Uniqueness of Optimal Cube Wrapping

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Abstract

Consider wrapping the unit cube with a square without stretching or cutting. Beebee demonstrated such a wrapping with a square of side length $2\sqrt{2}$, and proved that no smaller square can fulfill the task [1]. We show that Beebee's construction is the unique optimal wrapping, up to the symmetries of the unit cube and the placement of the junk material.

1 Introduction

Optimal cube wrapping with a square is one of the few solved instances regarding optimal wrappings of 3D objects. In the problem, the square must cover the entire surface of the unit cube, without being stretched or cut. Furthermore, the square is not allowed to go inside the unit cube, that is, the unit cube is a solid. Under those conditions, Beebee showed that the smallest possible square that can wrap the unit cube has side length $2\sqrt{2}$ [1].

In proving that $2\sqrt{2}$ is the minimum, Beebee observed that the surface distance between any point on the unit cube and its antipodal point is at least 2. With this observation, Beebee considered the point P' on the unit cube covered by the center of the square, and the point Q on the square covering the antipodal point of P'. It then follows, due to the fact that the square isn't stretched, that Q is at least 2 away from the center of the square. Because the farthest points on the square from the center are the four corners, the square must have side length at least $2\sqrt{2}$.

Beebee demonstrated that a square of side length $2\sqrt{2}$ suffices with a wrapping that uses the crease pattern shown in Figure 1 (left). The square region of side length 1 in the middle is used to cover the bottom facet of the unit cube. The four surrounding square regions are used to cover the four side facets (let's say that the unit square to the lower left of the central square covers the front facet, that to the lower right covers the right facet, etc). Finally, the four triangular regions at the corners are combined to cover the top facet. The placement of the rest of the square, the *junk material*, doesn't matter. We call this optimal construction the *standard wrapping*.

We show that the standard wrapping represents the

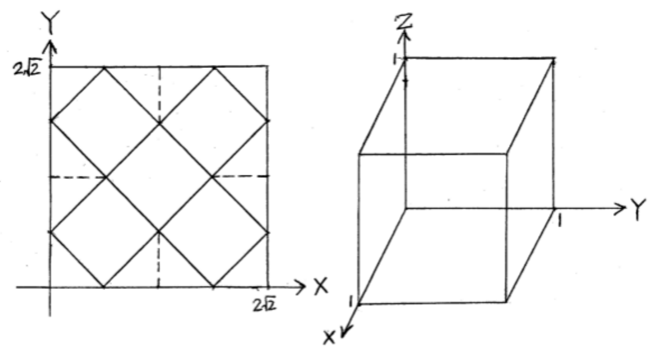

Figure 1: Square of side length $2\sqrt{2}$ with the standard wrapping crease pattern (left), and the unit cube (right)

unique wrapping of the unit cube with a square of side length $2\sqrt{2}$, up to the symmetries of the unit cube and the placement of the junk material. Given any optimal wrapping, we match it gradually with the standard wrapping. That is, we first show that the center of the square must coincide with the center of a facet of the unit cube, then show that the four corners of the square must coincide with the center of the opposite facet, etc.

Among the properties of a wrapping, our proof relies only on that paths and regions on the square can't be stretched. This implies, for example, that a region R on the square can cover a surface area no bigger than the area of R. Our proof also relies on a lemma similar to Beebee's observation on surface distance (Lemma 2): Given any point P' on the unit cube that is not the center of any facet, there is a point Q' on the unit cube such that the surface distance between P' and Q' is strictly greater than 2. This lemma is applied when we show that the center of the square must coincide with the center of a facet. To prove the lemma, we describe a general scheme to calculate the surface distance between an arbitrary pair of points.

It seems that Beebee's observation that the surface distance between any pair of antipodal points is at least 2 is only stated without proof in [1]. Our Lemma 2 includes a formal proof of this observation. Furthermore, it seems that Beebee's proof implicitly relies on the extra condition that the entire square is restricted to the surface of the unit cube. For example, he assumed that the center of the square coincides with some point on the surface. Our result does not require this condition.

