Optimal Control Analysis of Computer Virus Transmission

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Abstract

Computer virus has become a global problem and affecting many industries both developed and the developing countries.In this study, a deterministic computer virus model is formulated incorporating removal devices. The basic properties of the model is studied and the reproduction number is calculated. The steady states are studied and found to be stable. We analyze different properties with parameter change by carrying out the sensitivity analysis of the model. Time optimal control is included and Pontryagin's Maximum Principle is used to characterize the all necessary condition for controlling the spread of computer virus. The most effective strategy for controlling computer virus is the combination of all the three controls. Graphical illustrations are presented to show the effects.

Keywords: Computer Virus; Optimal Control; Pontryagin's Maximum Principle; Removable Devices; Sensitivity Analysis

1 Introduction

The study of Computer Virus and its control has been a challenge over the years. The Computer virus is defined as a piece of software that contains malicious code, which could propagate, be installed and cause damage to computer data without the authorised permission or knowledge of the user [15, 16, 33].

In the industries, we observe that though the impact of the virus and the damages it has caused have been reduced moderately due to the increasing awareness by the public and high technological inventions. Despite this, the problem still persists [3, 13, 19, 24, 26, 35]. In general, the importance of mathematical models is that it helps to analyse the steady states, minimize the disease spread, control the disease outbreak and characterize the model propagation [17, 37].

Also, our modern financial institutions, culture, infrastructures and information and communication technology now depends mostly on computer networks and internet connectivities [27]. As the rate of dependence on computer networks increase, we observe that cyber attacks are also on the increase [31]. In order to avert this, or reduce considerably the cyber attack rate, there is a need to formulate deterministic models which would capture if not most but at least very important parameters such that there would be a control measure to which the computer virus could be spread [26].

Mathematical modeling in recent times has played a vital role of providing important insights into many processes including population behavior and their controls. Again, for some year now, it has also become an indispensable vehicle with which dynamical behaviors of many systems are understood such as computers virus so that the appropriate decision concerning the right interventions are undertaken.

For instance, Jin investigated the significance of applying epidemiological models in computer virus protection and prevention, and discussed their implication in developing anti-virus technologies and policies [15]. Also Lopez and Cipolatti introduced a simplified theoretical model to describe a virtual virus propagation process in a set of interacting computers. They also considered the propagation mechanisms which are those related to the reception of messages through internet as well as the ones concerning the simple exchange of files using recording devices as compact disks or the commonly used floppy disks [10]. Zhu *et al.* considered the effect of removable devices on the transmission of computer virus [37]. Chen

et al., presented a mathematical model, referred to as the Analytical Active Worm Propagation (AAWP) model, which characterized the propagation of worms that employ random scanning. They compared the model with the Epidemiological model and Weaver's simulator. Their results showed that the model characterized the spread of worms effectively [3]. Furthermore, Omair and Samir also analyzed the efficiency of antivirus software and crashing of the nodes due to virus attack [26].

Our main goal is to construct a mathematical model on computer virus transmission and incorporating removal devices with some control strategies. The paper is arranged as follows, in Section 2, we present the model formulation and carry out the stability analysis of the model. In Section 3, we perform sensitivity analysis of parameters. In Section 4, time dependent control is incorporated in the model and analytical solution of the controls. The numerical solutions are presented in Section 5. Finally conclusion is drawn in Section 6.

2 The Model

The model sub-divides the total Computer population, denoted by N, into sub-populations of Susceptible computers (S), Exposed computers (E), Infected computers (I), Recovered computers (R). We assume that computers can be infected through electronic mails and internet access. But computers are not contributing or infected internet network. Let Susceptible removable device be D_S and Infected removable device be D_I. So that $N = S + E + I + R$ and $D_N = D_S + D_I$. The diagram and differential equations are given in Figure 1 and the following:

$$\begin{aligned}
\frac{d}{dt}S &= \Lambda - \frac{\beta_2 D_I S}{D_N} - \beta_1 SI - dS + \eta R \\
\frac{d}{dt}E &= \frac{\beta_2 D_I S}{D_N} + \beta_1 SI - (d+\mu)E \\
\frac{d}{dt}I &= \mu E - (d+\gamma+\alpha)I \\
\frac{d}{dt}R &= \gamma I - (d+\eta)R \qquad (1)\\
\frac{d}{dt}D_S &= \Lambda_d - \frac{\beta_2 D_S I}{N} + \sigma D_I - d_2 D_S, \\
\frac{d}{dt}D_I &= \frac{\beta_2 D_S I}{N} - (d_2+\sigma)D_I.
\end{aligned}$$

The β_1, β_2 are the transmission probabilities of computer virus, while the terms σ is the ingestion rate and γ is the recovery rate while α virus induced computer death. Computer recruitment rate is Λ, μ is progression from exposed to infected class.

Lemma 1. *The closed set*

$$D = (S, E, I, R, D_S, D_I) \in \mathbb{R}^6_+ : N \leq \frac{\Lambda}{d}, D_N \leq \frac{\Lambda_d}{d_2}$$

is positively invariant and attracting for the model [8].

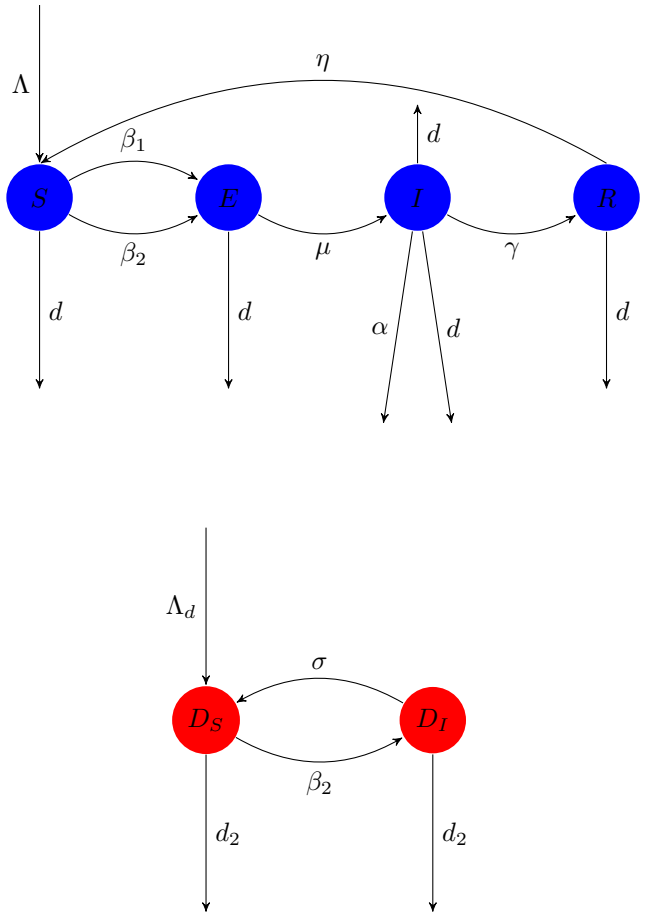

Figure 1: The computer virus transmission model diagram

Table 1: Computer virus transmission model notations

Parameters	Description
S	The number of Susceptible Computers.
E	The number of Exposed Computer.
I	The number of Infected Computers.
R	The number of Recovered Computers.
D_S	The number of Susceptible removable device.
D_I	The number of Infected Removable device.
Λ	Computer recruitment
Λ_d	Removable device recruitment
β_1	The infectivity contact rate at with network attacks occur.
β_2	The infectivity contact rate at with virus attacks removable device.
d	Natural death of Computer System.
d_2	Death rate of removable device.
α	Rate of Virus induced death.
γ	Recovery rate.
η	Waning rate of Computer.
μ	The Computer exposed rate.
σ	Ingestion rate.

