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# POLYHEDRAL DYNAMICS AND THE GEOMETRY OF SYSTEMS

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# PREFACE

One of the basic methodological problems of large-scale systems analysis is to define meaningful mathematical structures for the components or "pieces" comprising the system and to study their interconnections. Most of the fundamental advances in mathematical programming and optimal control theory, such as the Dantzig-Wolfe decomposition method, fast Fourier transforms, and generalized x-y functions have been special cases of this basic idea.

The results of this report provide a reasonably general mathematical framework within which the structure-connectivity question may be attacked by algebraic and geometric means. As the initial effort in what is projected to be a long-term research program, the current paper deals primarily with definitions, examples, and indications of the utility of the proposed methodology .

The results of this study should prove useful in a number of IIASA areas, particularly to the Energy, Water, and Ecology groups.

## SUMMARY

The report shows how a binary relation between two abstract sets may be geometrically interpreted as a simplicial complex. Standard and non-standard concepts from combinatorial topology are then employed to study the global connectivity structure of the complex. Classical notions such as homology are illustrated by examples chosen from various fields.

The connection between the standard differential equation definition of a dynamical system and the polyhedral dynamic set-up is explored in some detail. It is shown that the complex associated with a linear system provides a very illuminating paradigm within which new interpretations of open- and closed-loop control laws are possible. The report concludes with a discussion of topics for future investigation.

# Polyhedral Dynamics and the Geometry of Systems

## GLOBAL VS. LOCAL ANALYSIS IN SYSTEM THEORY

Beginning with the work of Newton in celestial mechanics, mathematical techniques for the analysis of systems have proceeded upon the basic principle that a detailed understanding of local system properties would lead (via the system's dynamical equations of motion) to a complete understanding of the global system structure and behavior. Obviously, this reductionist principle served well for several centuries in physics until the advent of quantum mechanics and relativity theory called it into question in connection with the study of the so-called "elementary" particles.

In more recent times, the unparalleled success of the reductionist point of view in classical physics has spawned the hope on the part of many biologists, sociologists, economists, and others that, by following the local path blazed by the pioneering physicists, they too would be rewarded not only with new conceptual insights, but also with ready-made operational tools "pre-tested", so to speak, by the physicists. Unfortunately, such a program, while still under way, has already met with some of the same obstacles encountered by the modern physicists and it now seems clear that, at best, local analyses will be only partially successful in answering many of the most pressing problems faced in the socio-economic sphere.

The failure of the local, calculus-based, tools to provide satisfactory answers to questions involving the global structure of systems has generated a renewed interest in the system theory community in the use of global mathematical techniques in systems analysis. Supposedly arcane (and useless) areas of mathematics such as group theory, invariant theory, Lie algebras, and differential geometry are now being used to probe the inner workings of complex systems and many new insights into the "holistic" structure of systems have been obtained in the past decade or so [12,13].

As an aside, it is amusing to note that this shift in emphasis from the local to the global corresponds to a swing of the intellectual pendulum back from Newtonian to Aristotelian physics. In his Politics, Aristotle states "in the order of Nature the State is prior to the household or the individual. For the whole must needs be prior to its parts." This view is in direct conflict with the post-medieval scientific method since it leads to a physics in which the significance of set members is explained in terms of the significance of the set (the whole). Modern physical theories, of course, do exactly the opposite; the whole is "explained" in terms of the (elementary) parts. The Aristotelian view dominated physical thought for many centuries until the modern experimentalist view, begun by Galileo and legitimatized by Newton, took over the stage. Now we see a revival of interest in the holistic theories, sending us back to that other Aristotelian notion of "moderation in all things".

Our goal in this report is to outline a mathematical approach based upon concepts from algebraic topology for the study of global system structure. The essence of our approach, introduced by Atkin in 1974 under the name of q-Analysis (here we propose to describe the theory as "polyhedral dynamics"), is to utilize the connective structure of the system in order to obtain a geometrical (and algebraic) representation of the system as a simplicial complex. Ideas and techniques of classical algebraic topology, together with some newer notions motivated by the system-theoretic context, are then used to provide new insights into the global connectivity structure of the system and to study the manner in which the individual system components interconnect to form the total structure. Following a discussion of the basic topological concepts introduced by Atkin [1,2,3,4] in a variety of frameworks, we examine the notions precisely by interpreting them in the context of linear systems. In this manner we hope to obtain a deeper understanding of the nature of feedback control laws, as well as an alternate viewpoint on the problem of controllability. In addition, we shall show that the standard duality results of linear system theory have a natural geometrical interpretation in the language of simplices and complexes. Finally, we present evidence to suggest that the majority of the global structural results given for linear systems may also be extended to nonlinear systems with only modest additional effort.

## BASIC CONCEPTS FROM ALGEBRAIC TOPOLOGY

In this section, we briefly review the background material from classical algebraic (combinatorial) topology which will be needed for our subsequent development. Much more detail can be obtained in any of the classic references in this area, for example [5,6], and the Appendix C of [1].

The general set-up for polyhedral dynamics, as initially conceived by Atkin [1], is to regard a system as a relationship between the elements of finite sets. To avoid, for the moment, hierarchical considerations [2,3,4], we assume that two finite sets $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ are given, together with a relation $\lambda \subset Y \times X$, i.e. $\lambda$ is a rule which associates elements of Y with those of X according to some criterion. For example, if $Y = \{1,2,3,4,5\}$, $X = \{0,1,2\}$, and $\lambda$ is the relation "-- is less than --", then $\lambda = \{(1,2)\}$, i.e. the subset of $Y \times X$ corresponding to the relation $\lambda$ is the single element set $\{(y_1, x_3)\}$. Associated with any such relation $\lambda$, we also have the inverse relation $\lambda^{-1} \subset X \times Y$, which is defined by the rule that if $(y,x) \in \lambda$, then $(x,y) \in \lambda^{-1}$. For example, if $\lambda$ represents the open proposition "-- is the child of --", then $\lambda^{-1}$ is the proposition "--is the parent of --". Clearly, regarding a system as a relation between two sets is a very general concept whose successful application hinges critically upon an adroit choice of the sets X and Y and the relation $\lambda$. However, it is a notion sufficiently broad, mathematically speaking, to support a surprising amount of geometrical structure as we now indicate.

In direct correspondence to the foregoing set-theoretic description of a system, we can obtain a geometrical representation of the relation $\lambda \subset Y \times X$ in the following manner. Let the elements $\{x_1, \ldots, x_n\}$ of the set X abstractly represent the vertices of a simplicial complex, while the elements of Y represent the simplices. Then the simplices actually forming the complex (denoted by $K_Y(X;\lambda)$) are defined by the relation $\lambda$. Thus the simplex $\sigma_{r-1} = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_r}\rangle$ is a member of $K_Y(X;\lambda)$ if and only if there exists some $y_j \in Y$ such that $(y_j, x_{i_s}) \in \lambda$ for all $s = 1, \ldots, r$. In this case, we denote the simplex $\langle x_{i_1}, x_{i_2}, \ldots, x_{i_r}\rangle$ by $y_j$. The dimension of K, dim K, is defined to be equal to the dimension of the highest dimensional simplex contained in K. Thus, assuming each element of Y is $\lambda$-related to at least one element of X, we see that $\lambda$ induces the simplicial complex $K_Y(X;\lambda)$, which geometrically represents the global picture of the relation. By interchanging the roles of X and Y, and using the inverse relation $\lambda^{-1}$, we also obtain the conjugate complex $K_X(Y;\lambda^{-1})$ representing the relation $\lambda^{-1}$.

