Medvedev's logic and products of converse well orders

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Abstract

We show that modal (and hence, intuitionistic) Medvedev's logic is the intersection of the logics of finite direct powers of ordinals with the converse ordering taken without the top element, and that the latter logics have the finite model property. Then we provide other semantic characterizations of Medvedev's logic and related systems in terms of various natural substructures of products of ordinals.

Keywords: modal logic, Noetherian order, Medvedev's logic, finite model property, direct product of ordinals

1 Introduction

In this paper we study modal logics of Noetherian (in other terms, converse well-founded) partially ordered sets which are substructures of direct products of converse well-ordered sets, in particular, the products without an upper part.

Well-known examples of this kind are Grz.2 (the Grzegorczyk modal logic extended with the axiom of weak directedness) and modal Medvedev's logic. If follows from [6] that Grz.2 is the logic of all Boolean cubes $(2, \geq)^n$ with $n < \omega$. Such cubes without the top element are called *Medvedev's frames*; the intuitionistic logic of all Medvedev's frames is the well-known *Medvedev's logic of finite problems* [7,8]. *Modal Medvedev's logic* Mdv is the modal logic of these structures [9,12]. Also, Grz.2 and Mdv can be characterized as the logics of finite subsets of ω ordered by the converse inclusion:

$$\mathrm{Grz.2} = \mathrm{Log}(P_\omega(\omega), \supseteq) = \mathrm{Log}\{(2, \geq)^n\} : n < \omega\}, \tag{1}$$
$$\mathrm{Mdv} = \mathrm{Log}(P_\omega(\omega)\setminus\{\emptyset\}, \supseteq) = \mathrm{Log}\{(2, \geq)^n \setminus \{\mathrm{top}\} : n < \omega\}. \tag{2}$$

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In spite of the similarity in the above semantic characterizations, logical properties of Grz.2 and Mdv are different. In particular, Grz.2 is given by a finite set of axioms, so in view of its completeness with respect to a class of finite frames, it is decidable. It is well known that both modal and intuitionistic Medvedev logics are not finitely axiomatizable [6,9]. Whether the modal or intuitionistic logics of the structures $\{(2,\geq)^n \setminus \{\text{top}\} : n < \omega\}$ are recursively axiomatizable is an old-standing open problem.

In Sections 3 and 4, we consider modal logics of similar structures where instead of 2 any ordinal is allowed. For a finite n, we define $\Gamma(n)$ and $\Delta(n)$ as the logics of the frames $(\omega,\geq)^n$ and $(\omega,\geq)^n \setminus \{\text{top}\}$, respectively. It is straightforward that these logics have the finite model property. In Theorem 3.3, we show that the frames $(\alpha,\geq)^n$ and $(\alpha,\geq)^n \setminus \{\text{top}\}$ have the same logics for any infinite α; consequently (Corollary 3.4),

$$\Gamma(n) = \mathrm{Log}\big\{\textstyle\prod_{i<n}(\alpha_i,\geq) : \alpha_i \text{ is an ordinal}\big\},$$

$$\Delta(n) = \mathrm{Log}\big\{\textstyle\prod_{i<n}(\alpha_i,\geq) \setminus \{\text{top}\} : \alpha_i \text{ is an ordinal}\big\}.$$

We observe that $\bigcap_{n<\omega}\Gamma(n) = \text{Grz.2}$ (this fact is straightforward from (1)) and that $\bigcap_{n<\omega}\Delta(n) = \text{Mdv}$ (Theorem 4.5).

As well as Mdv, each of the logics $\Gamma(n)$, $\Delta(n)$ is characterized by a recursive set of finite frames, so these logics are co-recursively enumerable. It is known [12] that Mdv.2 = Grz.2. We show that $\Delta(n).2 = \Gamma(n)$ for all $n < \omega$ (Theorem 4.7); consequently, if $\Delta(n)$ is recursively axiomatizable, then so is $\Gamma(n)$. However, decidability of $\Gamma(n)$ and $\Delta(n)$ is an open problem for $n > 1$.

In Section 5, we study the restrictions of the direct powers $(\omega,\geq)^n$ to several $A \subseteq \omega^n$ and the modal logics of these restrictions, which thus generalize the above logics $\Gamma(n)$ and $\Delta(n)$; we also consider Noetherian subframes of infinite powers of converse ordinals.

In Section 6, we discuss other semantic interpretations of the logics $\Gamma(n)$ and $\Delta(n)$.

2 Preliminaries

We assume the reader's familiarity with the basic notions in modal logic, which can be found, e.g., in [1] or [2]; we recall only some standard concepts.

Modal formulas are built from a countable set of propositional variables $p, q, \ldots$ by using $\bot$ and $\to$ (chosen as the primitive Boolean connectives) and the unary modal operator $\Diamond$. Other connectives are standard abbreviations; in particular, $\Box$ abbreviates $\neg\Diamond\neg$. By a *(modal) logic* we mean a normal propositional uni-modal logic.

A Kripke frame $\mathfrak{F}' = (W', R')$ is a *weak subframe* of a Kripke frame $\mathfrak{F} = (W, R)$ iff $\emptyset \neq W' \subseteq W$ and $R' \subseteq R$. A Kripke model $\mathfrak{M}' = (\mathfrak{F}', \theta')$ is a *weak submodel* of a Kripke model $\mathfrak{M} = (\mathfrak{F}, \theta)$ iff $\mathfrak{F}'$ is a weak subframe of $\mathfrak{F}$ and $\theta'(p) = \theta(p) \cap W'$ for all propositional variables p. If $R' = R \cap (W' \times W')$, then $\mathfrak{F}'$ and $\mathfrak{M}'$ are called the *restrictions of* $\mathfrak{F}$ and *of* $\mathfrak{M}$ *to* W', respectively. If moreover, W' is *upward closed w.r.t.* R (in other words, is an *upper cone*, i.e., $x \in W'$ and xRy implies $y \in W'$), then these restrictions are called a *generated* *subframe* and a *generated submodel*, respectively. If $w \in W$ and W' is the least upward closed set that contains w, then the resulting substructures are said to be *point-generated*. A frame is *rooted* if it is generated by one of its points.

$\mathrm{Log}(\mathfrak{F})$ denotes the modal logic of a frame $\mathfrak{F}$ (or of a class of frames). If L is a modal logic, L.2 denotes its extension with the formula $\Diamond\Box p \to \Box\Diamond p$, and L.3 with the formula $\Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$. Recall that $(W, R) \models \Diamond\Box p \to \Box\Diamond p$ iff $R^{-1} \circ R \subseteq R \circ R^{-1}$; in particular, if (W, R) is a finite partial order with a least element, the latter means that there is a top element. For a non-strict partial order $\mathfrak{F}$, the formula $\mathfrak{F} \models \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p)$ means that every point-generated subframe of $\mathfrak{F}$ is linear. Grz denotes the logic of the class of all Noetherian (in other words, converse well-founded) non-strict partial orders, and is called the *Grzegorczyk logic*. The following facts are well known (see, e.g., [2]): Grz is the logic of all (finite) Noetherian non-strict partial orders; Grz.2 is the logic of the class of all (finite) Noetherian non-strict partial orders with a greatest element; Grz.3 is the logic of all (finite) linear Noetherian non-strict partial orders.

The following proposition is standard, see, e.g., [2, Section 8.5].