2 Proof of Uniqueness

In this section, we prove that the standard wrapping represents the unique optimal cube wrapping by a square, up to the symmetries of the unit cube and the placement of the junk material, under the conditions that 1) the square is not stretched or cut, and 2) the square does not intersect the interior of the unit cube.

Let S be a square of side length $2\sqrt{2}$, and U the unit cube. We identify S and U with the regions $[0, 2\sqrt{2}]^2$ in $\mathbb{R}^2$ and $[0, 1]^3$ in $\mathbb{R}^3$, respectively, as in Figure 1. Let ∂U be the surface of the unit cube, and $V = \mathbb{R}^3 - (U - \partial U)$ the complement of the interior of the unit cube.

A given optimal wrapping can be identified with a continuous map $f : S \to V$ satisfying

$$\partial U \subset f(S). \tag{1}$$

For P,Q in S or V, let $d(P, Q)$ be the length of the segment connecting P and Q. For P',$Q' \in V$, let $d_V(P', Q')$ be the length of the shortest path through V between P' and Q'. For a 2-dimensional region R in S or V, let $a(R)$ be the area of R. That the square is not stretched or cut implies that

$$d(P, Q) \geq d_V(f(P), f(Q)) \tag{2}$$

for any pair of $P, Q \in S$, and that

$$a(R) \geq a(f(R)) \tag{3}$$

for any $R \subset S$.

For a path p in S or V, let $l(p)$ be its length. If p is a path in S, then $f(p)$ is a path in V. We have

$$l(p) \geq l(f(p)). \tag{4}$$

Notice that (4) can be deduced from an infinitesimal version of (2). In fact, (2) and (4) imply each other.

Our goal is to show that any continuous $f : S \to V$ satisfying (1), (2), (3) and (4) is essentially equivalent to the standard wrapping.

Although the range of f is V, our analysis ends up mostly focusing on the surface of the unit cube $\partial U \subset V$. Intuitively, mapping some part of the square outside of the unit cube is only going to make covering the surface more difficult, due to waste of material. Our next lemma provides some insights by showing that, for any P',$Q' \in \partial U$, the minimal path length from P' to Q' through V is achieved by, and only by, some path entirely in ∂U.

Lemma 1 *Given P',$Q' \in \partial U$ and any path $q \subset V$ between them that is not entirely in ∂U, there exists a path $p \subset \partial U$ between them such that $l(p) < l(q)$.*

Proof. Let π_z^0 be the operator that acts as the identity for points on or above the $z = 0$ plane, and acts as the perpendicular projection onto the $z = 0$ plane for points below the $z = 0$ plane. Let π_z^1 be the operator that acts as the identity for points on or below the $z = 1$ plane, and acts as the perpendicular projection onto the $z = 1$ plane for points above the $z = 1$ plane. Define π_x^0, π_x^1, π_y^0, and π_y^1 similarly. Lastly, let

$$\pi = \pi_x^0 \circ \pi_x^1 \circ \pi_y^0 \circ \pi_y^1 \circ \pi_z^0 \circ \pi_z^1.$$

Obviously, $\pi(P') = P'$, $\pi(Q') = Q'$, and π sends points of V to ∂U. Therefore, $p = \pi(q)$ is a path in ∂U between P' and Q'. Clearly, none of the six "projection" operators composing π can increase the path length. To see that $l(p) < l(q)$, consider point $P_1' \in q$ that is not in ∂U. Say P_1' is below the $z = 0$ plane. Because Q' is on or above the $z = 0$ plane, there must exist $P_2' \in q$ such that 1) P_2' is on the $z = 0$ plane, and 2) the section of q from P_1' to P_2' lies on or below the $z = 0$ plane. Because π eliminates the change in z-coordinate for this section of q, its length is strictly decreased. As a result, we have $l(p) < l(q)$. □

Now we describe a general scheme to calculate $d_V(P', Q')$ for P',$Q' \in \partial U$. This scheme is applied several times throughout the paper, including in the proof of Lemma 2.