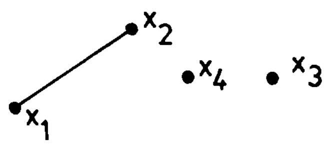
A simple example will help to clarify these matters. Let X = {bread, milk, stamps, shoes}, Y = {market, department store, bank, post office}; let $\lambda$ be the relation $(y_i, x_j) \in \lambda$ if and only if product $x_j$ can be purchased at facility $y_i$. Then clearly,

$$\lambda = \{(y_1,x_1),(y_1,x_2),(y_4,x_3),(y_2,x_4)\} \quad .$$

Thus, the simplices of $K_Y(X;\lambda)$ are

$$y_1 = \langle x_1, x_2\rangle, \quad y_2 = \langle x_4\rangle, \quad y_3 = \langle \phi\rangle, \quad y_4 = \langle x_3\rangle \quad .$$

(Note: the "empty" simplex $y_3$ does not belong to $K_Y(X;\lambda)$ unless we agree to "augment" the complex by the addition of $\phi$ as a vertex representing a (-1)-dimensional simplex.) Geometrically, $K_Y(X;\lambda)$ has the structure



showing that $K_Y(X;\lambda)$ is a multiply-connected complex consisting of the 1-simplex $y_1$, the two 0-simplices $y_2$ and $y_4$, and the (-1)-simplex $y_3$. As is obvious by inspection, this "system" displays a very low level of connectivity, a notion we shall make more precise in a moment.

A compact form with which to represent the relation $\lambda$ is by its <u>incidence</u> matrix $\Lambda$. Adopting the convention that the (i,j) entry of $\Lambda$ corresponds to the pair $(y_i,x_j)$, we set

$$\lambda_{ij} = \begin{cases} 1, & \text{if } (y_i,x_j) \in \lambda \\ 0, & \text{otherwise} \end{cases} .$$

Thus, we represent $K_Y(X;\lambda)$

$$\Lambda = \begin{array}{c|c} \lambda & X \\ \hline Y & \lambda_{ij} \end{array} ,$$

while the conjugate complex $K_X(Y;\lambda^{-1})$ has the representation

$$\begin{array}{c|c} \lambda^{-1} & Y \\ \hline X & \lambda_{ij}^{-1} \end{array} = \Lambda' \qquad \text{("' " denoting matrix transposition).}$$

A more complete picture of how K is connected is obtained by a study of the "homological" structure of the complex. Roughly speaking, we analyze how many "holes" K contains and their respective dimensions. To make these geometrical notions precise, we first present some background definitions and concepts, taken from the work in [1]. In what follows, we adopt the standard notation $\sigma_p$ to represent an arbitrary, but fixed, p-dimensional simplex (i.e. a simplex consisting of p+1 vertices).

<u>Chains and Boundaries</u>

We restrict the discussion to the case of a relation $\lambda$

between two finite sets X and Y; in particular $\lambda \subset Y \times X$ and $\lambda^{-1} \subset X \times Y$. Either of the two simplicial complexes $K_Y(X;\lambda)$, $K_X(Y;\lambda^{-1})$ possesses a finite dimension and a finite number of simplices.

We therefore take the case of such a complex, say $K_Y(X;\lambda)$, in which dim K = n; we assume that we have an orientation on K, induced by an ordering of the vertex set X, and that this is displayed by labelling the vertices $x_1, x_2, \ldots, x_k$, with $k \geq n+1$. We select an integer p such that $0 \leq p \leq n$ and we label all the simplices of order p as $\sigma_p^i$, $i = 1,2,\ldots,h_p$, where we suppose that there are $h_p$ p-simplices in K.

We now form the formal linear sum of these p-simplices and call any such combination a p-chain, allowing multiples of any one $\sigma_p$. We denote the totality of these p-chains by $C_p$ and one member of $C_p$ by $c_p$. Thus a typical p-chain

$$c_p = m_1\sigma_p^1 + m_2\sigma_p^2 + \ldots + m_{h_p}\sigma_p^{h_p} \quad ,$$

with each $m_i \in J$ where J is an arbitrary Abelian group. We can then regard this set $C_p$ as a group (an additive Abelian group) under the operation +, by demanding

$$c_p + c_p' = (m_1 + m_1')\sigma_p^1 + \ldots + (m_{h_p} + m_{h_p}')\sigma_p^{h_p}$$

together with the identity (zero) $0_p$ for which each $m_i = 0$. Combining every group $C_p$, for $p = 0,1,\ldots,n$, we obtain by the direct sum the chain group C., written

$$C_\cdot = C_0 \oplus C_1 \oplus \ldots \oplus C_n \quad .$$

Any element in $C_\cdot$ is of the form

$$c_\cdot = c_0 + c_1 + \ldots + c_n \quad .$$

With every p-chain $c_p$ we now associate a certain (p-1)-chain, called its boundary, and denoted by $\partial c_p$. We define $\partial c_p$ precisely in terms of the boundary of a simplex $\partial\sigma_p$, and if

$c_p = \sum_i m_i \sigma_p^i$ we take

$$\partial c_p = \sum_i m_i \partial \sigma_p^i \quad .$$

In other words, we require that $\partial$ should be a homomorphism from $C_p$ into $C_{p-1}$.

If a typical $\sigma_p$ is $\sigma_p = \langle x_1 x_2 \ldots x_{p+1} \rangle$ we define $\partial\sigma_p$ by

$$\partial\sigma_p = \partial\langle x_1 x_2 \ldots x_{p+1}\rangle = \sum_i (-1)^{i+1} \langle x_1 x_2 \ldots \hat{x}_i \ldots x_{p+1}\rangle$$

where $\hat{x}_i$ means that the vertex $x_i$ is omitted.

Figure 1 shows a geometric representation of a $\sigma_2 = \langle x_1 x_2 x_3 \rangle$ together with the orientation, and the induced orientations on the edges. In this case

$$\partial\sigma_2 = \partial\langle x_1 x_2 x_3\rangle$$

$$= (-1)^2 \langle x_2 x_3\rangle + (-1)^3 \langle x_1 x_3\rangle + (-1)^4 \langle x_1 x_2\rangle \quad ;$$

this means that

$$\partial\sigma_2 = \sigma_1^1 - \sigma_1^2 + \sigma_1^3 \quad ,$$

which is a 1-chain, a member of $C_1$.

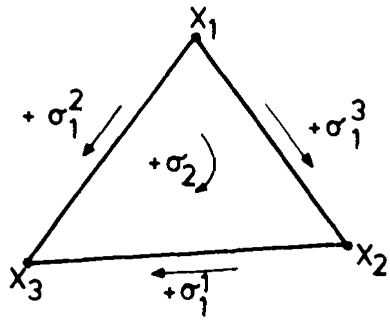



Figure 1. A 2-simplex with its faces oriented.