Proof. Adding the first four equations and then the last two equations of the model, we have:

$$\begin{aligned}\frac{dN}{dt} &= \Lambda - dN - \alpha I \\ \frac{dD_N}{dt} &= \Lambda_d - d_2 D_N\end{aligned}$$

Since $\frac{dN}{dt} \leq \Lambda - dN$ and $\frac{dD_N}{dt} \leq \Lambda_d - d_2 D_N$, it follows that $\frac{dN}{dt} \leq 0$ and $\frac{dD_N}{dt} \leq 0$ if $N \geq \frac{\Lambda}{d}$ and $D_S \geq \frac{\Lambda_d}{d_2}$ respectively.

Hence by Comparison theorem in [12],

$$\begin{aligned}\frac{dN}{dt} + dN &\leq \Lambda \\ \frac{d}{dt}(Ne^{dt}) &\leq \Lambda e^{dt}\end{aligned}$$

$\Longleftrightarrow$

$$\begin{aligned}N(t) &\leq N(0)e^{-dt} + \frac{\Lambda}{d}(1 - e^{-dt}) \\ D_N(t) &\leq D(0)e^{-d_2 t} + \frac{\Lambda_d}{d_2}(1 - e^{-d_2 t})\end{aligned}$$

Therefore, $N(t) \leq \frac{\Lambda}{d}$ if $N(0) \leq \frac{\Lambda}{d}$ and $D_S(t) \leq \frac{\Lambda_d}{d_2}$ if $D_S(0) \leq \frac{\Lambda_d}{d_2}$. Thus, the region D is positively invariant for the model.

Furthermore, if $N(0) > \frac{\Lambda}{d}$ and $D_S(0) > \frac{\Lambda_d}{d_2}$, then, one of the solution enters D in finite time i.e. $N(t) \longrightarrow \frac{\Lambda}{d}$ and $D_S(t) \longrightarrow \frac{\Lambda_d}{d_2}$ as $t \longrightarrow \infty$. We conclude that the region D attracts all solution in $\mathbb{R}^6_+$. □

3 Analysis of Steady States

3.1 Basic Reproduction Number R_0 of Model

The Virus-Free Equilibrium (VFE) of Equation (1) is computed as

$$\begin{aligned}\varepsilon_0 &= (S^*, E^*, I^*, R^*, D_S^*, D_I^*) \\ &= \left(\frac{\Lambda}{d}, 0, 0, 0, \frac{\Lambda_d}{d_2}, 0\right) \qquad (2)\end{aligned}$$

By the Van den Driessche and Watmough [5], the basic reproduction number R_0 of the computer-virus model is computed by using the Next Generation Matrix Method. It is given by:

$$R_0 = r(FV^{-1})$$

where $r(.)$ is the spectral radius. Therefore,

$$F = \begin{pmatrix} 0 & \frac{\Lambda d_2 \beta_1}{d\Lambda_d} & \frac{\Lambda\beta_3}{d} \\ 0 & 0 & 0 \\ 0 & \frac{d\beta_3\Lambda_d}{\Lambda d_2} & 0 \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{d+\mu} & 0 & 0 \\ \frac{\mu\sigma+\mu d_2}{(d+\alpha+\gamma)(d+\mu)(\sigma+d_2)} & \frac{1}{d+\alpha+\gamma} & 0 \\ 0 & 0 & \frac{1}{\sigma+d_2} \end{pmatrix}$$

$$FV^{-1} = \begin{pmatrix} \frac{\Lambda d_2(\mu\sigma+\mu d_2)\beta_1}{d(d+\alpha+\gamma)(d+\mu)(\sigma+d_2)\Lambda_d} & \frac{\Lambda d_2\beta_1}{d(d+\alpha+\gamma)\Lambda_d} & \frac{\Lambda\beta_3}{d(\sigma+d_2)} \\ 0 & 0 & 0 \\ \frac{d(\mu\sigma+\mu d_2)\beta_3\Lambda_d}{(d+\alpha+\gamma)\Lambda(d+\mu)d_2(\sigma+d_2)} & \frac{d\beta_3\Lambda_d}{(d+\alpha+\gamma)\Lambda d_2} & 0 \end{pmatrix} \qquad (3)$$

Since the second row in Equation (3) has zero entries, then we reduce the above matrix to:

$$FV^{-1} = \begin{pmatrix} \frac{\Lambda d_2(\mu\sigma+\mu d_2)\beta_1}{d(d+\alpha+\gamma)(d+\mu)(\sigma+d_2)\Lambda_d} & \frac{\Lambda d_2\beta_1}{d(d+\alpha+\gamma)\Lambda_d} \\ \frac{d(\mu\sigma+\mu d_2)\beta_3\Lambda_d}{(d+\alpha+\gamma)\Lambda(d+\mu)d_2(\sigma+d_2)} & \frac{d\beta_3\Lambda_d}{(d+\alpha+\gamma)\Lambda d_2} \end{pmatrix}$$

Now, we have that by the largest eigenvalue of the above matrix, the basic reproduction number for the model is given by:

$$R_0 = \frac{\Lambda^2\mu d_2^2\beta_1 + d^3\beta_2\Lambda_d^2 + d^2\mu\beta_2\Lambda_d^2}{d(d+\alpha+\gamma)\Lambda(d+\mu)d_2\Lambda_d}$$

Hence, we will establish the local and global stability of the VFE.

3.2 Local Stability of Virus-Free Equilibruim

Theorem 1. *The virus-free equilibrium ε_0 exists for all R_0 and is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.*

Proof. We compute the Jacobian matrix and evaluate it at VFE. Therefore we have:

$$J_{\varepsilon_0} = \begin{pmatrix} -d & 0 & -\beta_1\frac{\Lambda}{d} & \eta & 0 & -\beta_2\frac{\Lambda d_2}{\Lambda_d d} \\ 0 & -(d+\mu) & \beta_1\frac{\Lambda}{d} & 0 & 0 & \beta_2\frac{\Lambda d_2}{\Lambda_d d} \\ 0 & \mu & -(d+\gamma+\alpha) & 0 & 0 & 0 \\ 0 & 0 & \gamma & -(d+\eta) & 0 & 0 \\ 0 & 0 & -\beta_2\frac{\Lambda_d d}{\Lambda d_2} & 0 & -d_2 & \sigma \\ 0 & 0 & \beta_2\frac{\Lambda_d d}{\Lambda d_2} & 0 & 0 & -(d_2+\sigma) \end{pmatrix} \qquad (4)$$

We now compute the eigenvalues of the Jacobian matrix. From Matrix (4), we observed that the first and the fifth columns contain only the diagonal terms $-d$ and $-d_2$, which are the first two eigenvalues. To obtain the other eigenvalues, the Jacobian matrix (4) is reduced to a sub-matrix (5) as follows:

$$J'_{\varepsilon_0} = \begin{pmatrix} -(d+\mu) & \beta_1\frac{\Lambda}{d} & 0 & \beta_2\frac{\Lambda d_2}{\Lambda_d d} \\ \mu & -(d+\gamma+\alpha) & 0 & 0 \\ 0 & \gamma & -(d+\eta) & 0 \\ 0 & \beta_2\frac{\Lambda_d d}{\Lambda d_2} & 0 & -(d_2+\sigma) \end{pmatrix} \qquad (5)$$