Proposition 2.1 (Generated Subframe Lemma)

(i) *If $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}$, then* $\mathrm{Log}(\mathfrak{F}) \subseteq \mathrm{Log}(\mathfrak{F}')$.

(ii) *Let $\{\mathfrak{F}_i : i \in I\}$ be a family of generated subframes of a frame $\mathfrak{F}$, and let every x in $\mathfrak{F}$ belong to some $\mathfrak{F}_i$. Then* $\mathrm{Log}\,\mathfrak{F} = \mathrm{Log}\{\mathfrak{F}_i : i \in I\}$.

The following two propositions are also well known (see, e.g., [1, Proposition 2.14 and Lemma 3.20]).

Proposition 2.2 *If $\mathfrak{G}$ is a p-morphic image of $\mathfrak{F}$, then* $\mathrm{Log}(\mathfrak{F}) \subseteq \mathrm{Log}(\mathfrak{G})$.

Proposition 2.3 *Let $\mathfrak{F}$ be transitive, $\mathfrak{G}$ finite rooted. If* $\mathrm{Log}(\mathfrak{F}) \subseteq \mathrm{Log}(\mathfrak{G})$, *then $\mathfrak{G}$ is a p-morphic image of a point-generated subframe of $\mathfrak{F}$.*

Given frames $\mathfrak{F}_i = (W_i, R_i)$, $i \in I$, their *direct product* is the frame $\prod_I \mathfrak{F}_i = (W, R)$ where $W = \prod_{i\in I} W_i$, the Cartesian product of the sets W_i, and R is defined point-wise: xRy iff $x_i R_i y_i$ for all $i \in I$. Given a frame $\mathfrak{F}$, we write $\mathfrak{F}^n$ for its nth direct power.

Proposition 2.4 *If $(\mathfrak{F}_i)_{i\in I}$ is a non-empty family of frames and we have $\forall x \exists y\, (xR_i y)$ in every $\mathfrak{F}_i$, then for every $i \in I$ the ith projection is a p-morphism of $\prod_I \mathfrak{F}_i$ onto $\mathfrak{F}_i$.*

Proof. Immediate from the definition. □

Proposition 2.5 *If $(\mathfrak{F}_i)_{i\in I}$ and $(\mathfrak{G}_i)_{i\in I}$ are non-empty families of frames such that for every $i \in I$ there exists a p-morphism of $\mathfrak{F}_i$ onto $\mathfrak{G}_i$, then there exists a p-morphism of $\prod_I \mathfrak{F}_i$ onto $\prod_I \mathfrak{G}_i$.*

Proof. For each $i \in I$, pick a p-morphism π_i of $\mathfrak{F}_i$ onto $\mathfrak{G}_i$. For each f in $\prod_I \mathfrak{F}_i$ put $\pi(f)(i) := \pi_i(f(i))$. It is straightforward that π is the required p-morphism. □

Definition 2.6 Let $\mathfrak{M}'$ be a weak submodel of $\mathfrak{M}$, and let Ψ be a set of modal formulas. $\mathfrak{M}'$ is a *selective filtration of* $\mathfrak{M}$ *through* Ψ iff the following holds: whenever $\mathfrak{M}, w \models \Diamond\psi$ for some $w \in W'$ and $\Diamond\psi \in \Psi$, then there exists $v \in W'$ such that $wR'v$ and $\mathfrak{M}, v \models \psi$ (i.e., v witnesses that $\Diamond\psi$ holds at w in $\mathfrak{M}$).

The following fact was known since 1970s.

Proposition 2.7 (Selective Filtration Lemma) *Let* Ψ *be a set of formulas closed under taking subformulas, and let* $\mathfrak{M}'$ *be a selective filtration of* $\mathfrak{M}$ *through* Ψ*. Then for all* $w \in W'$ *and* $\varphi \in \Psi$*, we have*

$$\mathfrak{M}', w \models \varphi \;\Leftrightarrow\; \mathfrak{M}, w \models \varphi.$$

Proof. Induction on φ. □

Through the paper, we regularly consider a frame $\mathfrak{F}$ with the removed top element (provided the relation of the frame is an order and the top element exists), for which we reserve the notation $\mathfrak{F} \setminus \{\text{top}\}$. In particular, if α is an ordinal, $(\alpha, \geq)^n \setminus \{\text{top}\}$ denotes the direct power $(\alpha, \geq)^n$ without the n-sequence of 0's.

3 Products of converse ordinals

Below we consider frames $(\alpha, \geq)$ where α is an ordinal and $\geq$ its converse ordering, and their direct products $\prod_{i<n}(\alpha_i, \geq)$ with finite n.

It follows from Proposition 2.4 that the logics of such frames decrease by inclusion with increasing ordinals α. We are going to show that, in fact, these logics do not depend a particular α whenever α is infinite; moreover, they are characterized by upper finite cones of their frames (and thus have the finite model property).

For a formula φ and a Kripke model $\mathfrak{M} = (W, R, \theta)$, let $\|\varphi\|_{\mathfrak{M}} = \{x \in W : \mathfrak{M}, x \models \varphi\}$; in particular, $\|p\|_{\mathfrak{M}} = \theta(p)$.

Lemma 3.1 *Let* $\mathfrak{M} = (W, \geq, \theta)$ *be a Kripke model such that* $\leq$ *is a well-founded partial order on* W*. Let* Ψ *be a set of formulas closed under taking subformulas, and let* V *be a non-empty subset of* W *such that*

$$\{x \in W : \exists\, \varphi \in \Psi\ (x \textit{ is } \leq\textit{-minimal in } \|\varphi\|_{\mathfrak{M}})\} \subseteq V.$$

Then the restriction of $\mathfrak{M}$ *to* V *is a selective filtration of* $\mathfrak{M}$ *through* Ψ*.*

Proof. Suppose $\mathfrak{M}, w \models \Diamond\psi$ for some $w \in V$ and $\Diamond\psi \in \Psi$. Then the set $\{v \in W : w \geq v\} \cap \|\psi\|_{\mathfrak{M}}$ is non-empty. Let v be a $\leq$-minimal element of this set. Then v is a $\leq$-minimal element of $\|\psi\|_{\mathfrak{M}}$, and hence $v \in V$. By the construction, $w \geq v$ and $\mathfrak{M}, v \models \psi$. □

Remark 3.2 Lemma 3.1 remains true if $\leq$ is a well-founded pre-order (in this case, x is said to be $\leq$-*minimal* iff for all $y \in W$, $y \leq x$ implies $x \leq y$.)

Theorem 3.3 *For all* $\alpha \geq \omega$ *and* $n < \omega$*, we have:*

(i) $\mathrm{Log}((\alpha, \geq)^n) = \mathrm{Log}((\omega, \geq)^n) = \mathrm{Log}\{(m, \geq)^n : m < \omega\}$,

(ii) $\mathrm{Log}((\alpha, \geq)^n \setminus \{\mathrm{top}\}) = \mathrm{Log}((\omega, \geq)^n \setminus \{\mathrm{top}\}) = \mathrm{Log}\{(m, \geq)^n \setminus \{\mathrm{top}\} : m < \omega\}$.