The boundary of a chain can be seen as its image under the operator $\partial$, which is a map

$$\partial : C_p \to C_{p-1} \text{ for } p = 1,\ldots n \quad .$$

Not only is $\partial$ a homomorphism (it preserves the additive structure), but it is easily seen to be nilpotent--that is to say, $\partial(\partial c_p) = 0$ in $C_{p-2}$, or

$$\partial^2 = 0 \text{(the zero map)}.$$

In the case shown in Figure 1, we have

$$\begin{aligned}\partial^2\sigma_2 = \partial(\partial\sigma_2) &= \partial(\sigma_1^1 - \sigma_1^2 + \sigma_1^3)\\ &= \partial\langle x_2x_3\rangle - \partial\langle x_1x_3\rangle + \partial\langle x_1x_2\rangle\\ &= \langle x_3\rangle - \langle x_2\rangle - (\langle x_3\rangle - \langle x_1\rangle) + \langle x_2\rangle - \langle x_1\rangle\\ &= 0 \quad .\end{aligned}$$

Since $\partial: C_p \to C_{p-1}$ is a homomorphism, the image of $C_p$ under $\partial$ must be a subgroup of $C_{p-1}$; we denote this image $\partial C_p$ variously by im $\partial$ or by $B_{p-1}$ and, because $\partial$ is nilpotent, we see that

$$\partial B_{p-1} = 0 \text{ in } C_{p-2} \text{ or } \partial(\text{im } \partial) = 0 \quad .$$

Those p-chains $c_p \in C_p$ which are such that their boundaries vanish, that is $\partial c_p = 0$, are called p-cycles. They form a subgroup of $C_p$, being the kernel of the homomorphism $\partial$, and are usually denoted by the symbols $z_p$, the whole subgroup being $Z_p$. The members of $B_p$ (which is $\partial C_{p+1}$) are clearly cycles too, by the above, and so $B_p \subset Z_p$. In fact $B_p$ is a subgroup of $Z_p$.

The members of $B_p$ are called bounding cycles (they are cycles in an identical or trivial sense), and those members of $Z_p$ which are not members of $B_p$ can be identified as representatives of the elements of the factor group (or quotient group)

$Z_p/B_p$. The members of this factor group are of the form

$$z_p + B_p \quad ,$$

and, if we select one member, say $z_p$, out of this equivalence class, we can also denote it by $[z_p]$. When two p-cycles $z_p^1$ and $z_p^2$ differ by a p-boundary, then $z_p^1 - z_p^2 \in B_p$ and we say that $z_p^1$ and $z_p^2$ are homologous (often written as $z_p^1 \sim z_p^2$). This is a relation on the set of cycles and it is easy to see that it is an equivalence relation. The quotient set $Z_p/\sim$, under the relation of 'being homologous to', is the quotient group $Z_p/B_p$, the group structure being determined by the operation + on the members $z_p + B_p$. In this group structure, the set $B_p$ acts as the additive identity (the 'zero'), since

$$(z_p + B_p) + B_p = z_p + B_p$$

for all $z_p$.

This pth factor group $Z_p/B_p$ is what is called the pth homology group and denoted by $H_p$:

$$H_p = Z_p/B_p \quad , \quad p = 0,1,\ldots,n \quad .$$

The group of cycles $Z_p$ being mapped to zero by the homomorphism $\partial$ is what is known as the kernel of $\partial$ (written ker $\partial$) and so we find the alternative form

$$H_p = \ker \partial / \mathrm{im}\ \partial \quad .$$

The operation of $\partial$ on the graded group $C_.$ can be indicated by the sequence:

$$C_. = C_0 \oplus C_1 \oplus C_2 \oplus \ldots \oplus C_p \overset{\partial}{\oplus} C_{p+1} \ldots \oplus C_n \quad ,$$

together with the symbolic diagram of Figure 2 below.

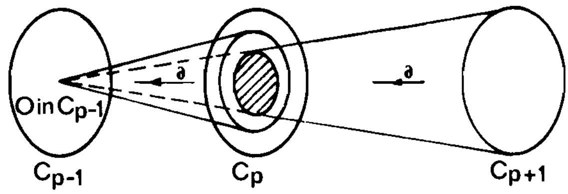



Figure 2. A nilpotent ∂ operating on a graded group C.

In this diagram $B_p$ is represented by the shaded bull's-eye in $C_p$; $Z_p$ is the inner ring surrounding this shaded portion.

When $H_p = 0$ there is only one equivalence class in the factor group and this is $B_p$; every $z_p \varepsilon B_p$; every cycle is a bounding cycle. When $H_p \neq 0$ there is more than one element in the factor group and so there must be at least one cycle which is not a bounding cycle at this level. In Figure 1 we have $H_1 = 0$ because the only 1-cycle is the combination $\sigma_1^1 - \sigma_1^2 + \sigma_1^3$ (and multiples thereof) and this is $\partial\sigma_2$. Because there is no $C_3$ there cannot be a $B_2$ (the $\sigma_2$ is not the boundary of anything) and since $\partial\sigma_2 \neq 0$, $Z_2$ is also empty. Under these conditions we also write $H_2 = 0$. When $H_p = 0$ we speak of the homology being trivial at the p-level; when we say that 'the homology is trivial', without specifying the values of p, we mean that $H_p = 0$ for all values of p other than p = 0. This latter group $H_0$ is never zero, except possibly when the complex K is augmented by inclusion of the simplex whose vertex set is empty.

We can see in Figure 1 that the homology is trivial, and also that $H_0 \neq 0$. For any $c_0$ is of the form

$$c_0 = m_1\langle x_1\rangle + m_2\langle x_2\rangle + m_3\langle x_3\rangle \quad ,$$

and taking the boundary of a point to be zero, it follows that

$c_0$ must be a 0-cycle, $c_0 \varepsilon Z_0$. But the vertices $x_1$, $x_2$, $x_3$ form part of an arc-wise connected structure in the sense that 1-chains $c_1$, $c_1'$ exist such that

$$\langle x_2 \rangle = \langle x_1 \rangle + \partial c_1$$

$$\langle x_3 \rangle = \langle x_1 \rangle + \partial c_1'$$

(in fact we need only take $c_1 = \sigma_1^3$ and $c_1' = \sigma_1^2$). Hence we have

$$c_0 = z_0 = (m_1 + m_2 + m_3)\langle x_1 \rangle + \partial(\text{some 1-chain}) \quad .$$

Hence the vertex $x_1$ acts like a special chosen 0-cycle $\hat{z}_0$, all the possible 0-cycles in the structure can be generated by writing

$$z_0 = m\hat{z}_0 + \partial(\text{some 1-chain}) \quad ,$$

and $\hat{z}_0$, consisting of a single point, cannot be the boundary of any 1-chain. Hence $\hat{z}_0 \not\in B_0$ and so $H_0 \neq 0$; in fact $H_0$ contains a single generator and, being an additive group, it is isomorphic therefore to the additive group J (which is generated by a single symbol, viz., 1). Thus we see that for the complex represented in Figure 1,

$$H_0 = J \quad ,$$

or, preferably, we should use the symbol for isomorphism and write $H_0 \cong J$.

The above argument shows that this structure is characteristic of the complex being arc-wise connected and we can therefore generalize it to give the result:

if K possesses k connected components then

$$H_0(K) = J \oplus J \oplus \ldots \oplus J$$

with k summands. This number k is also known as the zero-order Betti number of K and then it is written as $\beta_0$.

