Shapley's "2×2" Theorem for game forms*

Nikolai S. Kukushkin†

August 11, 2007

Abstract

If a finite two person game form has the property that every 2×2 fragment is Nash consistent, then no derivative game admits an individual improvement cycle.

1 Introduction

Shapley (1964) showed that if every 2×2 submatrix of a payoff matrix possesses a saddle point, then the whole matrix also possesses a saddle point. Such matrices were studied by Gurvich and Libkin (1990). The result does not extend even to bimatrix games.

The purpose of this note is to show that a strengthened version of Shapley's theorem holds for bimatrix game forms: If a finite two person game form has the property that every 2×2 fragment is Nash consistent, then no derivative game admits an individual improvement cycle (hence the whole game form is Nash consistent as well).

The notions of individual improvement paths and cycles are taken from Monderer and Shapley (1996). If a finite strategic game admits no improvement cycle, then every fragment possesses a Nash equilibrium. The converse does not hold even for two person games (Takahashi and Yamamori, 2002). For two person *game forms*, however, it happens to hold; whether it holds for more than two players remains an open question.

Nash consistency of a game form satisfying our condition can be derived from the existing literature (Vladimir Gurvich, personal communication): If the players are given arbitrary antagonistic preferences, then the derivative game possesses a saddle point by Shapley (1964); by Moulin (1976), the game form must be tight; by Gurvich (1988), it is Nash consistent. However, the absence of improvement cycles is a much stronger requirement. Several examples of such game forms are given in Kukushkin (2002). Theorem 1 from that paper describes a class of games with perfect information satisfying the requirement; the class contains Rosenthal's (1981, Example 3) centipede game (the last observation is due to Dave Furth).

Milchtaich (1996) introduced a more restrictive notion of a best response improvement path (cycle). Kukushkin (2004) obtained natural sufficient conditions for the absence of such cycles in a strategic game. Corollary 2 below shows that there is no difference between the two acyclicity notions as long as two-person game forms are considered.

Section 2 contains the basic definitions and the formulation of the main result; its proof is in Section 3. Possible extensions and open questions are presented in Section 4.

*Financial support from a Presidential Grant for the State Support of the Leading Scientific Schools (NSh-5379.2006.1), the Russian Foundation for Basic Research (grant 05-01-00942), and the Spanish Ministry of Education (project SEJ 2004-00968) is acknowledged. I thank Vladimir Gurvich for a fruitful discussion.

†Russian Academy of Sciences, Dorodnicyn Computing Center, 40, Vavilova, Moscow 119333 Russia
E-mail: ququns@pochta.ru

2 Formulations

A *finite game form* G is defined by a finite set of players N, a finite strategy set X_i for each $i \in N$, a finite set of outcomes A and a mapping $g\colon X_N \to A$, where $X_N = \prod_{i\in N} X_i$ is the set of strategy profiles. Once preferences of the players over the outcomes are specified, and we always assume this to be done with a list υ_N of ordinal utilities $\upsilon_i\colon A \to \mathbb{R}$, $i \in N$, a *derivative game* $G(\upsilon_N)$ emerges, in which the set of players is N, the strategy sets are X_i's and utilities are $u_i(x_N) = \upsilon_i(g(x_N))$.

A *strategic path* is a finite or infinite sequence $\{x_N^k\}_{k=0,1,\dots}$ of strategy profiles such that x_N^{k+1} and x_N^k differ in the choice of just one player. A *strategic cycle* is a strategic path $x_N^0, x_N^1, \dots, x_N^M$ such that $x_N^0 = x_N^M$ and $M > 0$. A strategic path (cycle) is an *improvement path* (*cycle*) in a derivative game $G(\upsilon_N)$ if $u_i(x_N^{k+1}) > u_i(x_N^k)$ whenever x_N^{k+1} and x_N^k differ in x_i. If, additionally, x_i^{k+1} is a best response to x_{-i}^k, we have a *best response improvement path* (*cycle*).