Also in the new matrix (5), we see that, the third column contains a diagonal term $-(d+\eta)$ which is the third eigenvalue. Therefore we reduce Matrix (5) to:

$$J''_{\varepsilon_0} = \begin{pmatrix} -(d+\mu) & \beta_1\frac{\Lambda}{d} & \beta_2\frac{\Lambda d_2}{\Lambda_d d} \\ \mu & -(d+\gamma+\alpha) & 0 \\ 0 & \beta_2\frac{\Lambda_d d}{\Lambda d_2} & -(d_2+\sigma) \end{pmatrix} \qquad (6)$$

Now let P_i, $i = 1, 2, 3$, be the eigenvalues of Matrix (6),

we have

$$\begin{aligned} & Det(J''_{\varepsilon_0} - PI) \\ = & \begin{vmatrix} -(d+\mu)-P & \beta_1\frac{\Lambda}{d} & \beta_2\frac{\Lambda d_2}{\Lambda_d d} \\ \mu & -(d+\gamma+\alpha)-P & 0 \\ 0 & \beta_2\frac{\Lambda_d d}{\Lambda d_2} & -(d_2+\sigma)-P \end{vmatrix} \\ = & 0. \end{aligned} \tag{7}$$

The eigenvalues of the characteristic Equation (7) are the zeros which satisfies the following equation below:

$$P^3 + P^2F_2 + P^1F_1 + F_0 = 0 \tag{8}$$

Where

$$\begin{aligned} F_2 &= 2d+\alpha+\gamma+\mu+\sigma+d_2 \\ F_1 &= \Lambda d_2(d+\alpha+\gamma)\Lambda_d(d+\mu)[P-R_0] \\ F_0 &= \Lambda_d d_2(d+\alpha+\gamma)(d+\mu)\Lambda[Q-R_0] \end{aligned} \tag{9}$$

with

$$\begin{aligned} P &= (d(2d+\alpha+\gamma+\mu)d_2 - \Lambda\mu\beta_1 + \Lambda^2\mu d_2^2\beta_1 \\ &\quad +d((d+\alpha+\gamma)(d+\mu)+(2d+\alpha+\gamma+\mu)\sigma \\ &\quad +d(d+\mu)\beta_3\Lambda_d^2))/(d(d+\alpha+\gamma)\Lambda(d+\mu)d_2\Lambda_d), \\ Q &= (\Lambda^2\mu d_2^2\beta_1 + (\sigma+d_2)(d(d+\alpha+\gamma)(d+\mu)-\Lambda\mu\beta_1) \\ &\quad -d\mu\beta_2^2 + d^3\beta_2\Lambda_d^2 + d^2\mu\beta_2\Lambda_d^2)/(d(d+\alpha+\gamma) \\ &\quad \Lambda(d+\mu)d_2\Lambda_d). \end{aligned}$$

Now, from the basic reproduction number deduced, we can make the following observations for F_1 and F_0, that is

$$F_1 = \begin{cases} >0 & when \quad R_0 < P < 1 \quad or \quad 1 < R_0 < P \\ <0 & when \quad P < R_0 < 1 \quad or \quad 1 < P < R_0, \end{cases}$$

$$F_0 = \begin{cases} >0 & when \quad R_0 < Q < 1 \quad or \quad 1 < R_0 < Q \\ <0 & when \quad Q < R_0 < 1 \quad or \quad 1 < Q < R_0, \end{cases}$$

Therefore, we see that when

$R_0 < 1$ provided that $R_0 < P$ and $R_0 < Q$, the virus-free equilibrium is locally and asymptotically stable, otherwise it is unstable.

The requirement of the real and negative eigenvalues ensuring stability is clearly satisfied by P.

Now, for the roots of the polynomial equation (8) by which the eigenvalues are obtained, we therefore make the following analysis based on the Routh-Hurwitz stability criteria [12],

Firstly, the coefficients $F_0, F_2 > 0$, that is, must be positive.

Secondly, for the eigenvalues to have real negative parts, i.e. $F_2F_1 > F_0$.

It is obvious that the coefficients $F_0 > 0$ and $F_2 > 0$. Also,

$$\begin{aligned} F_2F_1 - F_0 =& (d+\alpha+\gamma)\Lambda(d+\mu)d_2((2d+\alpha+\gamma+\mu \\ &+\sigma+d_2)(P-R_0)+(R_0-Q)) > 0, \end{aligned}$$

Therefore, by the Routh-Hurwitz criteria for stability, we conclude that the virus free equilibrium is locally asymptotically stable whenever $R_0 < 1$. □

3.3 Endemic Equillibrium

Endemic Equilibrium: In order to obtain the endemic equilibrium of the model *i. e.* the equilibrium where at least one of the infected components of the model is non-zero [25], we solve the system of equations at steady states and obtain:

Let $p = (d+\alpha+\gamma)$, $q = (d+\mu)$, $r = (d+\eta)$, $v = (d+\alpha)$ and $w = (\sigma + d_2)$.

$$\begin{aligned} S^* &= d\gamma p\Lambda^2\mu q d_2\Lambda_d - d^2q(dpr + (dp+v\eta)\mu)\beta_2\Lambda_d^2 \\ &\quad -\Lambda^2\mu(dpr+(dp+v\eta)\mu)d_2^2\beta_1/d^2\gamma p\Lambda\mu q d_2\Lambda_d \\ E^* &= \frac{r\left(\Lambda^2\mu d_2^2\beta_1 + d^3\beta_2\Lambda_d^2 + d^2\mu\beta_2\Lambda_d^2\right)}{d\gamma\Lambda\mu q d_2\Lambda_d} \\ I^* &= \frac{r\left(\Lambda^2\mu d_2^2\beta_1 + d^3\beta_2\Lambda_d^2 + d^2\mu\beta_2\Lambda_d^2\right)}{d\gamma p\Lambda q d_2\Lambda_d} \\ R^* &= \frac{\Lambda^2\mu d_2^2\beta_1 + d^3\beta_2\Lambda_d^2 + d^2\mu\beta_2\Lambda_d^2}{dp\Lambda q d_2\Lambda_d} \\ D_S^* &= \frac{w\Lambda_d(\alpha r\Lambda^2\mu d_2^2\beta_1 - d\gamma p\Lambda^2 q d_2\Lambda_d + d^2\alpha r q\beta_2\Lambda_d^2)}{r\Lambda^2\mu d_2^2\beta_1(\alpha w - d\beta_2) - d\gamma p\Lambda^2 q d_2 w\Lambda_d + d^2 rq(\alpha w - d\beta_2)\beta_2\Lambda_d^2} \\ D_I^* &= \frac{dr\beta_2\Lambda_d(\Lambda^2\mu d_2^2\beta_1 + d^2q\beta_2\Lambda_d^2)}{r\Lambda^2\mu d_2^2\beta_1(-\alpha w + d\beta_2) + d\gamma p\Lambda^2 q d_2 w\Lambda_d + d^2 rq(-\alpha w + d\beta_2)\beta_2\Lambda_d^2} \end{aligned}$$

Theorem 2. *The unique Endemic Equilibrium of the model* (1) *is globally asymptotically stable whenever* $R_0 > 1$.