## Betti Numbers and Torsion

The groups $C_p$, $Z_p$, $B_p$ already discussed are examples of finitely generated free groups, there being no linear dependencies between the generators of any of them. But this property of being 'free' is not necessarily true of the factor group $H_p$. Indeed, in general, we find that $H_p$ can be written as the direct sum of two parts, of which one is a free group and the other is not. To explain this idea, and to illustrate it by a practical example, we write our general $H_p$ in the form

$$H_p = G_p^0 \oplus \text{Tor}\, H_p \quad ,$$

where $G_p^0$ is to be a free group and Tor $H_p$ is to be called the torsion subgroup of $H_p$. Any element of Tor $H_p$, say h, is such that nh = 0 for some finite integer n (with 0 being the additive identity of the group $H_p$). In the context of boundaries and cycles this means that h can be written in the form $h = z_p + B_p$, because $h \in H_p$, and that there is an n such that

$$nh = nz_p + nB_p \quad ;$$

this element must be in $B_p$ (the zero of the factor group). But this means that, although $z_p \notin B_p$, it must be that $nz_p \in B_p$ for this particular value of n. This rather strange behavior of certain torsion cycles is the property which the subgroup Tor $H_p$ characterizes.

Members of the free group $G_p^0$ cannot behave in this way; if $z_p \in G_p^0$ and $z_p \notin B_p$ then $nz_p \notin B_p$ for any non-zero value of n. For this reason a free group is often called an infinite cyclic group, in contrast to the finite cyclic groups which go to make up Tor $H_p$. Thus $G_p^0$ will consist of summands of type J (the number of summands will equal the number of distinct generators of $G_p^0$) while Tor $H_p$ will consist of summands of type $J_m$ (the additive integers modulo m if J = integers) for some choices of m. This must be so because a group like $J_m$ is an additive Abelian group with the property that if $h \varepsilon J_m$ then $mh = 0$. If Tor $H_p$ contains a number of subgroups then each one will be isomorphic to some $J_m$, for a suitable m.

The number of generators of $G_p^0$ (the number of free generators of $H_p$) is called the pth Betti number of the complex K, sometimes written as $\beta_p$.

p-Holes

We have seen (cf. Figure 1) the case of a complex K possessing a trivial homological structure; in that example $H_1 = 0$ because the triangle $\sigma_2$ is filled in. If we cut out the inside of this $\sigma_2$, leaving only the edges, then we find that $H_1 = J$, because there is now a single generator in the shape of

$$\sigma_1^1 - \sigma_1^2 + \sigma_1^3 \quad ,$$

which is not the boundary of a $\sigma_2$, the $\sigma_2$ having been removed. Thus the single generator of $H_1$ represents the presence in K of a hole, bounded by 1-simplices (edges), what we shall call a 1-dimensional hole. If the complex K contained two hollowed-out triangles then $H_1$ would be isomorphic to the direct sum of J and J, written $H_1 = J \oplus J$. In a similar vein, if a geometrical representation of the complex K possessed a spherical hole (bounded by the surface of a sphere) we would find that $H_2$ would contain a single generator $\hat{z}_2 \varepsilon B_2$; and if we found that $H_2 = J \oplus J$ we could interpret it as meaning that K possessed two 2-dimensional holes.

In general then we wish to stress the interpretation of the free group $G_p^0$ as an algebraic representation of the occurrence of p-dimensional holes in the complex K; the precise number of these holes is given by the pth Betti number $\beta_p$. A geometrical representation of the complex - as far as $G_p^0$ is concerned - therefore looks like a sort of multi-dimensional Swiss cheese.

The q-connectivity analysis discussed in the next section is dedicated to showing us the structure of the "cheese" in between the holes. The possible interpretation of the torsion subgroup Tor $H_p$ is more elusive in this cheese-like context, but the following example [1] shows that it can have a very practical significance in another.

Example

Denote the faces of a gambler's die by the symbols $v^1$, $v^2$, $v^3$, $v^4$, $v^5$, $v^6$. Let these be the vertices of a 5-simplex and let K be this simplex together with all its faces; for example, a typical 1-simplex is the pair $\langle v^i\ v^j \rangle$ with $i \neq j$. Impose the induced orientation on K, induced by the natural ordering of the vertices. Now conduct a series of experiments in which the die is successively thrown until there is a repetition of a die-face; in this, interpret the sequence $\{v^i, v^j\}$ as the negative of the sequence $\{v^j, v^i\}$. The result of a series of successive throws is to observe an element in the graded chain group

$$C_{.} = C_0 \oplus C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \quad .$$

Notice that the boundary of the run <123> is the 1-chain <12>, <23>, <31>.

In the first place we expect the experimenter to be able to observe every possible distinct run and series of runs. It would then follow that in the graded chain group every cycle is a boundary and so

$$H_p = 0 \quad \text{for} \quad p = 1,2,3,4 \quad ;$$

thus the homology is trivial.

But now let us alter the arrangement so that the experimenter suffers the handicap of working with a laboratory assistant who sees to it (by doctoring the records) that, let us say, the run <123> never occurs - either by itself or as a face of any other run. This results in a drastic alteration of the complex K and its associated chain group. For example, the sequence <123456> never occurs, since it contains <123>. Furthermore, in the new complex K', there exists a cycle

$$z_1 = \langle 12\rangle + \langle 23\rangle + \langle 31\rangle$$

which is not a boundary. Hence the intervention of the assistant is reflected in an increase in the 1st Betti number $\beta_1$ from the value 0 to the value 1. The assistant is responsible for punching a hole in the complex; the homology group $H_1$ is now isomorphic to J.

Let us go further and alter the arrangements yet again. Suppose that the experiment is conducted by two fair-minded gamblers. They begin by noticing that the probabilities of distinct runs corresponding to typical simplices $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$ are 5/6, 5/9, 5/18, 5/54 and 5/324. Since they intend to bet on the experiment, our two gamblers agree to weight the simplices so as to even up the chances. They do this by introducing new (weighted) simplices as generators for the new chain group C'. These generators $\sigma_i'$ are related to the old generators $\sigma_i$ by the formulae

$$\sigma_1' = 54\sigma_1;\ \sigma_2' = 36\sigma_2;\ \sigma_3' = 18\sigma_3;\ \sigma_4' = 6\sigma_4;\ \sigma_5' = \sigma_5 \quad .$$

Now the homology has been altered yet again; for example,

$$54\{\langle 12\rangle + \langle 23\rangle + \langle 31\rangle\}$$

is in $Z_1'$ but not in $B_1'$, because the latter consists of multiples of $108 \sum_i \sigma_1^i$, 108 being the lowest common multiple of 36 and 54. Hence there exists a cycle $z_1$ such that $2z_1 \in B_1'$. This makes a contribution to $H_1$ of the summand $J_2$; $H_1$ now contains a torsion subgroup Tor $H_1$. In fact

$$H_1 = J_2 \oplus J_2 \oplus \ldots \oplus J_2 \quad ,$$

there being 10 summands in all. The other $H_p$ are not affected and $H_p = 0$ for $p = 2,3,4$.