A game form G is *acyclic* if no derivative game $G(\upsilon_N)$ admits an improvement cycle. A game form G is *Nash consistent* if every derivative game $G(\upsilon_N)$ possesses a Nash equilibrium. Since we only consider finite games, every acyclic game form is Nash consistent.

Lemma 2.1. *A 2×2 game form $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ is Nash consistent if and only if $\{a, d\} \cap \{b, c\} \neq \emptyset$.*

Proof. The sufficiency immediately follows from Gurvich (1988); the necessity, from Moulin (1976). Both are easy to check by themselves. Note that every Nash consistent 2×2 game form is acyclic. $\square$

A *fragment* G' of G is a game form with the same set of players N and nonempty subsets $\emptyset \neq X_i' \subseteq X_i$ for all $i \in N$. If G is acyclic, then so is every fragment of G; Nash consistency need not be "inherited" in this sense.

Remark. Shapley (1964) used the term "subgame," but since then it has become widely used in the literature on extensive games with a different meaning.

Theorem. *A finite two person game form G is acyclic if and only if every 2×2 fragment of G is Nash consistent.*

The necessity is straightforward; the sufficiency proof is deferred to the next section.

3 Proof

Till the end of the proof, we assume to the contrary that there is a strategic cycle $x_N^0, x_N^1, \dots, x_N^M = x_N^0$ which becomes an improvement cycle in a derivative game $G(\upsilon_N)$. Without restricting generality, we may assume that there is no shorter improvement cycle in any derivative game, hence the improvements of both players alternate along the cycle, hence $M = 2m$. Since every 2×2 fragment is Nash consistent, $m > 2$.

Without restricting generality, we assume $N = \{1, 2\}$. Since the cycle could be started from any position, we assume that $x_i^{2k+i} \neq x_i^{2k+i-1}$ for all k and both i. Moreover, since the roles of both players are symmetric, we often consider player 1 as the representative player. We denote $K = \{0, \dots, m-1\}$, $\Xi_i = \{x_N^{2k+i}\}_{k\in K}$ for each $i \in N$, and $\Xi = \Xi_1 \cup \Xi_2$.

Step 3.1. *If $i \in N$, $k, h \in K$, and $x_i^{2k+i} = x_i^{2h+i}$, then $k = h$.*

Proof. Suppose the contrary: there are $k > h$ such that $x_1^{2k+1} = x_1^{2k+2} = x_1^{2h+1} = x_1^{2h+2}$. If $u_2(x_N^{2k+2}) \geq u_2(x_N^{2h+2})\,[> u_2(x_N^{2h+1})]$, then $x_N^0, \dots, x_N^{2h+1}, x_N^{2k+2}, \dots, x_N^{2m} = x_N^0$ is a shorter improvement cycle in $G(\upsilon_N)$. If $u_2(x_N^{2h+2}) \geq u_2(x_N^{2k+2})\,[> u_2(x_N^{2k+1})]$, then $x_N^{2k}, x_N^{2k+1}, x_N^{2h+2}, \dots, x_N^{2k}$ is again a shorter improvement cycle in $G(\upsilon_N)$. $\square$

We denote $B = g(\Xi)$, $B_0 = g(\Xi_1) \cap g(\Xi_2)$, and, for each $i \in N$, $B_i = g(\Xi_i) \setminus B_0$ and $Y^i = g^{-1}(B_i) \cap \Xi$. By definition, $B_1 \cap B_2 = \emptyset$. We define υ_i^*: $\upsilon_i^*(a) = \max_{b \in B} \upsilon_i(b)$ for $a \in B_i$; $\upsilon_i^*(a) = \min_{b \in B} \upsilon_i(b)$ for $a \in B_{3-i}$; $\upsilon_i^*(a) = \upsilon_i(a)$ otherwise. It is easy to see that

$$\operatorname*{Argmax}_{b \in B} \upsilon_i^*(b) = B_i = \operatorname*{Argmin}_{b \in B} \upsilon_{3-i}^*(b) \tag{1}$$

and that $x_N^0, x_N^1, \ldots, x_N^{2m} = x_N^0$ is an improvement cycle in $G(\upsilon_N^*)$ as well.