Proof. (i). For all $m < \omega$, $(m, \geq)^n$ is a generated subframe of $(\omega, \geq)^n$, and the latter is a generated subframe of $(\alpha, \geq)^n$ since $\omega \leq \alpha$. It follows that

$$\mathrm{Log}((\alpha, \geq)^n) \subseteq \mathrm{Log}((\omega, \geq)^n) \subseteq \mathrm{Log}\{(m, \geq)^n : m < \omega\}.$$

So it remains to show that $\mathrm{Log}\{(m, \geq)^n : m < \omega\} \subseteq \mathrm{Log}(\alpha, \geq)^n$. To this end, suppose that a modal formula φ is satisfiable in $(\alpha, \geq)^n$ and show that φ is satisfiable in $(m, \geq)^n$ for some $m < \omega$.

Assume there is a model $\mathfrak{M}$ on the frame $(\alpha, \geq)^n$ satisfying φ at some point, in other words, $\|\varphi\|_{\mathfrak{M}} \neq \emptyset$. Let Ψ consist of all subformulas of φ. For each $\psi \in \Psi$, we let

$$U_\psi := \{s \in \alpha^n : s \text{ is } \leq\text{-minimal in } \|\psi\|_{\mathfrak{M}}\} \text{ and } U := \bigcup_{\psi \in \Psi} U_\psi \,.$$

Every antichain in $(\alpha, \leq)^n$ is finite (by generalized Dickson's lemma, see, e.g., [5, Theorem 2.10]), so every U_ψ is finite. Hence, since Ψ is finite, U is finite as well.

Further, consider the projections of U for all $i < n$ and their product:

$$U_i := \{\beta < \alpha : \exists x \in U \; x(i) = \beta\} \text{ and } V := \prod_{i<n} U_i \,.$$

Since $\|\varphi\|_{\mathfrak{M}} \neq \emptyset$, and so $U_\varphi \neq \emptyset$, the set U contains some s such that $\mathfrak{M}, s \models \varphi$. Clearly, $U \subseteq V$. By Lemma 3.1, the restriction $\mathfrak{M}'$ of $\mathfrak{M}$ to V is a selective filtration of $\mathfrak{M}$ through Ψ. By the Selective Filtration Lemma (Proposition 2.7), we get $\mathfrak{M}', s \models \varphi$.

This proves that φ is satisfiable in $(V, \geq)$. Finally, for each $i < n$, we let

$$m_i := |U_i| \text{ and } m := \max_{i<n} m_i \,.$$

Clearly, $(V, \geq)$ is isomorphic to $(\prod_{i<n} m_i, \geq)$, and the latter is a generated subframe of $(m, \geq)^n$. It follows that φ is satisfiable in $(m, \geq)^n$, as required.

(ii). The proof is analogous with the only modification at the step when we define U_i. Namely, let $\mathfrak{M}$ be a model on the frame $(\alpha, \geq)^n \setminus \{\mathrm{top}\}$ such that $\|\varphi\|_{\mathfrak{M}} \neq \emptyset$, and let the set U be defined as above. Now for each $i < n$, we let

$$U_i := \{\beta < \alpha : \exists x \in U \; x(i) = \beta\} \cup \{0\}.$$

We define the numbers m_i and m as above and observe that the frame of the restriction of $\mathfrak{M}$ to the set $\prod_{i<n} U_i \setminus \{\mathrm{top}\}$ is isomorphic to the frame $(\prod_{i<n} m_i, \geq) \setminus \{\mathrm{top}\}$. The latter is a generated subframe of $(m, \geq)^n \setminus \{\mathrm{top}\}$. By the same reasoning as above, φ is satisfiable in $(m, \geq)^n \setminus \{\mathrm{top}\}$, as required.

The proof is complete. □

We introduce the following notation for these logics:

$$\Gamma(n) := \mathrm{Log}((\omega, \geq)^n),$$
$$\Delta(n) := \mathrm{Log}((\omega, \geq)^n \setminus \{\mathrm{top}\}).$$

Corollary 3.4 *For all finite n, we have:*

(i) $\Gamma(n) = \mathrm{Log}\{\prod_{i<n}(\alpha_i, \geq) : \alpha_i$ *is an ordinal*$\}$,

(ii) $\Delta(n) = \mathrm{Log}\{\prod_{i<n}(\alpha_i, \geq) \setminus \{\mathrm{top}\} : \alpha_i$ *is an ordinal*$\}$.

Proof. Follows from Theorem 3.3 and Proposition 2.4. □

Corollary 3.5 *For all finite n,* $\mathrm{Log}(\mathfrak{F}) = \mathrm{Grz.3}$ *implies* $\mathrm{Log}(\mathfrak{F}^n) = \Gamma(n)$ *and* $\mathrm{Log}(\mathfrak{F}^n \setminus \{\mathrm{top}\}) = \Delta(n)$.

Proof. For every finite m, there is a point-generated subframe of $\mathfrak{F}$ isomorphic to $(m, \leq)$, and so a point-generated subframe of $\mathfrak{F}^n$ isomorphic to $(m, \leq)^n$. Hence, $\mathrm{Log}(\mathfrak{F}^n) \subseteq \Gamma(n)$ by Theorem 3.3. For the converse inclusion, observe that any point-generated subframe of $\mathfrak{F}^n$ is the direct product of n converse well-orders, and so $\mathrm{Log}(\mathfrak{F}^n) \supseteq \Gamma(n)$ by Corollary 3.4. The same reasoning proves that $\mathrm{Log}(\mathfrak{F}^n \setminus \{\mathrm{top}\}) = \Delta(n)$. □

Remark 3.6 One can generalize the above corollary for the direct product of n frames $\mathfrak{F}_i$ with $\mathrm{Log}(\mathfrak{F}_i) = \mathrm{Grz.3}$ for all $i < n$.

4 The logics $\Gamma(n)$, $\Delta(n)$, and Medvedev's logic

In this section, we study the connection between the logics $\Gamma(n)$ and Grz.2 and between the logics $\Delta(n)$ and Medvedev's logic.

4.1 The sequences $\Gamma(n)$ and $\Delta(n)$

Let Mdv denote (modal) *Medvedev's logic*, which is the logic of all non-empty finite subsets of ω endowed with their converse inclusion order:

$$\mathrm{Mdv} := \mathrm{Log}(\mathcal{P}_\omega(\omega)\setminus\{\emptyset\}, \supseteq).$$

The following fact follows from [6] (see also [13], [12]):

Proposition 4.1 $\mathrm{Log}(\mathcal{P}_\omega(\omega), \supseteq) = \mathrm{Grz.2}$.

Lemma 4.2 *For any $n < \omega$, there are modal formulas $\varphi^{\max}_{\leq n}$ and $\varphi^{\mathrm{ram}}_{\leq n}$ such that for any finite partial order $\mathfrak{F} = (W, \leq)$ with a least element,*

(i) $\mathfrak{F} \models \varphi^{\max}_{\leq n}$ *iff there exist $\leq n$ maximal points above each $x \in W$,*

(ii) $\mathfrak{F} \models \varphi^{\mathrm{ram}}_{\leq n}$ *iff there exist $\leq n$ immediate successors of each $x \in W$.*

Proof. (i). Put

$$\varphi^{\max}_{\leq n} := \bigwedge_{i<n} \Diamond \Box \big(p_i \wedge \bigwedge_{i \neq j < n} \neg p_j\big) \to \Box \Diamond \bigvee_{i<n} p_i \,.$$

If $\mathfrak{F}$ has at most n maximal points and the premise of $\varphi^{\max}_{\leq n}$ holds in a point w in a model on $\mathfrak{F}$, then at each maximal point one of p_i, $i < n$, is true; hence the conclusion $\varphi^{\max}_{\leq n}$ holds at w.