The gamblers' complex therefore possesses torsion which is expressed in $H_1$. It is thereby clear that the torsion can be introduced into $H_\cdot(K)$ in different ways, which give different summands $J_m$, by altering the odds on the outcome of the experiments. Thus $\sigma_1' = 48\sigma_1$ leads to 10 summands $J_3$, with

$$H_1 = J_3 \oplus J_3 \oplus \ldots \oplus J_3 \quad .$$

Cochains and Coboundaries

We can associate with a chain group $C_\cdot$ (with coefficients in J) a dual concept, namely that of mappings from $C_\cdot$ into J. In doing this we introduce the concept of a cochain, dual to that of a chain; every such cochain is a mapping from $C_\cdot$ into J:

$$c^p : C_p \to J \quad .$$

Precisely, we denote a p-cochain by $c^p$, and we also demand additivity

$$c^p(c_p + c_p') = c^p(c_p) + c^p(c_p') \quad .$$

We can build up any particular p-cochain $c^p$ in terms of a set of mappings from the p-simplices $\sigma_p$ into J. Hence, prior to the notion of a cochain we can have the notion of a cosimplex $\sigma^p$ which is simply a mapping

$$\sigma^p : \{\sigma_p^i\} \to J \quad ,$$

without any additive structure assumed. If there are $h_p$ p-simplices in K we can define a basis for the cosimplices as the set of $h_p$ mappings $\{\sigma_i^p,\ i = 1,2,\ldots,h_p\}$ where

$$\begin{aligned} \sigma_i^p(\sigma_p^j) &= 0 \quad \text{if} \quad i \neq j \\ &= 1 \quad \text{if} \quad i = j \quad . \end{aligned}$$

Then every cosimplex $\sigma^p$ is the sum of the $\sigma_i^p$, that is

$$\sigma^p = \sum_i \sigma_i^p \quad ,$$

and every p-cochain is a linear combination

$$c^p = \sum_i m_i \sigma_i^p \quad ,$$

together with the linearity condition. The zero cochain map (for any p) is the one defined by $m_i = 0$, for all values of i, and the whole set of p-cochains form an additive group $C^p$. Hence the graded cochain group is the direct sum

$$C^{\cdot} = C^0 \oplus C^1 \oplus \ldots \oplus C^n \quad ,$$

where n = dim K. To complete the duality, we can define a coboundary operator $\delta$ which is the adjoint of $\partial$. Adopting the inner product notation $(c_p, c^p)$ for the value (in J) of $c^p(c_p)$, we define $\delta$ by

$$(\partial c_{p+1}, c^p) = (c_{p+1}, \delta c^p) \quad ,$$

which shows that $\delta$: $C^p \to C^{p+1}$. It is also clear that $\delta$ is nilpotent, $\delta^2 = 0$, since

$$\begin{aligned} 0 = (0, c^p) &= (\partial^2 c_{p+2}, c^p) \\ &= (\partial c_{p+2}, \delta c^p) \\ &= (c_{p+2}, \delta^2 c^p) \qquad \text{for all choices of } c_{p+2}, \end{aligned}$$

and so $\delta^2 c^p$ must be the zero map. We now have the dual cohomology groups, $H^p(K;J)$ defined by

$$H^p = Z^p/B^p = \ker \delta / \mathrm{im}\ \delta \quad .$$

## POLYHEDRAL DYNAMICS

The previous section briefly reviewed a number of classical concepts from algebraic topology and their use in analyzing some features of the global connectivity of a simplicial complex K. In particular, we saw that knowledge of the homology groups enables us to determine the multidimensional holes of K. In addition, the torsion subgroups give information concerning the "twists" of the cycles of K and their dimensions.

Now we turn the focus of our attention to some non-classical aspects of the connective structure of K. Rather than studying the holes of the complex which, in essence, is the study of what is absent from K, we investigate the "material" which is actually present. In other words, we shall look at the chains of connection which form the fabric of the complex. In addition, we introduce measures which allow us to study how well any individual simplex is "integrated" into the total complex, thereby providing means for local analysis to complement the global picture obtained from the connectivity patterns. Finally, we inject a note of dynamism into the picture by means of the notion of a pattern on a complex. A pattern is basically a map which assigns a numerical value to each simplex of K. Pattern measures "traffic" which must be (i) determined by a vertex set and (ii) be graded on the simplices of K. Thus, we will see that a dynamical system can be mathematically structured as a complex in which there is a continual "flow" of numbers among the simplices. The connective structure of K then gives information as to various geometrical obstacles preventing free flow throughout the complex, as well as the dimension and nature of the various obstacles preventing such a flow. The concepts of Newtonian and Einsteinian forces will then make their natural appearance in order to explain the flow of patterns in any given situation.

Chains of Connection

Given two simplices $\sigma_i$ and $\sigma_j$ in a complex K, we say they are joined by a chain of connection if there exists a finite sequence of simplices $\sigma_{\alpha_1}, \sigma_{\alpha_2}, \ldots, \sigma_{\alpha_n}$, in K, such that

i) $\sigma_{\alpha_1}$ is a face of $\sigma_i$ ,

ii) $\sigma_{\alpha_n}$ is a face of $\sigma_j$ ,

iii) $\sigma_{\alpha_s}$ and $\sigma_{\alpha_{s+1}}$ share a common face, $s = 1,2,\ldots,n-1$.

Such a chain is said to be of length n-1. If

$$q = \min \{i, \alpha_1, \alpha_2, \ldots, \alpha_n, j\} ,$$

then we say the chain is a q-connectivity.

It is trivial to verify that the notion of q-connectivity is an equivalence relation upon the simplices of K. Thus, it is of interest to study the equivalence classes generated by this relation.

As a measure of the global connectivity pattern of the complex K, we introduce the first structure vector $\underset{\sim}{Q}$, whose entries are non-negative integers indicating the number of equivalence classes in K for each q, $q = 0,1,\ldots,\dim K$, i.e.

$$Q_i = \text{the number of } i\text{-connected components in } K,$$
$$i = 0,1,\ldots,\dim K.$$

Intuitively, one could imagine looking at the complex K through special glasses which only enable the viewer to see i-dimensional objects. With such glasses, the viewer would then see the complex K split into $Q_i$ disjoint pieces. Consequently, the q-connectivity vector

$$Q = (Q_{\dim K}, \ldots, Q_1, Q_0)$$

gives valuable information as to how the "pieces" comprising the relation $\lambda$ are connected to each other and at what dimensional levels these connections take place. One should note that all the simplices in a particular component need not have q-simplex interfaces in a pairwise fashion, but rather there will be multi-dimensional "tubes" of simplices which join the members of the component. These tubes embody the local structure of the complex and, therefore, of the relation $\lambda$.

A simple algorithm suitable for computing Q from the incidence matrix of $\lambda$ is given in Appendix A.

Eccentricity

While the structure vector Q provides valuable information concerning the global connectivity structure in K, it gives very limited information about the individual simplices comprising the complex. Since the simplices and vertices are identified with the elements of the sets X and Y, they are the items of primary physical concern and, as a result, it is of some importance to attempt to develop some measures indicating the degree to which each individual simplex integrates into the entire complex K.

Such a measure for a given simplex $\sigma$ should clearly take account of two important factors:

i) the dimension of $\sigma$, i.e. to how many distinct elements is the simplex $\lambda$-related; and

ii) the degree to which $\sigma$ is connected to the remainder of the complex, i.e. how well integrated $\sigma$ is into K.

It is fairly obvious that a high-dimensional $\sigma$ is, in some sense, more important to the understanding of K than is a low-dimensional simplex. However, for understanding the total complex, even if $\sigma$ is high-dimensional, it may still not be too important if it is only weakly-connected with the remainder of K. Thus, both of the requirements above must be taken into account in devising a measure of the importance of $\sigma$ to the entire complex.