Step 3.2. *For each $i \in N$, the set Y^i is a singleton (hence B_i is a singleton as well).*

Proof. First, we note that $B_i \neq \emptyset$ for each $i \in N$ by (1). Let $g(x_N^{2k+1}) \in B_1 \ni g(x_N^{2h+1})$ and $k > h$; note that $x_2^{2k+1} = x_2^{2k}$ and $x_2^{2h+1} = x_2^{2h}$. Applying Lemma 2.1 to the fragment $\{x_1^{2h+1}, x_1^{2k+1}\} \times \{x_2^{2h}, x_2^{2k}\}$, we obtain that either $g(x_1^{2k+1}, x_2^{2h}) \in B_1$ or $g(x_1^{2h+1}, x_2^{2k}) \in B_1$.

In the first case, $x_N^0, x_N^1, \ldots, x_N^{2h}, (x_1^{2k+1}, x_2^{2h}), x_N^{2k+2}, \ldots, x_N^{2m} = x_N^0$ is an improvement cycle in $G(\upsilon_N^*)$: $u_1^*(x_1^{2k+1}, x_2^{2h}) > u_1^*(x_N^{2h})$ because $g(x_N^{2h}) \notin B_1 \ni g(x_1^{2k+1}, x_2^{2h})$; $u_2^*(x_N^{2k+2}) > u_2^*(x_1^{2k+1}, x_2^{2h})$ because $g(x_N^{2k+2}) \notin B_1$. In the second case, $x_N^{2h+2}, \ldots, x_N^{2k}, (x_1^{2h+1}, x_2^{2k}), x_N^{2h+2}$ is an improvement cycle in $G(\upsilon_N^*)$ for similar reasons. In either case, we obtain a contradiction with the assumption that a shorter improvement cycle is impossible. □

Since $m > 2$, Step 3.2 immediately implies $B_0 \neq \emptyset$. We also see that each υ_i^* actually coincides with υ_i. Henceforth, we use the notation $Y^i = \{y_N^i\}$.

Step 3.3. *There is $i \in N$ such that $y_i^1 = y_i^2$.*

Proof. Since we can start the cycle anyplace, we assume that $y_N^2 = x_N^0$. Suppose to the contrary that $y_N^1 = x_N^{2k+1}$ with $0 < k < m-1$. Applying Lemma 2.1 to the fragment $\{x_1^0, x_1^{2k+1}\} \times \{x_2^0, x_2^{2k}\}$, we obtain that one of the following four alternatives must hold.

If $g(x_1^0, x_2^{2k}) = g(y_N^1)$, then $x_N^0, x_N^1, \ldots, x_N^{2k}, (x_1^0, x_2^{2k}), x_N^0$ is an improvement cycle in $G(\upsilon_N)$. If $g(x_1^0, x_2^{2k}) = g(y_N^2)$, then $x_N^{2k+1}, \ldots, x_N^{2m-1}, (x_1^0, x_2^{2k}), x_N^{2k+1}$ is an improvement cycle. If $g(x_1^{2k+1}, x_2^0) = g(y_N^1)$, then $x_N^0, (x_1^{2k+1}, x_2^0), x_1^{2k+2}, \ldots, x_N^{2m} = x_N^0$ is an improvement cycle. If $g(x_1^{2k+1}, x_2^0) = g(y_N^2)$, then $x_N^1, x_N^2, \ldots, x_N^{2k+1}, (x_1^{2k+1}, x_2^0), x_N^1$ is an improvement cycle.

As in the proof of Step 3.2, we have a contradiction with the assumption that a shorter improvement cycle is impossible. □

We are approaching a final contradiction. Supposing, without restricting generality, that $y_2^1 = y_2^2$, we pick $a \in \operatorname{Argmax}_{b \in B_0} \upsilon_1(b)$; by definition, there is $k \in K$ such that $g(x_N^{2k}) = a$. Since $u_1(x_N^{2k+1}) > u_1(x_N^{2k})$, we must have $x_N^{2k+1} = y_N^1$; but then $x_N^{2k} = y_N^2$ by Steps 3.3 and 3.1, hence $g(x_N^{2k}) \notin B_0$.