If $\mathfrak{F}$ has at least $n+1$ maximal points $w_0, \dots, w_n$, put $\theta(p_i) = \{w_i\}$ for all $i < n$ and consider the model $(\mathfrak{F}, \theta)$; at the least element of $\mathfrak{F}$ (which exists by our assumption), the premise of $\varphi^{\max}_{\leq n}$ holds, and the conclusion is false, since all p_i, $i < n$, are false at w_n.

(ii). This statement is due to [3] (see also [2, Proposition 2.41]): there are intuitionistic formulas expressing the property on finite posets; the formulas $\varphi^{\mathrm{ram}}_{\leq n}$ are their Gödel–Tarski translations. □

Lemma 4.3 *For any n, $0 < n < \omega$,*

(i) $\varphi^{\max}_{\leq 1}$ *belongs to* $\Gamma(n)$,

(ii) $\varphi^{\max}_{\leq n}$ *belongs to* $\Delta(n)$ *but not to* $\Delta(n+1)$,

(iii) $\varphi^{\mathrm{ram}}_{\leq n}$ *belongs to* $\Delta(n)$ *and* $\Gamma(n)$ *but not to* $\Delta(n+1)$ *and* $\Gamma(n+1)$.

Proof. Clear. □

Proposition 4.4 *For all $n < \omega$,*

(i) $\Delta(1) = \Gamma(1) = \mathrm{Grz.3}$,

(ii) $\mathrm{Grz} \subset \Delta(n) \subset \Gamma(n)$ *if* $n \geq 2$,

(iii) $\Delta(n+1) \subset \Delta(n)$ *and* $\Gamma(n+1) \subset \Gamma(n)$,

(iv) $\Delta(n) \nsubseteq \Gamma(n+1)$ *and* $\Gamma(n) \nsubseteq \Delta(2)$.

Proof. Grz is valid in any Noetherian poset, so we have $\mathrm{Grz} \subseteq \Delta(n)$ in (ii). The logic Grz.3 is the logic of all Noetherian linearly ordered sets. This proves (i).

For $\Delta(n) \subseteq \Gamma(n)$, note that $(\omega^n, \geq)$ is a p-morphic image of the frame $\mathfrak{F} = (\omega, \geq)^n \backslash \{\mathrm{top}\}$; e.g., the map π defined by letting $(\pi s)(i) = \max(s(i), 1)$, for all s in $\mathfrak{F}$ and $i < n$, is a p-morphism of $\mathfrak{F}$ onto $(\omega \backslash 1, \geq)^n$, and the latter frame is obviously isomorphic to $(\omega, \geq)^n$. Also $(\omega, \geq)^n$ is isomorphic to a generated subframe of $(\omega, \geq)^{n+1}$, e.g., to the subframe consisting of elements $s \in \omega^{n+1}$ with $s(0) = 0$, and similarly for the frames without their top elements, thus proving $\Delta(n+1) \subseteq \Delta(n)$ and $\Gamma(n+1) \subseteq \Gamma(n)$ in (iii).

Furthermore, by Lemma 4.2(i),(ii), the formula $\varphi^{\max}_{\leq 1}$ is in $\Gamma(n) \backslash \Delta(n)$ whenever $n \geq 2$, whence it follows $\Delta(n) \neq \Gamma(n)$ in (ii) and $\Gamma(n) \nsubseteq \Delta(2)$ in (iv). Also by Lemma 4.2(ii), the formula $\varphi^{\max}_{\leq n}$ is in $\Delta(n) \setminus \Delta(n+1)$ and $\Gamma(n) \setminus \Gamma(n+1)$ whenever $n \geq 1$, whence it follows $\Delta(n+1) \neq \Delta(n)$ and $\Gamma(n+1) \neq \Gamma(n)$, and so $\mathrm{Grz} \neq \Delta(n)$, thus completing the proof of (ii) and (iii). Finally, by Lemma 4.2(iii), the formula $\varphi^{\mathrm{ram}}_{\leq n}$ is in $\Delta(n) \setminus \Gamma(n+1)$ whenever $n \geq 1$, which completes the proof of (iv) and also provides another way to see $\mathrm{Grz} \neq \Delta(n)$.

The proposition is proved. □

Theorem 4.5 $\bigcap_{n<\omega} \Delta(n) = \mathrm{Mdv}$ *and* $\bigcap_{n<\omega} \Gamma(n) = \mathrm{Grz.2}$.

Proof. The inclusion $\bigcap_{n<\omega} \Delta(n) \subseteq \mathrm{Mdv}$ is immediate from the fact that every frame $(2, \geq)^n \setminus \{\mathrm{top}\}$ is a point-generated subframe of $(\omega, \geq)^n \setminus \{\mathrm{top}\}$. By the same argument and Proposition 4.1, we get $\bigcap_{n<\omega} \Gamma(n) \subseteq \mathrm{Grz.2}$ as well.

Let us check the inclusion $\mathrm{Mdv} \subseteq \bigcap_{n<\omega} \Delta(n)$. For this, it suffices to construct, for each $n < \omega$, a p-morphism σ_n of $(\mathcal{P}_\omega(\omega) \backslash \{\emptyset\}, \supseteq)$ onto $(\omega, \geq)^n \setminus \{\mathrm{top}\}$.

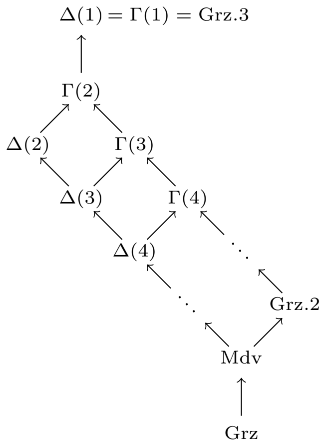

Fig. 1. Inclusions between logics $\Gamma(n)$ and $\Delta(n)$

Define it as follows. Partition ω into n infinite disjoint subsets X_i, $i < n$, and let $\sigma_n(A) = (|A \cap X_0|, \dots, |A \cap X_{n-1}|)$, for all $A \in \mathcal{P}_\omega(\omega) \setminus \{\emptyset\}$. It is easy to see that σ_n is a surjective p-morphism. Indeed, $A \supseteq B$ clearly implies $|A \cap X_i| \geq |B \cap X_i|$ for all $i < n$, and thus $\sigma_n(A) \geq \sigma_n(B)$; also if $\sigma_n(A) = (|A \cap X_0|, \dots, |A \cap X_{n-1}|) \geq (b_0, \dots, b_{n-1})$ then letting B_i consisting of the first b_i elements of $A \cap X_i$, we have $A \supseteq B$ and $\sigma_n(B) = (b_0, \dots, b_{n-1})$; and the surjectivity is obvious by the same reason.

Finally, we have $\mathrm{Grz.2} \subseteq \bigcap_{n<\omega} \Gamma(n)$ since every $(\omega, \geq)^n$ is a Noetherian poset with a top element.

The proof is complete. □

The diagram of strict inclusions between the considered logics is shown on Figure 1.