By Lemma 1, we only need to consider paths entirely in ∂U. If P' and Q' are on the same facet, then $d_V(P', Q')$ is obviously just $d(P', Q')$. Otherwise, we evaluate cases based on the different possible sequences of facets visited on a path from P' to Q'. For convenience, let the back, front, left, right, bottom, and top facets be the facets of U that lie on, respectively, the $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 1$ planes. For example, if a path starts at $(\frac{1}{2}, \frac{1}{2}, 0)$, heads straight to $(1, \frac{1}{2}, 0)$, then $(1, 0, \frac{1}{2})$, and finally arrives at $(\frac{1}{2}, 0, \frac{1}{2})$, we say that the path visits facets in the sequence of bottom, front, and left (see Figure 2).

Clearly, the shortest path from P' to Q' wouldn't visit the same facet more than once. If the path exits, say, the top facet at point P_1', and later reenters the top facet at point P_2', then it could achieve a smaller length by heading straight from P_1' to P_2' instead.

Consequently, there are finitely many different possible sequences of visited facets. For each sequence of facets F_1, F_2, …, F_k, we evaluate the shortest path from P' to Q' that visits the facets according to this sequence by reducing it to a 2-dimensional problem, as follows. First place F_1 in a plane. Suppose that facets up till F_i have been placed in the plane, we place F_{i+1} in the plane so that 1) F_i and F_{i+1} share the same edge as the one they share in $\mathbb{R}^3$, and 2) F_i and F_{i+1} don't completely overlap. Notice that there is a unique way of placing F_{i+1}. Let Γ be the 2-dimensional region formed

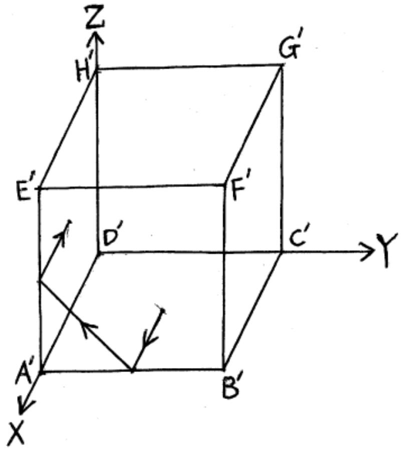

Figure 2: Example of a path that visits the facets in the sequence of bottom, front, and left

by those k facets. Notice that $P' \in F_1$ and $Q' \in F_k$ each corresponds to a point in Γ. Then, the minimal path length visiting this sequence of facets equals the length of the shortest path in Γ from P' to Q', which can be calculated easily.

After we calculate the shortest path length for each possible sequence of visited facets, we can calculate $d_V(P', Q')$ as being the smallest of those values.

There are dozens of cases. Fortunately, we don't always need exact answers for our arguments. Moreover, symmetries and additional observations also greatly reduce the number of cases.

The next lemma, generalizing Beebee's observation discussed in the introduction, is later used to show that the center of the square must coincide with the center of some facet.

Lemma 2 *For any $P' \in \partial U$, there exists $Q' \in \partial U$ such that $d_V(P', Q') \geq 2$. If P' is not the center of any of the facets, then Q' can be chosen so that the inequality is strict.*

Proof. Let A', B', C', D', E', F', G', and H' be the vertices of U as shown in Figure 2. Without loss of generality, suppose $P' = (x, y, 0)$ is on the bottom facet. Let $Q' = (1-x, 1-y, 1)$ be the antipodal point of P'. We apply our scheme described above. Any surface path from P' to Q' visits the bottom facet at the beginning, the top facet at the end, and at least one side facets in between.

Suppose that the total number of facets visited is 3. Without loss of generality, suppose that the facet visited between the bottom and the top facets is the front facet. Figure 3 shows the 2-dimensional problem this case reduces to. The minimal path length in Γ between P' and Q' is $\sqrt{2^2 + (1-2y)^2} \geq 2$. The inequality is strict unless $y = \frac{1}{2}$.