Proof. If $R_0 > 1$, then there exist a unique Endemic equilibrium. We therefore consider the non-linear Lyapunov function $\mathcal{V}$ such that

$$\begin{aligned} \mathcal{V} =& S^*\left[\frac{S}{S^*} - \ln\frac{S}{S^*}\right] + E^*\left[\frac{E}{E^*} - \ln\frac{E}{E^*}\right] + I^*\left[\frac{I}{I^*} - \ln\frac{I}{I^*}\right] \\ &+ R^*\left[\frac{R}{R^*} - \ln\frac{R}{R^*}\right] + D_S^*\left[\frac{D_S}{D_S^*} - \ln\frac{D_S}{D_S^*}\right] + D_I^*\left[\frac{D_I}{D_I^*} - \ln\frac{D_I}{D_I^*}\right], \\ \dot{\mathcal{V}} =& \left[1-\frac{S^*}{S}\right]\dot{S} + \left[1-\frac{E^*}{E}\right]\dot{E} + \left[1-\frac{I^*}{I}\right]\dot{I} + \left[1-\frac{R^*}{R}\right]\dot{R} \\ &+ \left[1-\frac{D_S^*}{D_S}\right]\dot{D_S} + \left[1-\frac{D_I^*}{D_I}\right]\dot{D_I}, \\ \dot{\mathcal{V}} =& \left[1-\frac{S^*}{S}\right]\left[\Lambda - \frac{\beta_2 D_I S}{D_N} - \beta_1 SI - dS + \eta R\right] \\ &+ \left[1-\frac{E^*}{E}\right]\left[\frac{\beta_2 D_I S}{D_N} + \beta_1 SI - h_1 E\right] \\ &+ \left[1-\frac{I^*}{I}\right][\mu E - h_2 I] + \left[1-\frac{R^*}{R}\right][\gamma I - h_3 R] \\ &+ \left[1-\frac{D_S^*}{D_S}\right]\left[\Lambda_d - \frac{\beta_2 D_S I}{N} + \sigma D_I - d_2 D_S\right] \\ &+ \left[1-\frac{D_I^*}{D_I}\right]\left[\frac{\beta_2 D_S I}{N} - h_4 D_I\right], \end{aligned} \tag{10}$$

where

$$\begin{aligned} h_1 &= d+\mu, \\ h_2 &= d+\gamma+\alpha, \\ h_3 &= d+\eta, \\ h_4 &= d_2+\sigma. \end{aligned}$$

By expanding (10), and rearranging the expressions, we

have

$$\begin{aligned}\dot{\mathcal{V}} =& dS^*\left[1-\frac{S}{S^*}+\frac{\beta_2}{d}\left[\frac{D_S I}{D_I^* N}-\frac{D_S I}{D_I N}\right]+\frac{\Lambda}{d}\left[\frac{1}{S^*}-\frac{1}{S}\right]\right]\\ &+h_1E^*\left[1-\frac{E}{E^*}\left[1-\frac{\mu}{h_1}\left[1-\frac{I^*}{I}\right]\right]+\frac{\beta_2}{h_1}\left[\frac{D_I S}{D_N E^*}-\frac{D_I S}{D_N E}\right]\right]\\ &+h_2I^*\left[1-\frac{I}{I^*}\left[1-\frac{\gamma}{h_2}\left[1-\frac{R^*}{R}\right]\right]+\frac{\beta_1}{h_2}I\left[\frac{S^*}{I^*}-\frac{E^*S}{EI^*}\right]\right]\\ &+h_3R^*\left[1-\frac{R}{R^*}\left[1-\frac{\eta}{h_3}\left[1-\frac{S^*}{S}\right]\right]\right]\\ &+d_2D_S^*\left[1-\frac{D_S}{D_S^*}+\frac{\beta_2}{d_2}\left[\frac{I}{N}\left[1-\frac{D_S}{D_S^*}\right]\right]+\frac{\Lambda_d}{d_2}\left[\frac{1}{D_S^*}-\frac{1}{D_S}\right]\right]\\ &+h_4D_I^*\left[1-\frac{D_I}{D_I^*}\left[1-\frac{\sigma}{h_4}\left[1-\frac{D_S^*}{D_S}\right]\right]+\frac{\beta_2}{h_4}\left[\frac{I}{N}\left[\frac{D_S}{D_I^*}-\frac{D_S}{D_I}\right]\right]\right]\end{aligned}$$

We now have that, since the arithmetic mean exceeds the geametric mean value [9, 30], then

$$\begin{aligned}\left[1-\frac{S}{S^*}+\frac{\beta_2}{d}\left[\frac{D_S I}{D_I^* N}-\frac{D_S I}{D_I N}\right]+\frac{\Lambda}{d}\left[\frac{1}{S^*}-\frac{1}{S}\right]\right] &\leq 0\\ \left[1-\frac{E}{E^*}\left[1-\frac{\mu}{h_1}\left[1-\frac{I^*}{I}\right]\right]+\frac{\beta_2}{h_1}\left[\frac{D_I S}{D_N E^*}-\frac{D_I S}{D_N E}\right]\right] &\leq 0,\\ \left[1-\frac{I}{I^*}\left[1-\frac{\gamma}{h_2}\left[1-\frac{R^*}{R}\right]\right]+\frac{\beta_1}{h_2}I\left[\frac{S^*}{I^*}-\frac{E^*S}{EI^*}\right]\right] &\leq 0,\\ \left[1-\frac{R}{R^*}\left[1-\frac{\eta}{h_3}\left[1-\frac{S^*}{S}\right]\right]\right] &\leq 0,\\ \left[1-\frac{D_S}{D_S^*}+\frac{\beta_2}{d_2}\left[\frac{I}{N}\left[1-\frac{D_S}{D_S^*}\right]\right]+\frac{\Lambda_d}{d_2}\left[\frac{1}{D_S^*}-\frac{1}{D_S}\right]\right] &\leq 0,\\ \left[1-\frac{D_I}{D_I^*}\left[1-\frac{\sigma}{h_4}\left[1-\frac{D_S^*}{D_S}\right]\right]+\frac{\beta_2}{h_4}\left[\frac{I}{N}\left[\frac{D_S}{D_I^*}-\frac{D_S}{D_I}\right]\right]\right] &\leq 0.\end{aligned}$$

Since the parameters of the model (1) are greater then or equal to zero, therefore we have that $\dot{\mathcal{V}} \leq 0$ for $R_0 > 1$. Hence it follows from LaSalle's Invariance Principle [17] that every solution of the equation in the model (1) approaches the Endemic Equilibrium as $t \longrightarrow \infty$ whenever $R_0 > 1$. □

4 Sensitivity Analysis

In order to investigate the above model robustness, due to uncertainties associated with the estimation of certain parameter values, it is important and useful to carry out a sensitivity analysis to investigate how sensitive the basic reproduction number is with respect to these parameters. It will also give us insight to know the parameters that have high impact or cause most reduction on the virus transmission, that is, in R_0 and therefore determine the control measure that is most effective in the control of the Computer virus transmission [25, 32].

To carry out this analysis, we compute the normalized forward sensitivity index of the reproduction number with respect to these parameters. This is also referred to as the ratio of the relative change in the variable change in the parameter [4, 25].

Definition 1. *The normalized forward sensitivity index of a variable h, that depends differentially on a parameter m, is defined as:*

$$\Pi_m := \frac{\partial h}{\partial m} \times \frac{m}{h}.$$

4.1 Sensitivity Indices of R_0

We derive the sensitivity of R_0 corresponding to the following parameters:

$$\begin{aligned}\Pi_\alpha &:= -\frac{\alpha}{d+\alpha+\gamma},\\ \Pi_\gamma &:= -\frac{\gamma}{d+\alpha+\gamma},\\ \Pi_{\beta_1} &:= \frac{\Lambda^2\mu d_2^2\beta_1}{\Lambda^2\mu d_2^2\beta_1 + d^2(d+\mu)\beta_2\Lambda_d^2},\\ \Pi_{\beta_2} &:= \frac{d^2(d+\mu)\beta_2\Lambda_d^2}{\Lambda^2\mu d_2^2\beta_1 + d^2(d+\mu)\beta_2\Lambda_d^2},\\ \Pi_\mu &:= \frac{d\Lambda^2\mu d_2^2\beta_1}{(d+\mu)(\Lambda^2\mu d_2^2\beta_1 + d^2(d+\mu)\beta_2\Lambda_d^2)},\\ \Pi_\eta &:= \Pi_\sigma := 0,\end{aligned}$$

Using parameter values from Table 2, (it should be stated that these parameters are chosen for illustrative purpose only, and may not necessarily be realistic in terms of epidemiological interpretations), we calculate the sensitivity indices of R_0 based on the following parameters $\mu, \beta_1, \beta_2, \alpha, \eta, \sigma, \gamma$. The parameters are therefore, arranged from the most sensitive to least. The most sensitive parameter is proportion of the natural death rate $\beta_2 = 0.8901$. While the least of the sensitivity parameters are the η and $\sigma = 0.0000$. An increase (or decrease) in β_2 by 10% increases (or decreases) the R_0 by 8.91%. Similarly increasing (or decreasing) the rate of recovery γ by 10% decreases (or increases) the R_0 by 3.33%. From the sensitivity analysis, it is clear that control efforts should be targeted towards the rate at which the infectivity contact rate at which the virus attacks removable device (β_2).