A measure which satisfies both of the points discussed above is the eccentricity of $\sigma$, denoted ecc($\sigma$). If we denote

$\hat{q}$ = dimension of $\sigma$ ,

$\check{q}$ = dimension of the highest-dimensional simplex with which $\sigma$ shares a face, i.e. $\check{q}$ is the largest q value for which $\sigma$ is in a component containing some other simplex,

then we define

$$\mathrm{ecc}(\sigma) = \frac{\hat{q} - \check{q}}{\check{q} + 1} \quad .$$

Clearly, $\hat{q} - \check{q}$ is a measure of the unusual, "non-conforming" nature of $\sigma$; however, $\hat{q} - \check{q} = 2$ is presumably more revealing if $\check{q} = 1$ than if $\check{q} = 10$. Hence, we use the ratio above as the measure of eccentricity, rather than the absolute difference $\hat{q} - \check{q}$.

As an example of the ideas of q-connectivity and eccentricity, consider the following hypothetical predator-prey ecosystem. Let the predator set be given by

$$Y = \{\text{Man, Lions, Elephants, Birds, Fish, Horses}\}, \\ = \{y_1, y_2, y_3, y_4, y_5, y_5\}$$

while the set of prey is

$$X = \begin{Bmatrix} \text{Antelopes, Grains, Pigs, Cattle,} \\ \text{Grass, Leaves, Insects, Reptiles} \end{Bmatrix}$$

$$= \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\} \quad .$$

We define a relation $\lambda$ on $Y \times X$ by saying that $(y_i, x_j) \in \lambda$ if and only if predator $y_i$ feeds on prey $x_j$. A plausible incidence matrix for this relation is

$\Lambda =$

| $\lambda$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| $y_1$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| $y_2$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $y_3$ | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| $y_4$ | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| $y_5$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $y_6$ | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |

.

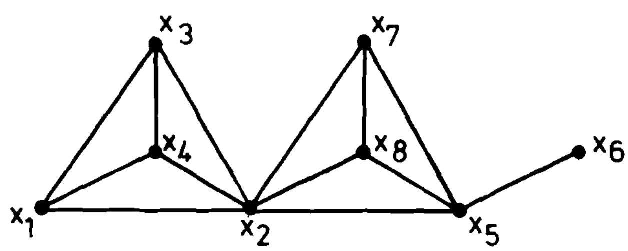
Thus, if we consider the complex $K_Y(X;\lambda)$, we have

$\langle x_1 x_2 x_3 x_4 \rangle$ is a $\sigma_3$ whose name is $y_1$,

$\langle x_1 x_3 \rangle$ is a $\sigma_1$ whose name is $y_2$,

and so on.

The geometrical representation of $K_Y(X;\lambda)$ is




Already, the geometry suggests that the vertex $\langle x_2 \rangle$, consisting of Grains, is going to be critical in the analysis of this eco-structure.

Referring to the algorithm given in Appendix A, the connectivity vector Q is formed. We have

$$
\begin{array}{llll}
\text{at} \quad q = 3 , & Q_3 = 2 , & \{y_1\}, \{y_4\} \\
q = 2 , & Q_2 = 2 , & \{y_1\}, \{y_4\} \\
q = 1 , & Q_1 = 3 , & \{y_1, y_2\}, \{y_3\}, \{y_4 y_6\} \\
q = 0 , & Q_0 = 1 , & \{\text{all}\} .
\end{array}
$$

Thus

$$Q = \overset{3}{(2}\ 2\ 3\ \overset{0}{1)} .$$

The eccentricities of the simplices $y_1 - y_6$ are

$$
\begin{array}{lll}
\text{ecc } y_1 = 1 , & \text{ecc } y_2 = 0 , & \text{ecc } y_3 = 1 , \\
\text{ecc } y_4 = 1 , & \text{ecc } y_5 = 0 , & \text{ecc } y_6 = 0 .
\end{array}
$$

Hence, we see that there is a great deal of homogeneity in K, no simplex exhibiting a significant degree of eccentricity. In other words, all of the predators are well integrated into the ecosystem.

Patterns and Dynamics

An essential feature characterizing most of modern system theory is the notion of a dynamic. Interesting as they are for some purposes, static processes are of limited utility when it comes to modelling most situations in economics, sociology, biology, and so on. Consequently, we now turn our attention to the development of concepts which inject a note of dynamism into the heretofore static geometric picture of a system given above.

The basic device used to incorporate a system dynamic into the structural analysis already developed is the idea of a pattern. We conceive of a pattern as being a mapping which assigns a number to each simplex of the complex K at each moment in time, i.e.

$$\Pi : K \to k^N ,$$

where N = dim K, k being a suitable number field. We also note that since the complex K is graded by the dimensionality of its component simplices, the pattern $\Pi$ is also graded. Thus, we may write

$$\Pi = \Pi_0 + \Pi_1 \oplus \ldots \oplus \Pi_N \quad ,$$

with each $\Pi_i$ being a map defined only upon simplices of dimension i. Thus, the numbers themselves acquire a "dimension" defined by the dimension of the simplex with which they are associated by $\Pi$.

The system dynamics is now identified with a <u>change of pattern $\delta\Pi$</u>, i.e. with a distribution of the numbers among the simplices,

$$\delta\Pi = \delta\pi_0 \oplus \delta\Pi_1 \oplus \ldots \oplus \delta\Pi_N \quad .$$

The existence of the complex K induces the notion of a <u>basic</u> pattern on K, namely that which associates a "1" with every simplex in K. Changes in this basic pattern are then interpreted in one of two ways:

i) <u>Newtonian</u> - we regard the complex K, itself, as being fixed. Then changes in the pattern, $\delta\Pi$, are interpreted as <u>stresses</u> or <u>forces</u> on the simplices of K. Thus, if $\delta\Pi_t \neq 0$, we have a t-force in K with $\delta\Pi_t > 0$ being a force of attraction, while $\delta\Pi_t < 0$ is regarded as a force of repulsion. From the Newtonian point of view, K is regarded as a static framework under stress.

ii) <u>Einsteinian</u> - an alternate approach to interpreting $\delta\Pi$ is to regard $\delta\Pi$ as defining a new complex backcloth by addition or deletion of vertices. In other words, the original geometry of K is changed to accommodate the change of pattern $\delta\Pi$ or, conversely, a change in the geometry may induce a pattern change $\delta\Pi$.

Let us explore the Einsteinian interpretation a bit further. Since the numbers associated with each simplex have a "natural" dimension equal to that of its simplex, a <u>free</u> change of pattern at level q is possible only if: (i) another simplex of dimension $\geq$ q exists in K and (ii) the two simplices in question belong to the same q-connected component of K.

Point (ii) explicitly indicates the relevance of our previous q-analysis to the dynamics of the process. If we define the unit vector

$$U = (1,1,\ldots,1) \quad ,$$

then the system obstruction vector is defined as

$$\hat{Q} = Q - U \quad .$$

Thus the non-zero components of $\hat{Q}$ indicate those q-levels in K for which a free change of pattern is not always possible, i.e. $\hat{Q}_q > 0$ implies the existence of a geometrical obstruction to the free change of $\Pi$. For a detailed mathematical discussion of this point, see [3].

