is an* $\mathbb{S}$*-expansion of* $\tilde{S}$*.*
- *(ii) A consistent set u is an L-extension in* $\Vdash^2_S$ *iff* $St(u)$ *is a ground* $\mathbb{S}$*-expansion of* $\tilde{S}$*.*

Thus, L-expansions are representable as objective subsets of modal expansions, while L-extensions correspond in this sense to ground expansions. Another important fact that can be obtained from the proof of this theorem is that this correspondence between biconsequence relations and modal nonmonotonic logics is *reversible* in the sense that, for any modal theory T and any modal logic in the above range, $\mathbb{S}$-expansions of T are representable as L-expansions of some biconsequence relation.

4.1 A note on MKNF

An earlier version of MBNF was presented by Lifschitz in [Lifschitz, 1991]. The system was called MKNF because its first modal operator was termed an operator of knowledge. Though it involved the same semantic interpretation, the definition of a model was different. Without going into details here, it can be shown that these models are exactly L-expansions of BNF-consequence relations satisfying the following additional rule:

$$(Cut)\qquad \frac{a|b\ \ c|d \Vdash A \qquad A\ a|b\ \ c|d \Vdash B}{a|b\ \ c|d \Vdash B}$$

In the context of MBNF, the rule means that provable conclusions should be believed. In fact, the rule remedies one of the puzzling features of MBNF, namely that propositions provable on the basis of some set of beliefs, non-beliefs and negations by failure have, in turn, almost no impact on our beliefs or on what could be negated by failure. In this sense, MBNF is not a system of *introspective* reasoning.

Another possible constraint that follows from the informal interpretation of the first modal operator as an operator of knowledge is the following condition saying, in effect, that knowledge implies truth:

$$(Reflexivity)\qquad A|\ \ |\Vdash A$$

However, adding this condition would make any BNF-sequent equivalent to some $\bot$-sequent.

Proposition 4.4 *A BNF-consequence relation satisfies (Cut) and (Reflexivity) if and only if*

$$a|b\ \ c|d \Vdash A \quad\equiv\quad a|b\ \ A\ \ c|d \Vdash \bot$$

As a result, MKNF would be reducible to biconsequence relations.

5 Default consequence relations

By Proposition 2.2, the two components of an L-model are determined, in effect, by different subsystems of coherent BNF consequence relations. L-expansions are determined by sequents of the form $|A\ \ a|b \Vdash \bot$, while world components are determined by sequents of the form $|\ \ a|b \Vdash A$. In both cases we seek for consequences of negation as failure assignments: belief propositions that follow from such assignments characterize L-expansions, while objective consequences determine world components of L-models. Now, it turns out that both kinds of sequents generate default consequence relations in the sense of [Bochman, 1994].

Default consequence relations were defined in [Bochman, 1994] as consequence relations involving sequents of the form $a\ \ b \Vdash A$. Such sequents can be seen as special cases of BNF-sequents. Note, however, that there is actually a number of possibilities for embedding default sequents into BNF-sequents, and in this way different default consequence relations can be obtained.

A most interesting default consequence relation for our present study can be defined as follows:

$$a\ \ b \Vdash^a A \quad\equiv\quad |A\ \ a|b \Vdash \bot$$

When $\Vdash$ is coherent, this default consequence relation satisfies all the properties of an autoepistemic consequence relation as defined in [Bochman, 1994]. The latter has been shown to be adequate for representing autoepistemic logic ([Moore, 1985]), where Moore's stable expansions are representable as sets satisfying the condition $u = \mathrm{Cn}(u\ \ \overline{u})$. Such sets were termed *expansions* of the corresponding default consequence relation. Now, the following theorem shows that expansions of $\Vdash^a$ are exactly L-expansions of $\Vdash$.

Theorem 5.1 *If* $\Vdash$ *is a coherent BNF-consequence relation, then* $\vdash^a$ *is an autoepistemic consequence relation. Moreover, consistent expansions of* $\vdash^a$ *coincide with consistent L-expansions of* $\Vdash$*.*

The theorem can be seen as a generalization of Theorem 4.4 from [Lin and Shoham, 1992] giving a representation of autoepistemic logic in GK.