5 Simulations for Computer Virus Model

In this section, we illustrated the effects of the changes of some basic parameters that may influence the transmission dynamics of the Computer virus model. In order to investigate the graphical trend of these changes of parameters in the model (1), we illustrate these by focusing on the transmission dynamics of the each sub-class of the model with respect to changes in some of its basic parameter values such as β_1 and μ.

In the course of this investigations, we studied the dynamical flow of the trend of the following graphs below. And Hence, we draw some conclusions based on the result obtained under the graphs.

Table 2: Sensitivity analysis of R_0

Parameters	Descriptions	Sensitivity
β_2	The infectivity contact rate at which virus attacks removable device	0.8901
γ	Rate of Recovery	-0.3333
α	Rate of Virus induced Computer death	-0.1667
β_1	The infectivity contact rate at with network attacks occur	0.1099
μ	Exposed rate	0.0824
η	Waning rate of Computer	0.0000
σ	Rate of Ingestion	0.0000

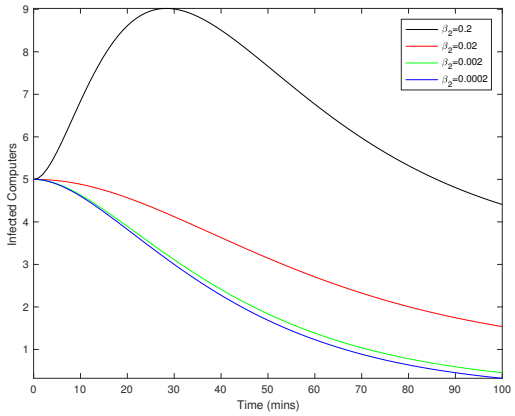

Figure 2: Model (1) Effects caused by changes in parameter β_2 in Infected class for $\Lambda = 0.8$,, $\beta_1 = 0.0002$, $\beta_2 = 0.2$, $\alpha = 0.01$, $\eta = 0.03$, $\gamma = 0.02$, $\mu = 0.01$, $\Lambda_d = 0.6$, $d = 0.03$, $d_2 = 0.005$, $\sigma = 0.002$

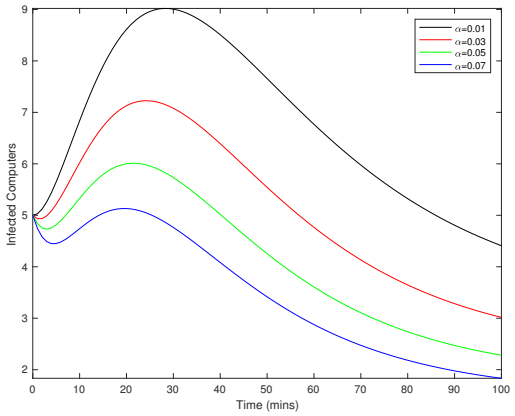

Figure 4: Model (1) Effects caused by changes in parameter α in Infected class for $\Lambda = 0.8$,, $\beta_1 = 0.0002$, $\beta_2 = 0.2$, $\alpha = 0.01$, $\eta = 0.03$, $\gamma = 0.02$, $\mu = 0.01$, $\Lambda_d = 0.6$, $d = 0.03$, $d_2 = 0.005$, $\sigma = 0.002$

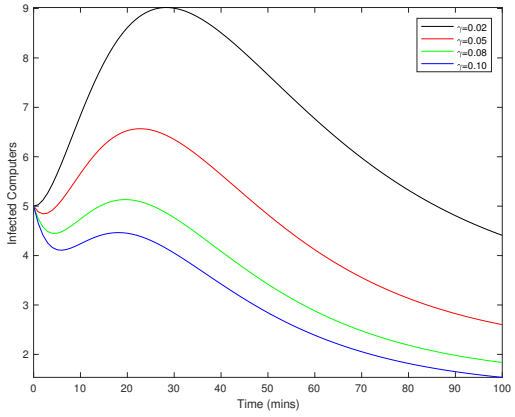

Figure 3: Model (1) Effects caused by changes in parameter γ in Infected class for $\Lambda = 0.8$,, $\beta_1 = 0.0002$, $\beta_2 = 0.2$, $\alpha = 0.01$, $\eta = 0.03$, $\gamma = 0.02$, $\mu = 0.01$, $\Lambda_d = 0.6$, $d = 0.03$, $d_2 = 0.005$, $\sigma = 0.002$

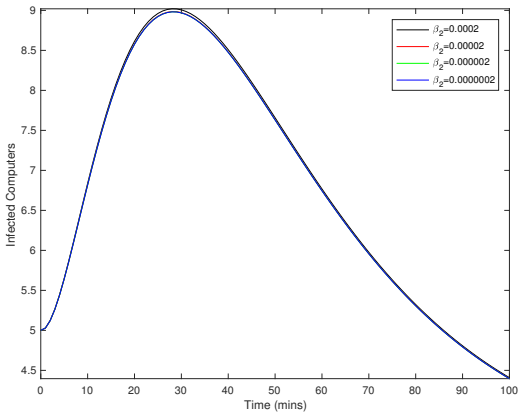

Figure 5: Model (1) Effects caused by changes in parameter β_1 in Infected class for $\Lambda = 0.8$,, $\beta_1 = 0.0002$, $\beta_2 = 0.2$, $\alpha = 0.01$, $\eta = 0.03$, $\gamma = 0.02$, $\mu = 0.01$, $\Lambda_d = 0.6$, $d = 0.03$, $d_2 = 0.005$, $\sigma = 0.002$

Studying the behavior of the mathematical model (1) for the transmission and spread of Computer virus through numerical simulations illustrated above, it was

observed that the most sensitive parameter in the control of the virus spread are the infectivity contact rate, the virus attacks removable device β_2 rate. Although some other parameters such as γ, the rate of recovery, α, virus induced computer death rate and β_1 the infectivity contact rate at with network attacks occur, are also quite sensitive to the model control. These graphs obtained also confirm the results in the sensitivity analysis.

From Figure 2, it is shown that, decrease in the rate at which the virus attacks removable device, decreases the rate at which the computer systems get infected with time and as a result decreases R_0. Figures 3 and 4 showed decrease in the infected class as a result of increase in the recovery rate and virus induced computer deaths, which is γ and α respectively, which also led to a reduction in R_0. It is also shown in Figure 5 that the reduction of the infectivity contact rate at which network attacks occur β_1 slightly affects the transmission dynamics of the model by reducing the Infected class which also ascertain the sensitivity analysis.