Returning now to the Einsteinian interpretation of $\delta\Pi$, we see that it amounts to saying that the only changes of $\Pi$ that can arise are those which the geometry of the system permits. In other words, the geometry of the complex is altered from stage to stage so that all pattern changes are free. Thus, the only allowable pattern changes are those free changes which the geometry permits.

## LINEAR SYSTEMS

With the previous pages as prologue, we now turn to the question how the polyhedral dynamics methodology interfaces with more traditional concepts of mathematical system theory. In particular, we shall be concerned in this section with illustrating the use of polyhedral dynamics for analyzing the geometrical structure of linear systems. It will be seen that the severe restriction of linearity enables us to gain a number of new insights into important aspects of linear system theory and that the polyhedral dynamics concept suggests a number of new directions for future research.

To fix our notations, we regard a linear dynamical system $\Sigma$ as being equivalent to a triple of constant matrices $\Sigma = (F,G,H)$, connected through the dynamical equations

$$\Sigma: \quad \begin{aligned} dx/dt &= Fx(t) + Gu(t) \quad , \\ y(t) &= Hx(t) \quad . \end{aligned}$$

Here x is the n-dimensional state vector, u is an m-dimensional input vector, and y is a p-dimensional output vector. The matrices F,G,H, are of sizes $n \times n$, $n \times m$, $p \times n$, respectively, with entries in some field k. Further mathematical details arising from such a set-up may be found in the texts [7,8]; for present purposes, it suffices to think of $\Sigma$ as being a "machine" which transforms the inputs u(t) into the outputs y(t) by means of the intermediate "internal" variable x(t). The matrices F,G,H, then prescribe the internal structure of $\Sigma$, together with the resrtrictions upon how $\Sigma$ is allowed to interact with the outside world.

Our first task in attempting to interpret the above set-up in the context of polyhedral dynamics is to identify appropriate sets X and Y. To make headway on this problem, we take our cue from the approach used in the theory of differential forms to treat ordinary differential equations of the above type [9]. The differential forms analysis makes a sharp distinction between the "state" or "position" at a given instant and the instantaneous "change" or "velocity" at the point. In fact, they are regarded as conjugate objects. Since our earlier discussion has stressed the role of conjugate relations obtained from a given relation $\lambda$ by interchanging the roles of the sets X and Y, it seems reasonable to consider choosing the sets X and Y to consist of the states $\{x^i\}$ and the differentials or co-states $\{dx^i\}$. For the sake of definiteness,

$$X = \{x^1, x^2, \ldots, x^n\} = \text{simplices} \quad ,$$
$$Y = \{dx^1, dx^2, \ldots, dx^n\} = \text{vertices} \quad .$$

Having selected X and Y in the above manner, we turn to the definition of the relation $\lambda \subset Y \times X$. Since the elements of X and Y both refer only to the internal variable x, it is evident that the definition of $\lambda$ will not involve the external interaction matrices G and H, but will be confined to the internal coupling structure present in F. Thus, we define $\lambda$ by the rule

$$(dx^i, x^j) \in \lambda \rightleftarrows f_{ij} \neq 0 \quad .$$

Reversing the roles of X and Y, we immediately obtain the defining rule for the conjugate relation $\lambda^{-1}$ as

$$(x^i, dx^j) \in \lambda^{-1} \rightleftarrows f_{ji} \neq 0 \quad .$$

Thus, we see that the incidence matrix $\Lambda$ (or $\Lambda^{-1}$) is obtained from F by the rules

$$[\Lambda]_{ij} = \begin{cases} 1 \ , & f_{ij} \neq 0 \\ 0 \ , & f_{ij} = 0 \end{cases}$$

and

$$[\Lambda^{-1}]_{ij} = \begin{cases} 1 \ , & f_{ji} \neq 0 \\ 0 \ , & f_{ji} = 0 \end{cases} \quad .$$

(Recall: $\Lambda^{-1}$ denotes the incidence matrix for the conjugate relations and does not mean the inverse of $\Lambda$ in the usual sense.)

The foregoing definitions have been introduced to make it particularly simple to make contact with the usual system dynamics. Since we have already defined the pattern to be a mapping assigning a number from some field k to each simplex at each time, we now see that in the above linear system set-up, the general notion of a pattern is nothing more than the actual numerical realization of the state vector at each instant in time, i.e.

$$\Pi \ : \ X \to k^n$$
$$[\Pi(x^i)](t) \to x^i(t) \quad .$$

Hence, as discussed above in a more abstract context, the dynamics of the process are contained in the pattern $\Pi$ and how it changes over the complex and not in the underlying geometrical structure of the complex itself, although the geometry does determine how the pattern $\Pi$ can change. This is an important distinction which must be clearly kept in mind.

In order to fix the basic notions, we consider a prototypical example. Consider a single-input system $\Sigma$ given in control canonical form, i.e. m = 1,

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \alpha_i \in k \ .$$

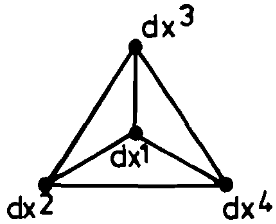
Assume that all $\alpha_i \neq 0$. It is easily seen that the incidence matrix for the relation $\lambda$ is

$$\Lambda = \begin{array}{c|ccccc} \lambda & dx^1 & dx^2 & dx^3 & \cdots & dx^n \\ \hline x^1 & 0 & 1 & 0 & \cdots & 0 \\ x^2 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & & \\ x^{n-1} & 0 & 0 & 0 & \cdots & 1 \\ x^n & 1 & 1 & 1 & \cdots & 1 \end{array}$$

while that for the conjugate relation $\lambda^{-1}$ is

$$\Lambda^{-1} = \Lambda' \ .$$

Geometrically, we may visualize (for n=4) the relation λ as




a tetrahedron representing the 3-simplex $x^4$, together with the three 0-simplices $x^1$, $x^2$, and $x^3$.

The structure vector Q is easily computed for this complex form Λ. We have

| | | | | |
|---|---|---|---|---|
| at level | $q = n - 1$ , | $Q_{n-1} = 1$ | $\{x^n\}$ | , |
| | $q = n - 2$ , | $Q_{n-2} = 1$ | $\{x^n\}$ | , |
| | $\vdots \quad \vdots$ | $\vdots$ | | |
| | $q = 1$ , | $Q_1 = 1$ | $\{x^n\}$ | , |
| | $q = 0$ , | $Q_0 = 1$ | {all} | . |

Thus,

$$Q = (\overset{n-1}{1},1,\ldots,\overset{0}{1}) \quad .$$

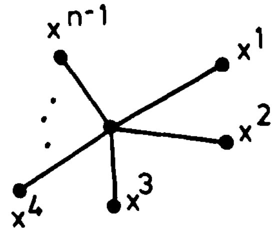
The conjugate complex, generated by the incidence matrix Λ', has the following connectivity structure:

| | | | | |
|---|---|---|---|---|
| at level | $q \geq 2$ , | $Q^{-1}_{\geq 2} = 0$ , | | |
| | $q = 1$ , | $Q^{-1}_1 = n - 1$ , | $\{dx^2\},\{dx^3\},\ldots,\{dx^4\}$ | , |
| | $q = 0$ , | $Q^{-1}_0 = 1$ , | {all} | . |

Hence,

$$Q^{-1} = (\overset{1}{n-1},\overset{0}{1}) \quad ,$$

and we have the geometrical representation




consisting of (n-1) 1-simplices $dx^2, \ldots, dx^n$, and the 0-simplex $dx^1$.