Another interesting default consequence relation arises when we treat BNF-sequents of the form $a|A\ \ |b \Vdash \bot$ as representing default sequents:

$$a\ \ b \Vdash^r A \quad\equiv\quad a|A\ \ |b \Vdash \bot$$

Such an interpretation corresponds in fact, to that used in [Lifschitz, 1994, Lin and Shoham, 1992] for representing default logic If ⊩ is strongly coheren ⊩^{r} is what is called in [Bochman 1994] a reflexive default consequence relation The latter has been shown to he adequate for representing default logic [Reiter 1980], where *extensions* correspond to sets satisfying the condition $u = \mathbb{C}n(\emptyset \ \overline{u})$ Now we have

Theorem 5 2 /f ⊩ is a *strongly coherent BNF consequence relation then* ⊩^{r} *is a reflexive default consequence relation Moreover consistent extensions of* ⊩^{r} *coincide with L extensions of* ⊩

1 he reader may notice an apparent discrepancy between the above two representation results especially if compared with the relevant results from [Lifschilz 1994 Lin and Shoham 1992] where extensions vvere actually shown to correspond to L-expansions as well as with the results from [Truszczynski 1991a 1991b] about im-- biddmg of default logic into modal nonmonotonic logics where exensions were shown to correspond to S-expansions[1] This discrepancy reveals an important though subtle general point about the correspondence between existing nonmonotonic formalisms

The above mentioned results may. cricle an impression of a broad equivalence between the relevant objects from different nonmonotonic formalisms 1 his Impression can even be strengthened with Iheorm 43 (1)above that establishes a correspondence between L-expansions in MBNF and S-expansions in modal nonmonotonic logics as well as with the result from [Bochman 1991] saving that S-expansions can be characterizedd precisely as extensions of *modal* default consequence relations Still the impression is slightly misleading In fact as far as only objective propositions are concerned there are two kinds of nonmonotonic objects here One kind includes I expansions of MBNF and objective kernels of S-expansions of modal nonmonotonic logics The other more specific, kind includes extensions of default logic objective subsets of ground S-expansions and S-extensions (cf Theorem 4 3 above and [Bochman 1991]) The difference between these two kinds of objects can be easily demonstrated if we notice for example that an L-expansion can be a subset of anothtr L-expansion, which is impossible e g for (L-)extensions

The following general result (cf also [Lifschitz and Woo, 1992]) sh ows that the distinct ion is due to the presence of positive orrurences of not in MBNF-formulas

Theorem 5 3 *Let i* $\Vdash_S$ *the least BNF consequence relation containing a set* S *of BNF-sequents of the form* $a|b \quad |d \Vdash A$ *Then any L.-expansion of* $\Vdash_S$ *is an L extension*

This result implies in particular that Truszczynski's translation of default theories generates only modal theories all expansions of which are ground Such a translation is not reversible—objective subsets of expansions are not representable, in general, as extensions of some default theory

[4]Note that the translation used by Truszczynski is actually a special case of the unimodal translation given above

Finally note that the following definition

$$a \ b \Vdash^c A \equiv | \ a|b \Vdash A$$

also determines, a default consequence relation This consequence relation involves BNF-sequents that are responsible for world components of L-models in the sense that, if u is an L expansion then $(\imath, u)$ is an L-model if and only if $\mathbb{C}n^c(u \ \overline{u}) \subseteq \imath$

Thus the main conclusion that can be made from the above results is that default consequence relations arc sufficiently expressive to capture the major nonmonotonic objects of MBNF In this sense MBNF is reducible to the former

6 Conclusions

I his study can be seen as a contribution to a (future) general theory of nonmonotonic reasoning (. orrespondencies established between logic programming default logic and various modal nonmonotonic logics including those, statt d above strongly indicate that all these formalisms give rise to essentially the same nonmonotonic constructions Moreover our results show that the rorre spondence between ihese formalisms can be extended to the h vcl of underlying *reasoning systems* and hence does not depend on particular nonmonotonic objects chosen

Among other nonmonotonic systems, MBNF is clearlv one of the most expressive formalisms: More ov< r as we have seen, it is even loo powerful in lis expressive capa bililies for the objects considered a& its intended models However tilt list of piausiblf nonmonotonic objeets is by no means closed and there are indications that this framework or its fragments (such as bicousequence n-la tion& introduced above), could be appropriate for inanv of them However this is a subject for another studv

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