By observing the numerical simulations, the control strategy should be targeted at decreasing β_2 by properly doing a thorough system scanning using an antivirus software before the system can be used. Adequate and frequent checks should be done always and ensure infected external devices such as flash drives should not be used on susceptible computers. By doing this, R_0 is greatly reduced and hence the virus drastically dies out.

6 Analysis of Optimal Control

In this section, we make use of Pontryagin's Maximum Principle so that we can obtain the essential conditions for the optimal control of the for computer virus model. Time dependent controls are incorporated the model in order to determine the best strategy for controlling the computer virus.For this reason, we consider the following objective functionals,

$$J(u_1,u_2,u_3)=\int_0^{t_F}\left(b_1E+b_2I+b_3D_I+\frac{c_1}{2}u_1^2+\frac{c_2}{2}u_2^2+\frac{c_3}{2}u_3^2\right)dt. \tag{11}$$

where c_1, c_2 and c_3 deal with the weighting constants for providing intensive public education on the use of removable device(prevention), effort on public campaign of how to maintain virus free on computers use (prevention) and treatment of infected computers with viruses (treatment) respectively. The cost corresponding to with prevention and treatment mechanisms are assumed to have a nonlinear character. Hence, we explore an optimal control u_1^*, u_2^* and u_3^* in a way that, $J(u_1^*, u_2^*, u_3^*) = \min J(u_1, u_2, u_3), \Gamma = \{(u_1, u_2, u_3)|0 \le u_i \le 1, i = 1, 2\}$.

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda-(1-u_1)\frac{\beta_2D_1S}{D_N}-(1-u_2)\beta_1SI-dS+\eta R\\
\frac{dE}{dt} &= (1-u_1)\frac{\beta_2D_1S}{D_N}+(1-u_2)\beta_1SI-(d+\mu)E\\
\frac{dI}{dt} &= \mu E-(d+u_3\gamma+\alpha)I\\
\frac{dR}{dt} &= u_3\gamma I-(d+\eta)R\\
\frac{dD_S}{dt} &= \Lambda_d-(1-u_1)\frac{\beta_2D_SI}{N}+\sigma D_1-d_2D_S\\
\frac{dD_I}{dt} &= (1-u_1)\frac{\beta_2D_SI}{N}-(d_2+\sigma)D_I. \qquad (12)
\end{aligned}$$

The necessary conditions that an optimal solution has to satisfy emanate from Pontryagin Maximum Principle [2, 25]. The principle actually converts (11)-(12) into a type of problem which principally aimed at minimizing pointwise a Hamiltonian H, with respect to u_1, u_2 and u_3.

$$\begin{aligned}
H &= b_1E+b_2I+b_3D_I+c_1u_1^2+c_2u_2^2+c_2u_3^2\\
&\quad +N_S\{\Lambda-(1-u_1)\frac{\beta_2D_1S}{D_N}-(1-u_2)\beta_1SI-dS+\eta R\}\\
&\quad +N_E\{(1-u_1)\frac{\beta_2D_1S}{D_N}+(1-u_2)\beta_1SI-(d+\mu)E\}\\
&\quad +N_I\{\mu E-(d+u_3\gamma+\alpha)I\}\\
&\quad +N_R\{u_3\gamma I-(d+\eta)R\}\\
&\quad +N_{D_S}\{\Lambda_d-(1-u_1)\frac{\beta_2D_SI}{N}+\sigma D_1-d_2D_S\}\\
&\quad +N_{D_I}\{(1-u_1)\frac{\beta_2D_SI}{N}-(d_2+\sigma)D_I\} \qquad (13)
\end{aligned}$$

where $N_S, N_E, N_I, N_R, N_{D_S}$ and N_{D_I} denote the adjoint variables or also referred to as co-state variables. The system of equations are arrived at by considering the right partial derivatives of the Hamiltonian (13) with respect to the associated state variable.

Theorem 3. *Given optimal controls u_1^*, u_2^*, u_3^* and solutions S, E, I, R, D_S, D_I of the associated state system (11)-(12) that minimize $J(u_1, u_2, u_3)$ over Γ. Thus, there exists adjoint variables*
$N_S, N_E, N_I, N_R, N_{D_S}, N_{D_S}$ satisfying

$$\frac{-d\lambda_i}{dt}=\frac{\partial H}{\partial i}$$

where $i = S, E, I, R, D_S, D_I$ and with transversality conditions

$$\begin{aligned}
N_S(t_f) &= N_E(t_f)=N_T(t_f)\\
&= N_R(t_f)=N_{D_S}(t_f)=N_{D_I}=0 \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
u_1^* &= \min\left\{1,\max\left(0,\frac{\frac{\beta_2D_1S}{D_N}(N_E-N_S)+\frac{\beta_2D_1S}{D_N}(N_{D_I}-N_{D_S})}{2c_1}\right)\right\},\\
u_2^* &= \min\left\{1,\max\left(0,\frac{\beta_1SI(N_E-N_S)}{2c_2}\right)\right\}, \qquad (15)\\
u_3^* &= \min\left\{1,\max\left(0,\frac{\gamma(N_I-N_R)}{2c_3}\right)\right\} \qquad (16)
\end{aligned}$$

Proof. Corollary 4.1 of Fleming and Rishel [7] gives the appropriate condition of possible existence of an optimal control as a result of convexity of the integrand of J with respect to u_1, u_2 and u_3, a *priori* boundedness of

the state control solutions, and the *Lipschitz* characteristics of the state system with regard to the state variables. The Hamiltonian function computed at the optimal control provides the governing adjoint variables. Therefore, the adjoint equations can be rearrange as

$$\begin{aligned}
-\frac{dN_S}{dt} &= dN_S + (1-u_1)\frac{\beta_2 D_1}{D_N}(N_S - N_E) \\
&\quad +(1-u_1)\frac{\beta_2 D_S I}{N^2}(N_{D_I} - N_{D_S}) \\
&\quad +(1-u_2)\beta_1 I(N_S - N_E) \\
-\frac{dN_E}{dt} &= -b_1 + (d+\mu)N_E - \mu N_I \\
&\quad +(1-u_1)\frac{\beta_2 D_S I}{N^2}(N_{D_I} - N_{D_S}) \\
-\frac{dN_I}{dt} &= -b_2 + (d+\alpha)N_I + (1-u_2)\beta_1 S(N_S - N_E) \\
&\quad +u_3\gamma(N_I - N_R) \\
&\quad +(1-u_1)\frac{\beta_2 D_S(N-I)}{N^2}(N_{D_S} - N_{D_I}) \\
-\frac{dN_R}{dt} &= dN_R + \eta(N_R - N_S) \\
&\quad +(1-u_1)\beta_2 D_S I(N_{D_1} - N_{D_S}) \\
-\frac{dN_{D_S}}{dt} &= d_2 N_{D_S} + (1-u_1)\frac{\beta_2 D_1 S}{D_N^2}(N_E - N_S) \\
&\quad +\frac{\beta_2 I}{N}(N_{D_S} - N_{D_I}) \\
-\frac{dN_{D_I}}{dt} &= -b_3 + (d_2+\sigma)N_{D_I} \\
&\quad +(1-u_1)\frac{\beta_2 S(D_N - D_I)}{D_N^2}(N_S - N_E) \\
&\quad \frac{\frac{\beta_2 D_1 S}{D_N}(N_E - N_S) + \frac{\beta_2 D_1 S}{D_N}(N_{D_I} - N_{D_S})}{2c_1}
\end{aligned}$$