The eccentricities for the two complexes are easily seen to be

$$\mathrm{ecc}(x^i) = \begin{cases} 0 , & i \neq n \\ n-1 , & i = n , \end{cases}$$

$$\mathrm{ecc}(dx^i) = \begin{cases} 1 , & i \neq 1 \\ 0 , & i = 1 , \end{cases}$$

indicating that only the simplex $x^n$ "stands out" in the complex $K_X(Y;\lambda)$, while only the simplex $dx^1$ is "antisocial" in the conjugate complex.

The preceding discussion shows that the system matrix F determines the geometry of the internal structure of $\Sigma$. We now investigate that of the role of the input matrix G and the ouput matrix H.

From the standpoint of "Newtonian" inputs, or forces, it is easy to see from the systems dynamics that an applied input u(t) will directly influence the vertex $dx^i$ if and only if at least one entry in the ith row if G is non-zero. In geometrical terms, we regard an open-loop input u(t) as being a force which is exerted upon the vertices of the complex $K_X(Y;\lambda)$, with the entries of G determining which vertices are affected directly by u(t), and by what magnitude.

The situation becomes far more interesting, however, if we consider feedback inputs of the type $u(t) = -Kx(t)$, K being a fixed matrix. In this case, we see that the new system dynamics become

$$\dot{x} = (F-GK)x \quad ,$$

which, by the above rule for generating the relations $\lambda$, defines a new complex. Consequently, we are justified in interpreting feedback as being an "Einsteinian" input, in the sense that it changes the actual geometry of the problem. In this context, we see that the system input matrix G plays a far more central role in that its entries determine how the internal geometry of $\Sigma$ may be altered by means of feedback, or Einsteinian, inputs. Thus, we have the interpretations

feedback input $\leftrightarrow$ changes of system geometry

open-loop inputs $\leftrightarrow$ induced forces on a fixed geometry.

Turning now to the consideration of the observation matrix H, we have

$$y_i(t) = h_{i1}x_1(t) + h_{i2}x_2(t) + \ldots + h_{in}x_n(t) \quad , \qquad i = 1,2,\ldots,p \quad .$$

However, by virtue of our definition of the $x^i$'s as simplices of the complex, together with the discussion given earlier of chains and the chain group, we see that each system output $y_i(\cdot)$ is an element of the chain group $C_{\cdot}$, i.e. each $y_i(\cdot)$ is a linear combination (over the field k) of simplices (the $x^i$) of various dimensions.

Our previous interpretations of the pattern $\Pi$ as being the mapping which "evaluates" the argument at time t show that the actual numerical output, $y_i(t)$, is an element of the co-chain group, i.e.

$$[\Pi \circ y_i(\cdot)] \in C^{\cdot} \quad .$$

Thus,

$$\begin{aligned} [\Pi \circ y](x) &= Hx(t) \\ &= y(t) \quad . \end{aligned}$$

These considerations show that the elements of H determine the specific co-chain used to generate the systems output. Another way of looking at this is to regard H as determining the subgroup of the chain group which is used to generate the output co-chain.

In geometrical terms, H determines which "pieces" of the internal geometry are reflected in the system output, an intuitively satisfying role for the observation matrix to play.

## DISCUSSION

The preceding results give rise to a number of important (and non-standard) questions regarding the geometric structure of linear systems. Some of the issues which come to mind immediately are:

i) Given fixed F and G, what geometric changes are possible by application of feedback? Such a question is intimately related to standard controllability and pole-shifting results. For some preliminary work in a related direction we recommend [10] which treats the controllability problem from the structural rather than numerical point of view, i.e. utilizing only the zero/non-zero structure of F and G rather than their precise numerical entries.

ii) Given the structure vectors Q and $Q^{-1}$, is it possible to reconstruct a system matrix F giving rise to this structure? If so, is F unique, and if not, how can we characterize all possible F generating Q and $Q^{-1}$?

iii) How may we generalize classical stability concepts to apply to the structural setting? Some of the work on "connective" stability, as reported in [11], provides preliminary results in this direction.

iv) What is the meaning of cycles in the system-theoretic context and how should one interpret the homology groups and the Betti numbers? Clearly, these important topological invariants provide vital information concerning the geometric obstructions to a free flow of patterns throughout a given complex. However, more work is needed to make the notions precise.

v) In what manner may the above interpretations of X,Y and $\lambda$ be extended to non-linear systems? A re-examination of the basic definitions shows that the linearity assumptions on the system dynamics play a very minor role in the final results. The only place where we see linearity entering in any essential fashion is in the interpretation of the system outputs as elements of the chain group; however, even here there seems to be a possibility to allow nonlinearity by defining a nonlinear analog of the chain group, or alternatively we might ask whether nonlinearity is equivalent to <u>graded</u> linearity via $C_{.} = \{C_p\}$. In short, most of the basic definitions and results seem capable of extension to broad classes of non-linear systems with modest additional effort.

iv) Since application of feedback inputs may be used to change the geometry of $\Sigma$, we are led to pose a new class of control processes in which the criterion function is chosen to measure aspects of system structure. It is of considerable interest to investigate how the standard "optimal" feedback law associated with quadratic cost functions modifies the geometry of $\Sigma$ and ask whether another law might be preferable if the criteria were modified to include various structural costs. In addition, we might also consider control processes for which some of the admissible inputs consist of addition and/or deletion of vertices from the complex. All of these questions call for further study.

vii) Passing to the complex $K_Y(X:\lambda^{-1})$, we immediately see the relevance of the preceding set-up to the usual duality results of standard linear system theory [8]. The intriguing question that now arises is whether or not the added geometrical insight provided by the simplicial set-up will serve as sufficient mathematical inspiration to create a duality theory for nonlinear systems.

viii) What interpretation can be given to the homotopy group elements [2] either in $K_X(Y;\lambda)$ or in $K_Y(X:\lambda^{-1})$? This essentially <u>local</u> view of the geometrical structure might be relevant to conventional noise in $\Sigma$ [3].

# APPENDIX A

## Algorithm for Q-Analysis

If the cardinalities of the sets Y and X are m and n, respectively, the incidence matrix $\Lambda$ is an $(m \times n)$ matrix with entries 0 or 1. In the product $\Lambda\Lambda'$, the number in position $(i,j)$ is the result of the inner product of row i with row j of $\Lambda$. This number equals the number of 1's common to rows i and j in $\Lambda$. Therefore, it is equal to the value $(q+1)$, where q is the dimension of the shared face of the simplices $\sigma_p$, $\sigma_r$ represented by rows i and j. Thus, the algorithm is:

(1) For $\Lambda\Lambda'$ (an $m \times m$ matrix),

(2) Evaluate $\Lambda\Lambda' - \Omega$, where $\Omega$ is an $m \times m$ matrix all of whose entries are 1,

(3) Retain only the upper triangular part (including the diagonal) of the symmetric matrix $\Lambda\Lambda' - \Omega$. The integers on the diagonal are the dimensions of the $Y_i$ as simplices. The Q-analysis then follows by inspection.

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