4.2 Remarks on axiomatization and decidability

By Lemma 4.3 (and Proposition 4.4), we have

$$\mathrm{Grz.2} + \varphi^{\mathrm{ram}}_{\leq n} \subseteq \Gamma(n) \ \text{ and } \ \mathrm{Grz} + \varphi^{\max}_{\leq n} + \varphi^{\mathrm{ram}}_{\leq n} \subseteq \Delta(n).$$

By Theorem 3.3, each of the logics $\Gamma(n)$, $\Delta(n)$ is characterized by a recursive set of finite frames, so these logics are co-recursively enumerable; hence, they are decidable if they are recursively axiomatizable.

Question 4.6 *Are the logics $\Gamma(n)$, $\Delta(n)$, $2 \leq n < \omega$, finitely axiomatizable? recursively axiomatizable?*

Theorem 4.7

(i) $\mathrm{Mdv.2} = \mathrm{Grz.2}$,

(ii) $\Delta(n).2 = \Gamma(n)$ *for all* $n < \omega$.

Proof. (i) This fact is known, see Propositions 13 and 9 in [12].

(ii) As $\Delta(n) \subseteq \Gamma(n)$, we have $\Delta(n).2 \subseteq \Gamma(n).2 = \Gamma(n)$.

In [10], it was shown that if a transitive logic L has the finite model property, then so does L.2. Hence $\Delta(n).2$ is the logic of the class of finite rooted $\Delta(n)$-frames with a greatest element.

Assume that a modal formula φ is refutable in a model $\mathfrak{M}$ on one of such frames $\mathfrak{F}$. Since $\mathfrak{F}$ validates $\Delta(n)$, it follows from Proposition 2.3 that $\mathfrak{M}$ is a p-morphic image of a model $\mathfrak{N}$ based on a point-generated subframe of $(\omega, \geq)^n \setminus \{\text{top}\}$. This p-morphism maps all maximal points of $\mathfrak{N}$ to the greatest point of $\mathfrak{M}$, and so the valuations of variables coincide at these points. Add a top point to $\mathfrak{N}$ with the same valuation and denote the new model by $\mathfrak{N}'$. Clearly, $\mathfrak{N}$ is a selective filtration of $\mathfrak{N}'$ through the set of all modal formulas. The formula φ is refutable in $\mathfrak{N}$ since $\mathfrak{M}$ is a p-morphic image of $\mathfrak{N}$. By the Selective Filtration Lemma (Proposition 2.7), φ is refutable in $\mathfrak{N}'$. Now it remains to observe that the frame of $\mathfrak{N}'$ is a $\Gamma(n)$-frame. This proves the inclusion $\Gamma(n) \subseteq \Delta(n).2$. ☐

Corollary 4.8 *If $\Delta(n)$ is finitely (recursively) axiomatizable, then so is $\Gamma(n)$.*

5 Substructures of finite powers

In this section, we study the restrictions of the direct powers $(\omega, \geq)^n$ to several $A \subseteq \omega^n$ and the modal logics of these restrictions, which thus generalize the logics $\Gamma(n)$ and $\Delta(n)$ considered above.

5.1 The logics $\Gamma(n, m)$

For an ordinal α, the *αth level* of a Noetherian frame $\mathfrak{F} = (X, \geq)$ consists of all points of $\mathfrak{F}$ having the rank α in the well-founded frame $(X, \leq)$.

The following fact is obvious.

Lemma 5.1 *The kth level of $(\omega, \geq)^n$ consists precisely of sequences s such that $\sum_{i<n} s(i) = k$, so its size is the number of (weak) compositions $\binom{k+n-1}{n-1}$.*

Let $\mathfrak{P}_{n,m}$ denote the restriction of the frame $(\omega, \geq)^n$ to its levels $\geq m$; so the universe of $\mathfrak{P}_{n,m}$ is $\left\{s \in \omega^n : \sum_{i<n} s(i) \geq m\right\}$. Four first frames for the case of $n = 2$ are depicted on Figure 2.

For $1 \leq n < \omega$, $0 \leq m < \omega$, let

$$\Gamma(n, m) := \mathrm{Log}(\mathfrak{P}_{n,m}).$$

Let also

$$\begin{aligned} \Gamma(\omega, m) &:= \mathrm{Log}\{\mathfrak{P}_{n,m} : n < \omega\}, \\ \Gamma(n, \omega) &:= \mathrm{Log}\{\mathfrak{P}_{n,m} : m < \omega\}, \\ \Gamma(\omega, \omega) &:= \mathrm{Log}\{\mathfrak{P}_{n,m} : m, n < \omega\}. \end{aligned}$$

Thus $\Gamma(\omega, m) = \bigcap_{n<\omega} \Gamma(n, m)$, $\Gamma(n, \omega) = \bigcap_{m<\omega} \Gamma(n, m)$, and $\Gamma(\omega, \omega) = \bigcap_{n,m<\omega} \Gamma(n, m)$. For sake of completeness we may let $\Gamma(0, 0) :=$ the logic of a trivial frame consisting of a reflexive singleton, axiomatized by $p \leftrightarrow \Diamond p$.

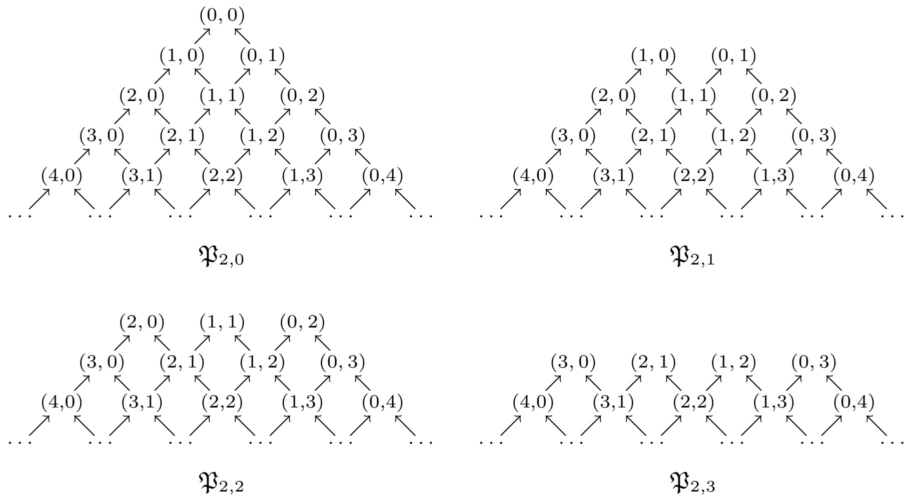

Fig. 2. Frames $\mathfrak{P}_{2,m}$

We restate the previous definitions and facts proved above:

(i) $\Gamma(1, m) = \mathrm{Grz.3}$ for all $m \le \omega$,

(ii) $\Gamma(n, 0) = \Gamma(n)$ and $\Gamma(n, 1) = \Delta(n)$,

(iii) $\Gamma(\omega, 0) = \mathrm{Grz.2}$ and $\Gamma(\omega, 1) = \mathrm{Mdv}$.

The following theorem generalizes Proposition 4.4.