Solving for the values of u_1^*, u_2^* and u_3^* with respect to the constraints, the characterization (15-16) can be arrived at as

$$\begin{aligned}
0 &= \frac{\partial H}{\partial u_1} \\
&= -2c_1 + \frac{\beta_2 D_1 S}{D_N}(N_E - N_S) + \frac{\beta_2 D_1 S}{D_N}(N_{D_I} - N_{D_S}) \\
0 &= \frac{\partial H}{\partial u_2} = -2c_2 + \beta_1 SI(N_E - N_S) \\
0 &= \frac{\partial H}{\partial u_3} = -2c_3 + \gamma(N_I - N_R).
\end{aligned}$$

Thus, we have (see for example Lenhart and Workman [18])

$$\begin{aligned}
u_1^* &= \frac{\frac{\beta_2 D_1 S}{D_N}(N_E - N_S) + \frac{\beta_2 D_1 S}{D_N}(N_{D_I} - N_{D_S})}{2c_1} \\
u_2^* &= \frac{\beta_1 SI(N_E - N_S)}{2c_2} \\
u_3^* &= \frac{\gamma(N_I - N_R)}{2c_3}.
\end{aligned}$$

Making use of standard control arguments which considers the bounds on the controls, we make the conclusion that

$$\begin{cases} 0 & \text{If } \xi_i^* \le 0 \\ \xi_i^* & \text{If } 0 < \xi_i^* < 0 \\ 1 & \text{If } 0 < \xi_i^* \ge 0 \end{cases}$$

For $i \in 1, 2, 3$ and where

$$\begin{aligned}
\xi_1^* &= \frac{\frac{\beta_2 D_1 S}{D_N}(N_E - N_S) + \frac{\beta_2 D_1 S}{D_N}(N_{D_I} - N_{D_S})}{2c_1} \\
\xi_2^* &= \frac{\beta_1 SI(N_E - N_S)}{2c_2} \\
\xi_3^* &= \frac{\gamma(N_I - N_R)}{2c_3}
\end{aligned}$$

□

The next section shall be focused on the detailed discussion on the numerical simulation solution results which is hinged on the optimality of the model taking into account of different kind of strategies of the optimal controls $u_1 u_2$ and u_3, the parameter selections and meanings from various strategies.

7 Numerical Simulations

The numerical simulation solutions for this model is undertaken using MATLAB 10.0 version. The optimality system, which comprise the state system and the adjoint system, was worked out to obtain the optimal control solution. The optimality system solution was calculated using a fourth-order Runge-Kutta iterative scheme. The adjoint equations were also worked out by the backward fourth-order Runge-Kutta scheme applying the preceding solutions of the state equations hinged on the transversality conditions Equation (14). The controls results obtained were updated using a convex combination of the previous controls and the value obtained from the characterizations. This activity was carried on and the iterations were terminated if the values of the unknowns at the former iterations were similar to the ones arrive at the current iteration [2, 18]. Table 1 shows parameters and values used in the numerical simulation of the computer virus model. The following weight constants were considered: $b_1 = 10, b_2 = 30, b_3 = 90$ and $c_1 = 100, c_2 = 120, c_3 = 300$.

7.1 Prevention (u_1, u_2) of Computer Virus and Removable Infected Device

The prevention control u_1 on the intensive public education on the use of removable device (intensive public education) and the prevention control u_2 (public campaign on computer virus free)are used to optimise the objective function J, and at the same time the control (u_3) is set to zero. Figure 6(a) depicts that the number of removable infected device infected D_I is significant different from application of control and without control presence. This control strategy only brings the number of infected removable device down but cannot control it as the number of

Table 3: Description of variables and parameters of the model

Parameter	Value	Ref.
β	0.65 day^{-1}	Assumed
σ	0.95	Assumed
η_1	1/(365x60) day^{-1}	[6]
η_2	0.44	[11]
η_3	0.6	[6]
π	1/14	[6]
γ	1/14	[6]
ρ_1	0.75	[6]
ρ_2	0.4	[11]
ρ_3	0.95 day^{-1}	assumed
δ	0.0085 day^{-1}	assumed
μ	0.055 day^{-1}	assumed

infected removable device increases after the intervention as shown in Figure 6(a). In Figure 6(b) there is a substantial difference also exists between the case controlled and without controlled case.

The positive impact on the control strategy suggests that giving intensive public campaign on the computer virus free is effective however, it does not completely controlled the number of exposed computers E. The Figure 6(c) shows the number of infected computer with virus and this strategy suggests that both the controlled and without controlled case are rising after the intervention. This condition is expected since there is no effective strategy on treatment of infected computers as shown in Figure 6(c). The control profile is depicted in Figure 6(d) as control $u3$ is set to zero.The control $u1$ is initially set to 50% for 20 days which is then increased to 100% for the rest of the intervention. While control $u2$ is initially also set to 50% for 6 days then rise up to 100% for the rest of the intervention as shown in Figure 6(c).

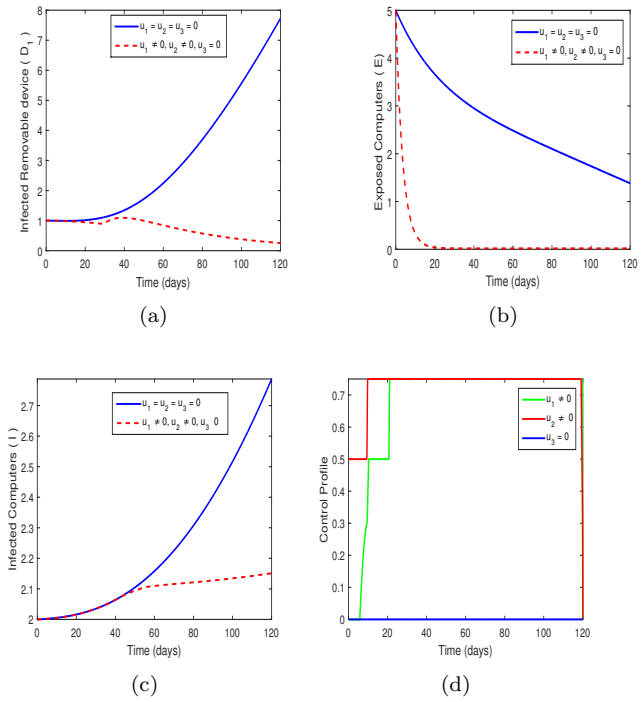

Figure 6: Simulations of the model showing the effect of computer virus and infected removable device prevention on transmission

7.2 Prevention (u_1) and Treatment (u_3) of Computer Virus and Removable Device

Prevention and treatment control u_1, u_3 (intensive public education on the use of removable device and treatment of infected computers) are used to optimize the objective function J, and while control (u_2) is set to zero. Figure 7(a) shows a significant difference between the use of control and that of without control. This strategy suggests that the number of infected removal devices are minimized but infected removable device D_1 increases. In Figure 7(b) there is a negative impact on the control strategy since the presence of the control has no effect on reducing the number of exposed computers E to infected computers. There is no control strategy designed to effectively ensure that intensive public education on computer virus are carried on. Figure 7(b) depicts the number of infected computers I and there is a significant difference between the controlled case and without the application of control. However, the presence of the control strategy has a minimal effect since both controlled and without control increasing after the intervention. In Figure 7(c) the control u_2 is set to zero while control u_1 is set to 25% for the beginning and relatively kept same for the entire intervention. The control u_3 is also for a start is kept at 100% for 60 day and then reduce to 24% which is then maintained for the rest of the intervention.