4 Extensions

Corollary 1. *Let G be a finite two person game form. If no antagonistic derivative game $G(\upsilon, -\upsilon)$ admits an improvement cycle, then G is acyclic.*

Proof. Since the necessity in Lemma 2.1 was proven with a reference to Moulin (1976), where antagonistic utilities were considered, we obtain that every 2×2 fragment of G is Nash consistent. Now our Theorem applies. □

Corollary 2. *Let G be a finite two person game form. If no derivative game $G(\upsilon_N)$ admits a best response improvement cycle, then G is acyclic.*

Proof. If G is not acyclic, it contains a fragment $\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ with $\{a, d\} \cap \{b, c\} = \emptyset$. We define $\upsilon_1(a) = \upsilon_1(d) = 1$ and $\upsilon_1(x) = 0$ for $x \notin \{a, d\}$; $\upsilon_2(b) = \upsilon_2(c) = 1$ and $\upsilon_2(x) = 0$ for $x \notin \{b, c\}$. Clearly, the fragment becomes a best response improvement cycle in $G(\upsilon_N)$. □

When there are more than two players, the straightforward analogue of our Theorem does not hold.

Example. Let us consider a three person $2 \times 2 \times 2$ game form with four outcomes, where player 1 chooses rows, player 2 columns, and player 3 matrices:

$$\begin{bmatrix} a & c \\ a & b \end{bmatrix} \quad \begin{bmatrix} a & d \\ b & b \end{bmatrix}.$$

Applying Lemma 2.1, we immediately see that every 2×2 fragment is Nash consistent. On the other hand, let us consider the following utilities: $\upsilon_1(c) = \upsilon_1(d) = 2$, $\upsilon_1(b) = 1$, $\upsilon_1(a) = 0$; $\upsilon_2(b) = \upsilon_2(c) = 2$, $\upsilon_2(a) = 1$, $\upsilon_2(d) = 0$; $\upsilon_3(a) = \upsilon_3(d) = 1$, $\upsilon_3(b) = \upsilon_3(c) = 0$. It is easily checked that, at each strategy profile, there is a single player capable of improvement:

$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix},$$

hence there is no Nash equilibrium. Therefore, the game form is not even Nash consistent.

Hypothesis. *A finite game form G is acyclic if and only if every fragment of G is Nash consistent.*

References

Gurvich, V.A., 1988. Equilibrium in pure strategies. Doklady Akademii Nauk SSSR 303, 789–793 (in Russian; English translation in Soviet Mathematics. Doklady 38, 597–602).

Gurvich, V.A., and L.O. Libkin, 1990. Absolutely determined matrices. Mathematical Social Sciences 20, 1–18.

Kukushkin, N.S., 2002. Perfect information and potential games. Games and Economic Behavior 38, 306–317.

Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48, 94–110.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games and Economic Behavior 13, 111–124.

Monderer, D., and L.S. Shapley, 1996. Potential games. Games and Economic Behavior 14, 124–143.

Moulin, H., 1976. Prolongements des jeux à deux joueurs de somme nulle. Bulletin de la Société Mathématique de France, Supplémentaire Mémoire No. 45.

Rosenthal, R.W., 1981. Games of perfect information, predatory pricing and the chain-store paradox. Journal of Economic Theory 25, 92–100.

Shapley, L.C., 1964. Some topics in two-person games. In: Advances in Game Theory (Annals of Mathematical Studies 52) M. Dresher, L.S. Shapley, and A.W. Tucker, eds. Princeton University Press, Princeton, 1–28.

Takahashi, S., and T. Yamamori, 2002. The pure Nash equilibrium property and the quasi-acyclic condition. Economics Bulletin 3, No. 22, 1–6. Available from http://economicsbulletin.vanderbilt.edu/2002/volume3/EB-02C70011A.pdf