Theorem 5.2

(i) $\mathrm{Grz} \subseteq \Gamma(\omega, \omega)$,

(ii) $\Gamma(n, m) \supseteq \Gamma(n', m')$ *if* $n \le n' \le \omega$ *and* $m \le m' \le \omega$,

(iii) $\Gamma(n, \omega) \nsubseteq \Gamma(n + 1, 0)$,

(iv) $\Gamma(n, m) \nsubseteq \Gamma(n', m')$ *if* $\binom{m+n-1}{n-1} < \binom{m'+n'-1}{n'-1}$.

Proof. Item (i) is clear, and for (ii), it suffices to consider only the cases of $n' = n + 1$ and $m' = m + 1$.

We have $\Gamma(n, m) \supseteq \Gamma(n + 1, m)$ since the frame $\mathfrak{P}_{n,m}$ is isomorphic to the generated subframe of $\mathfrak{P}_{n+1,m}$ consisting of s with $s(0) = 0$. To show $\Gamma(n, m) \supseteq \Gamma(n, m+1)$, we shall construct a p-morphism of $\mathfrak{P}_{n,m+1}$ onto $\mathfrak{P}_{n,m}$.

To simplify our construction, we first replace $\mathfrak{P}_{n,m}$ with the substructure $\mathfrak{Q}_{n,m}$ of $\mathfrak{P}_{n,m+1}$ consisting of s with $s(0) > 0$. The shift ρ defined by letting $(\rho s)(0) = s(0) + 1$, and $(\rho s)(i) = s(i)$ if $0 < i < n$, is an isomorphism of $\mathfrak{P}_{n,m}$ onto $\mathfrak{Q}_{n,m}$.

Now we construct a map π that gives, for all $m < \omega$, a p-morphism of $\mathfrak{P}_{n,m+1}$ onto $\mathfrak{Q}_{n,m} \subseteq \mathfrak{P}_{n,m+1}$. For any $s \in \omega^n \setminus \{(0, \ldots, 0)\}$, define $\pi(s)$ as follows: if $s(0) > 0$, let $\pi s = s$; otherwise, i.e., if $s(0) = 0$, let

$$(\pi s)(i) := \begin{cases} 1 & \text{if } i = 0, \\ s(i) - 1 & \text{if } i = \min \operatorname{supp}(s), \\ s(i) & \text{otherwise,} \end{cases}$$

where $\operatorname{supp}(s) := \{i < n : s(i) \neq 0\}$, the *support* of s. For brevity, let also $i_s := \min \operatorname{supp}(s)$. Note that $s \geq t$ implies $\operatorname{supp}(s) \supseteq \operatorname{supp}(t)$ and so $i_s \leq i_t$.

Let us verify that π is as required. Obviously, π maps $\mathfrak{P}_{n,m+1}$ onto $\mathfrak{Q}_{n,m}$ and leaves all the points of $\mathfrak{Q}_{n,m}$ fixed. It is easy to see that π preserves the levels: $\sum_{i<n} (\pi s)(i) = \sum_{i<n} s(i)$. We must show that π is a p-morphism, i.e., a homomorphism with the lifting property. This is a routine check, which however we write down for the sceptic reader.

Pick $s \geq t$ and show $\pi s \geq \pi t$. For the case of $t(0) > 0$, we have $s(0) > 0$ and so $\pi s = s \geq t = \pi t$. Consider now the case of $t(0) = 0$. We have then $(\pi t)(0) = 1$. But if $s(0) > 0$ then $(\pi s)(0) = s(0) \geq 1$, and if $s(0) = 0$ then $(\pi s)(0) = 1$, thus we have $(\pi t)(0) \leq (\pi s)(0)$ in any case. Further, if $s(0) > 0$ then $(\pi s)(i) = s(i) \geq t(i) \geq (\pi t)(i)$ for all $i \in n \setminus \{0\}$. If $s(0) = 0$, the same relationship $(\pi s)(i) = s(i) \geq t(i) \geq (\pi t)(i)$ holds for all $i \in n \setminus \{0, i_s\}$; moreover, $i_s \leq i_t$. But if $i_s = i_t$ then $(\pi s)(i_s) = s(i_s) - 1 \geq t(i_s) - 1 = (\pi t)(i_s)$, and if $i_s < i_t$ then $(\pi t)(i_s) = 0$ since $0 < i_s < i_t$; thus we have $(\pi s)(i_s) \geq (\pi t)(i_s)$ in any case.

Pick now $\pi s \geq t'$ with $t' \in \operatorname{ran}(\pi)$, i.e., with $\pi t' = t'$, and find t with $s \geq t$ and $\pi t = t'$. If $s(0) > 0$ then $\pi s = s$, so it suffices to let $t = t'$. If $s(0) = 0$, define t by letting $t(0) = 0$, $t(i_s) = t'(i_s) + 1$, and $t(i) = t'(i)$ if $i \in n \setminus \{i_s\}$. Note that $i_t = i_s$. We have: $s(0) = t(0) = 0$, so $(\pi s)(0) = (\pi t)(0) = 1$, and $t'(0) = 1$ (since $1 \leq t'(0) \leq (\pi s)(0) = 1$); also $s(i_s) = (\pi s)(i_s) + 1 \geq t(i_s) + 1 = t(i_s)$ and $(\pi t)(i_s) = t(i_s) - 1 = t'(i_s) + 1 - 1 = t'(i_s)$; finally, if $i \in n \setminus \{0, i_s\}$ then $s(i) = (\pi s)(i) \geq t'(i) = t(i)$.

So we have proved that π is a p-morphism of $\mathfrak{P}_{n,m+1}$ onto $\mathfrak{Q}_{n,m}$.

Finally, for (iii), note that $\varphi^{\text{ram}}_{\leq n} \in \Gamma(n, \omega) \setminus \Gamma(n + 1, 0)$, and for (iv), that $\varphi^{\max}_{\leq N} \in \Gamma(n, m) \setminus \Gamma(n', m')$ where $N = \binom{m+n-1}{n-1}$ by Lemma 5.1. $\square$

Corollary 5.3

(i) $\Gamma(n, m) \supset \Gamma(n', m)$ *if* $n < n'$,

(ii) $\Gamma(n, m) \supset \Gamma(n, m')$ *if* $m < m'$.

Proof. Theorem 5.2(ii)–(iv). $\square$

The diagram of (non-strict) inclusions between the logics $\Gamma(n, m)$ is shown on Figure 3.