7.3 Prevention (u_2) and Treatment (u_3) of Computer Virus

In this strategy, the prevention control strategy u_2 and treatment control u_3 (intensive public campaign on computer virus free and treatment) are used to optimize the objective function J. Figure 8(a) shows the number of infected removable device D_I and there is a significant difference between the controlled and without controlled. The results in Figure 8(a) suggests that there is a negative impact on the control strategy since controlled case is higher than without controlled case. Therefore, this control mechanism has no effect on controlling the number of infected removal devices D_I. Again, in Figure 8(b) there is a substantial difference between the application of control and without the use of control. The positive effect of this strategy suggests that the control strategy is effective during the entire intervention and is able to control the number of exposed computer $E$to virus. However, the uncontrolled case rise up at the end of the intervention as

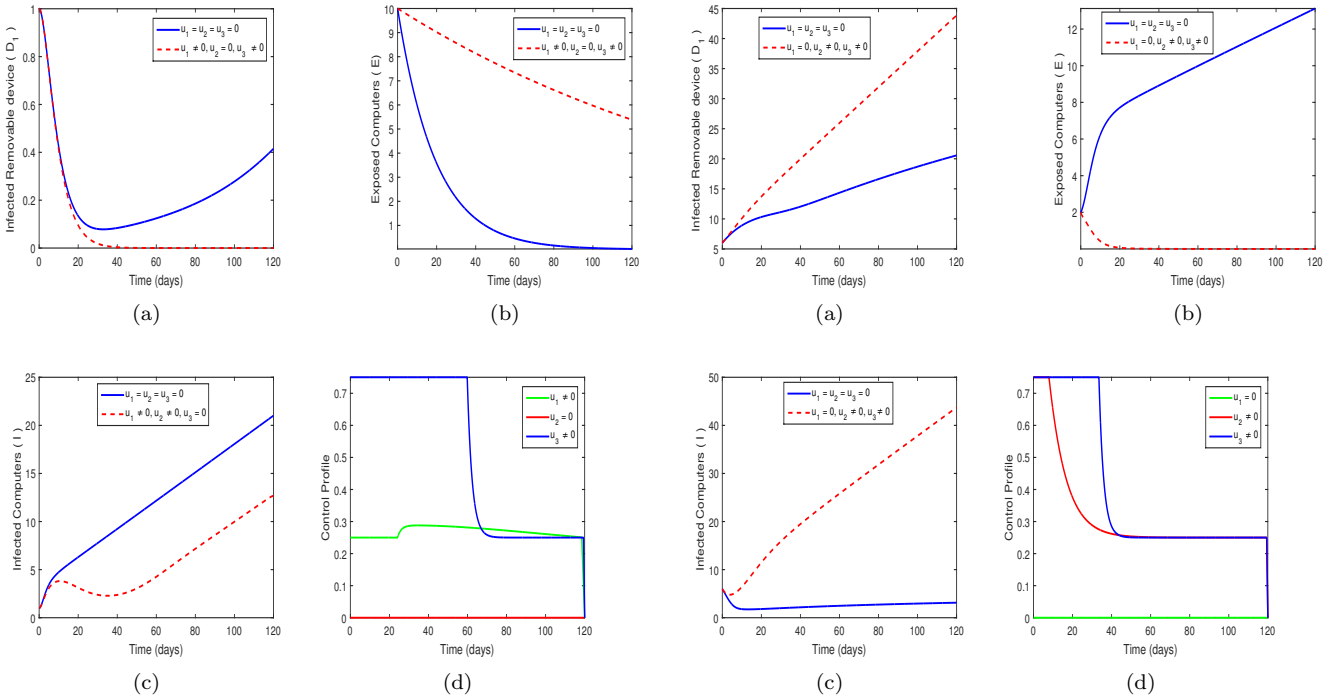

Figure 7: Simulations of the model showing the effect of computer virus and infected removal device prevention and treatment on transmission

Figure 8: Simulations of the model showing the effect of computer virus and infected removal devices prevention and treatment on transmission

shown in Figure 8(b). In Figure 8(c) there is a significant difference between the use of control and without control application. The negative impact as shown in Figure 8(b) suggests that this control strategy is not very robust in reducing the the number of infected computers. The control profile of this strategy is shown in Figure 8(d) and control u_1 is set to zero. The control u_2 is initially set to 100% for 18 days and gradually reduce to 27% which is then maintain throughout the rest of the intervention. Similarly, the control u_3 is at 100% for 34 days and gradually minimize to 27% for the rest of the intervention.

7.4 Prevention (u_1),(u_2) and Treatment (u_3) of Computer Virus and Infected Removal Devices

In this strategy all the control strategies u_1, u_2, u_3 (intensive education on infected removal devices, campaign on computer virus free and treatment of infected computers) are used simultaneously to optimize the objective function J. Figure 9(a), shows that there is significantly different between the controlled case and without controlled case. This positive impact suggests that the control mechanism is very effective and robust in controlling the number of infected removal devices. This maybe attributable to the combination of the other controls for their effectiveness. A similar pattern is depicted in Figure 9(b) as there is a vast positive difference between controlled case and without application of control. This also infers that the strategy is effective in controlling the number of exposed computers E. In fact, the number of exposed computers are totally brought under control with the combination of all the controls as shown in Figure 9(b). Figure 9(c) shows the number of infected computers I and there is a significant difference between the use of control and without control. The positive effect indicates that the control strategy is effective as both controlled and without controlled are brought under effective control as shown in Figure 9(c). The control profile for this strategy is depicted in Figure 9(d) as all the controls are used at the same time. Control u_1 is initially kept at 100% for 6 days then reduce to 28% which is maintained throughout the rest of the intervention. Similarly, control u_2 is also kept at 100% for 10 days for the beginning and then reduce to 28% which is kept for the rest of the intervention. While control u_3 is maintained at 100% for 25 days which is also reduce to 28 day and kept constantly for the rest of the intervention.

8 Conclusion

In this work, a computer virus model incorporating a removal device of deterministic type was formulated. The basic properties of the model was investigated then the stability analysis of the model was studied. The steady states found to be stable.The reproduction number R_0 was calculated. Time dependent controls was incorporated into the model and Pontryagin's Maximum Princi-

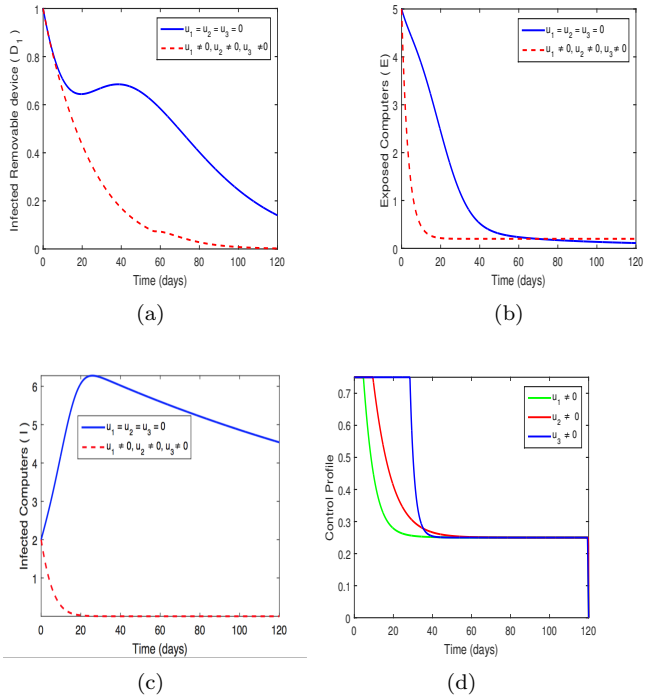

Figure 9: Simulations of the model showing the effect of infected computer virus and infected removal devices prevention and treatment only on transmission

ple was used to determine all the necessary conditions for controlling the spread of computer virus. The numerical simulation carried out on the control suggests that the best strategy in controlling the spread of computer virus is the use of all the three controls at the same time. The implication of the result seems to suggest that all effort must given to all the strategies designed without relaxing.

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