Question 5.4 *What about the inclusions of the logics that are not under the scope of Theorem 5.2? E.g., is* $\Gamma(3, 1) \subseteq \Gamma(2, 2)$*?*

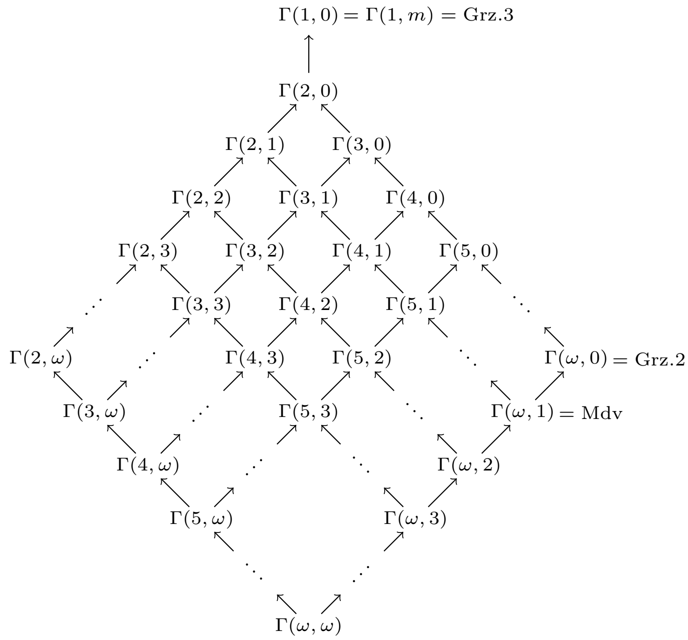

Fig. 3. Inclusions between logics $\Gamma(n, m)$

As we have Mdv $= \Gamma(\omega, 1) \supseteq \Gamma(\omega, 2) \supseteq \Gamma(\omega, 3) \supseteq \ldots \supseteq \Gamma(\omega, \omega) \supseteq$ Grz, these logics can be regarded as "sub-Medvedev"; the next proposition below confirms this view. Given cardinals $\kappa < \lambda$, let $\mathcal{P}_{\lambda,\kappa}(X) := \mathcal{P}_\lambda(X) \setminus \mathcal{P}_\kappa(X) = \{A \subseteq X : \kappa \leq |A| < \lambda\}$.

Proposition 5.5 *For all $m < \omega$,*

$$\Gamma(\omega, m) = \mathrm{Log}(\mathcal{P}_{\omega,m}(\omega), \supseteq) = \mathrm{Log}\{(\mathcal{P}_{n+1,m}(n), \supseteq) : n < \omega\}.$$

Proof. The second equality is clear since the finite frames $(\mathcal{P}_{n+1,m}(n), \supseteq)$ are point-generated subframes of $(\mathcal{P}_{\omega,m}(\omega), \supseteq)$ and cover it.

Also each $(\mathcal{P}_{n+1,m}(n), \supseteq)$ is isomorphic to a point-generated subframe of $\mathfrak{P}_{n,m}$ (via characteristic functions), whence we get $\subseteq$ in the first equality. To prove that $\supseteq$ holds as well, one use the maps σ_n constructed in the proof of Theorem 4.5. Recall that, for a partition of ω into n infinite subsets X_i, $i < n$, we let $\sigma_n(A)(i) := |A \cap X_i|$ for all $i < n$ and $A \in \mathcal{P}_\omega(\omega)$. Then σ_n is a p-morphism of $(\mathcal{P}_\omega(\omega), \supseteq)$ onto $(\omega, \geq)^n$; moreover, it is easy to see that it preserves the levels, i.e., the ranks of points (where, of course, the kth level of $(\mathcal{P}_\omega(\omega), \supseteq)$ consists of sets of size k), hence its restriction to $\mathcal{P}_{\omega,m}(\omega)$ is a p-morphism onto $\mathfrak{P}_{m,n}$. □

Related systems, the intuitionistic logics of $(\mathcal{P}(\omega) \setminus \mathcal{P}(m), \supseteq)$, $0 < m < \omega$, were considered in [13]; none of them are finitely axiomatizable. We conjecture that the logics $\Gamma(\omega, m)$ are not finitely axiomatizable as well.

5.2 The logics of monotone sequences

Given $n < \omega$, let

$$D^{\leq}(n) = \{s \in \omega^n : s(0) \leq \ldots \leq s(n-1)\},$$
$$D^{\geq}(n) = \{s \in \omega^n : s(0) \geq \ldots \geq s(n-1)\}.$$

Note that $D^{\leq}(2) = \leq$ and $D^{\geq}(2) = \geq$ (where $\leq$ and $\geq$ are on ω).

It is easy to see that for any n, the frames $(D^{\leq}(n), \geq)$ and $(D^{\geq}(n), \geq)$ are isomorphic under the natural isomorphism taking $(s(0), \ldots, s(n-1))$ to $(s(n-1), \ldots, s(0))$, hence it suffices to consider only one of them.

Theorem 5.6 $\mathrm{Log}(D^{\leq}(n), \geq) = \Gamma(n)$.

Proof. To see the inclusion $\Gamma(n) \subseteq \mathrm{Log}(D^{\leq}(n), \geq)$, define $\pi : \omega^n \to D^{\leq}(n)$ by letting for all $i < n$,

$$(\pi s)(i) := \max_{j \leq i} s(j).$$

Clearly, π is surjective, and it leaves all points of $D^{\leq}(n)$ fixed: $\pi s = s$ for all $s \in D^{\leq}(n)$. Also $s \geq t$ clearly implies $\pi s \geq \pi t$. And if $\pi s \geq t'$ for some $t' \in D^{\leq}(n)$, letting $t(i) = t'(i)$ if $s(i) = \pi s(i)$, and $t(i) = 0$ otherwise, we have $s \geq t$ and $\pi t = t'$. Thus π is a p-morphism of $(\omega^n, \geq)$ onto $(D^{\leq}(n), \geq)$, which proves the inclusion above.

To prove the converse inclusion $\mathrm{Log}(D^{\leq}(n), \geq) \subseteq \Gamma(n)$, for each $k < \omega$, let $S_k \subseteq D^{\leq}(n)$ be the subframe generated by the point $(k, k \cdot 2, \ldots, k \cdot n)$, and let $C_k \subseteq S_k$ be the "n-dimensional cube" of size k^n consisting of all points between the points $(0, k, k \cdot 2, \ldots, k \cdot (n-1))$ and $(k, k \cdot 2, k \cdot 3, \ldots, k \cdot n)$:

$$C_k := \{s \in \omega^n : k \cdot i \leq s(i) \leq k \cdot (i+1) \text{ for all } i < n\}.$$

Define $\pi_k : S_k \to C_k$ by letting for all $i < n$,

$$(\pi_k s)(i) := \max(s(i), k \cdot i).$$

It is easy to see that π_k leaves all points of C_k fixed, and it is a p-morphism of $(S_k, \geq)$ onto $(C_k, \geq)$, whence it follows $\mathrm{Log}(S_k) \subseteq \mathrm{Log}(C_k)$. But $(C_k, \geq)$ are isomorphic to $(n^k, \geq)$, whence $\bigcap_{k<\omega} \mathrm{Log}(C_k, \geq) = \Gamma(n)$ by Theorem 3.3, and $(S_k, \geq)$ are point-generated subframes of $(D^{\leq}(n), \geq)$ with $\bigcup_{k<\omega} S_k = D^{\leq}(n)$, whence $\bigcap_{k<\omega} \mathrm{Log}(S_k, \geq) = \mathrm{Log}(D^{\leq}(n), \geq)$ by Proposition 2.1(ii). Therefore,

$$\mathrm{Log}(D^{\leq}(n), \geq) = \bigcap_{k<\omega} \mathrm{Log}(S_k, \geq) \subseteq \bigcap_{k<\omega} \mathrm{Log}(C_k, \geq) = \Gamma(n),$$

as required. □

A similar result can be obtained for the logics $\Delta(n)$; for this, however, it does not suffice to remove only the top element from the frame of monotone sequences as the residual has its own top (e.g., in $D^{\leq}(2)$, under removing $(0,0)$ the new top $(0,1)$ appears). The precise statement of such a result on $\Delta(n)$ is hence more complicated, and we postpone this for the further work.

5.3 Noetherian substructures of infinite powers

The structures consisting of all finite subsets and of all finite sequences can be naturally identified with certain substructures of infinite powers. Hence, Mdv and, more generally, the logics $\Gamma(\omega, m)$ considered above, can be also regarded as instances of logics of such substructures.

Consider the frame consisting of all eventually zero sequences endowed with the point-wise order; it can be regarded as a subdirect infinite power (while all sequences form the usual direct power). Given ordinals α, β, define

$$E(\alpha, \beta) := \{f \in \alpha^\beta : |\mathrm{supp}(f)| < \omega\} \text{ where } \mathrm{supp}(f) := \{i < \beta : f(i) \neq 0\}.$$

As above, we consider the point-wise order on $E(\alpha, \beta)$: for $f, g \in \alpha^\beta$, we let $f \leq g$ iff $f(i) \leq g(i)$ for all $i < \beta$; so the top element of $(E(\alpha, \beta), \geq)$ is the β-sequence of 0's.

Theorem 5.7 *For all infinite ordinals α, β,*

$$\mathrm{Log}((E(\alpha, \beta), \geq) \setminus \{\mathrm{top}\}) = \mathrm{Mdv} \textit{ and } \mathrm{Log}(E(\alpha, \beta), \geq) = \mathrm{Grz.2}.$$

Proof. Given $f \in \alpha^\beta$, the subframe of the frame $(E(\alpha, \beta), \geq)$ generated by the point f is isomorphic to $(\alpha^{|\mathrm{supp}(f)|}, \geq)$, and likewise for the frame without the top element. As $\alpha, \beta \geq \omega$, Theorem 3.3 gives the conclusion. □

6 Other characterizations of $\Gamma(n)$ and $\Delta(n)$

The logics $\Gamma(n)$ and $\Delta(n)$ admit different semantic interpretations.

1. By the *standard translation* argument (see, e.g., [1, Section 2.4]), the logics $\Gamma(n)$ and $\Delta(n)$ are fragments of the *n-adic* (i.e., relation variables are n-ary) second order logic over natural numbers with the standard ordering. Propositional variables p are interpreted as n-ary predicates on ω, the order on tuples in $(\omega, \geq)^n$ is interpreted via conjunctions $\bigwedge_{i<n} x_i \geq y_i$.

2. The logics $\Gamma(2)$ and $\Delta(2)$ can be considered in the context of interval temporal logic (see, e.g., [4]). Let W be the set of closed segments $[m, n]$ of integer numbers containing a fixed integer (e.g., 0):

$$W := \{[m, n] : m \leq 0 \leq n\} \text{ where } [m, n] := \{k \in \mathbb{Z} : m \leq k \leq n\}.$$

It is immediate that $\mathrm{Log}(W, \supseteq)$ is $\Gamma(2)$ and $\mathrm{Log}(W \setminus \{[0, 0]\}, \supseteq)$ is $\Delta(2)$ according to the fact that $(W, \supseteq)$ is isomorphic to $(\omega, \geq)^2$; the isomorphism is defined by letting $(m, n) \mapsto [-m, n]$.

3. For the first time, the logics $\Gamma(n)$ and $\Delta(n)$ has been appeared in studies of modal logics of model-theoretic relations undertaken in a recent paper [11]. Referring the reader to that paper for related concepts and results, we prove the following characterization of the logics $\Gamma(2^n)$ and $\Delta(2^n)$, announced there.

Theorem 6.1 *Let $n < \omega$, and let τ be a signature consisting of n unary predicates and possibly some constants. The robust modal logic of the class of models of τ with the submodel relation is $\Gamma(2^n)$ if τ has at least one constant, and $\Delta(2^n)$ otherwise.*

Proof. Let $\sqsupseteq$ denote the *submodel relation* on models of a given signature: $\mathfrak{A} \sqsupseteq \mathfrak{B}$ iff $\mathfrak{B}$ is a submodel of the model $\mathfrak{B}$. Let also $\simeq$ be the isomorphism (equivalence) relation and $[\mathfrak{A}]_\simeq$ the equivalence class $\{\mathfrak{A}' : \mathfrak{A}' \simeq \mathfrak{A}\}$, i.e., the isomorphism type of $\mathfrak{A}$. We define $[\mathfrak{A}]_\simeq \sqsupseteq [\mathfrak{B}]_\simeq$ iff there exist $\mathfrak{A}' \simeq \mathfrak{A}$ and $\mathfrak{B}' \simeq \mathfrak{B}$ such that $\mathfrak{A}' \sqsupseteq \mathfrak{B}'$. By [11, Theorem 29], the robust modal logic of the submodel relation on the class $\mathfrak{K}$ of all models of a given signature τ is $\mathrm{Log}\{(\mathrm{Sub}_\simeq(\mathfrak{A}), \sqsupseteq_\simeq) : \mathfrak{A} \in \mathfrak{K}\}$ where $\mathrm{Sub}_\simeq(\mathfrak{A})$ is $\{[\mathfrak{B}]_\simeq : \mathfrak{A} \sqsupseteq \mathfrak{B}\}$.

Fix an enumeration $P_0, \dots, P_{n-1}$ of the unary predicates in τ. Let $\mathfrak{A}$ be a model of τ with the universe A, and let W be the set of cardinals $\leq |A|$. For each $B \subseteq A$ define a map $s_B : \mathcal{P}(n) \to W$ by letting, for all $X \subseteq n$,

$$s_B(X) := |\{a \in B : \mathfrak{A} \vDash P_i(a) \text{ iff } i \in X\}|.$$

Note that the submodels of $\mathfrak{A}$ given by subsets B and B' are isomorphic iff $s_B(X) = s_{B'}(X)$ for all $X \subseteq n$; we let $s_B \sim s_{B'}$ iff this is the case. Hence we can define a map $f : \mathrm{Sub}_\simeq(\mathfrak{A}) \to W^{\mathcal{P}(n)}$ by letting $f([B]_\simeq) := [s_B]_\sim$, and this map is injective.

Moreover, note that each model $\mathfrak{B}$ of τ is a submodel of a model $\mathfrak{A}$ of τ such that $s_{\mathfrak{A}}(n)$ has the greatest possible value $|A|$. Thus $\mathrm{Sub}_\simeq(\mathfrak{B})$ forms a point-generated subframe of $\mathrm{Sub}_\simeq(\mathfrak{A})$. Therefore, by Proposition 2.1, it suffices to handle the case of such models $\mathfrak{A}$. In this case, f is a bijection between $\mathrm{Sub}_\simeq(\mathfrak{A})$ and $W^{\mathcal{P}(n)}$ providing τ contains at least one constant; the top element relates to the least submodel of $\mathfrak{A}$ consisting of its constants (and the bottom element, of course, to the whole model $\mathfrak{A}$). Moreover, f is an isomorphism of $(\mathrm{Sub}_\simeq(\mathfrak{A}), \sqsupseteq_\simeq)$ and $(W, \geq)^{2^n}$ where $\leq$ is the usual ordering of cardinals. For the case without constants, the corresponding structure is $(W, \geq)^{2^n} \setminus \{\mathrm{top}\}$; maximal elements relate to the singleton submodels of $\mathfrak{A}$. Since $\leq$ on W is a well-order (under the axiom of choice), the statement now follows from Theorem 3.3. $\square$

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