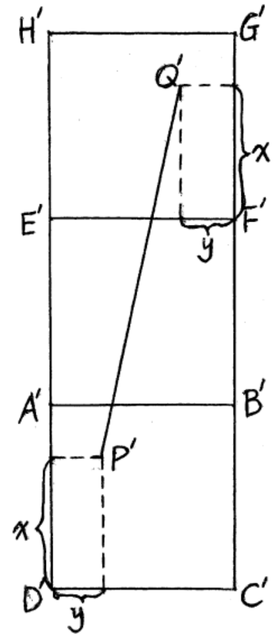

Figure 3: The equivalent 2-dimensional problem for finding the shortest surface path from $P' = (x, y, 0)$ to $Q' = (1-x, 1-y, 1)$ through the bottom, front, and top facets

Suppose that the total number of facets visited is 4. Without loss of generality, suppose that the facets visited between the bottom and the top facets are the front and the left facets. Figure 4 shows the 2-dimensional problem this case reduces to. The minimal path length in Γ between P' and Q' is at least $\sqrt{(x+y)^2 + (3-x-y)^2} \geq \frac{3}{\sqrt{2}} > 2$.

Finally, Suppose that the total number of facets visited is at least 5. Without loss of generality, suppose that the three facets visited right after the bottom facet are the front, the left, and the back facets. Let P_1' be the point through which the path exits the front facet and enters the left facet, and P_2' be the point through which the path exits the left facet and enters the back facet (see Figure 5). By looking at the x- and y- coordinates of the points, we see that $d(P', P_1') \geq y$, $d(P_2', Q') \geq 1-y$, and $d(P_1', P_2') \geq 1$. Moreover, the three equalities can't hold simultaneously, because at least one of the three pairs of points have distinct z-coordinates, yielding the strict inequality. So in this case, the minimal path length between P' and Q' is greater than 2.

Combining all the cases, we see that, by taking Q' to be the antipodal point of P', we have $d_V(P', Q') \geq 2$. The inequality is strict unless P' is on at least one of the $x = \frac{1}{2}$, $y = \frac{1}{2}$, and $z = \frac{1}{2}$ planes (see the first case).

To finish the proof, we need to pick Q' a little differently when P' is on exactly one of the $x = \frac{1}{2}$, $y = \frac{1}{2}$, and $z = \frac{1}{2}$ planes (if P' is on two of those three planes, then

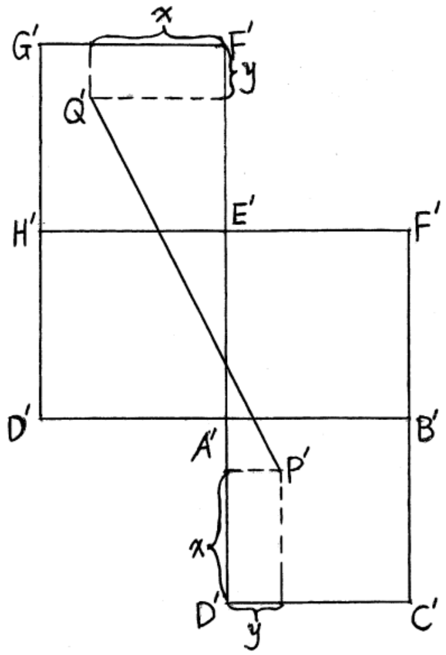

Figure 4: The equivalent 2-dimensional problem for finding the shortest surface path from $P' = (x, y, 0)$ to $Q' = (1-x, 1-y, 1)$ through the bottom, front, left, and top facets

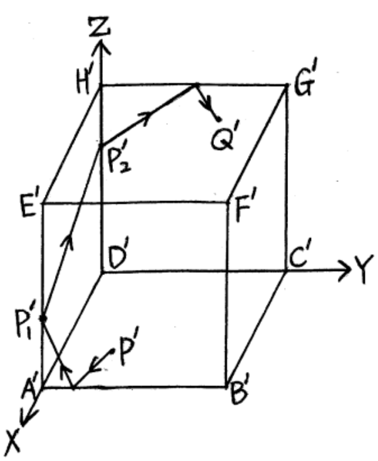

Figure 5: A surface path from $P' = (x, y, 0)$ to $Q' = (1-x, 1-y, 1)$ through the bottom, front, left, back, and top facets. P'_1 is the point where the path exits the front facet and enters the left facet, and P'_2 is the point where the path exits the left facet and enters the back facet.

P' is the center of some facet). Without loss of generality, suppose that $P' = (x, \frac{1}{2}, 0)$, where $x \neq \frac{1}{2}$. We pick $Q' = (1-x, \frac{1}{2}+\epsilon, 1)$, $\epsilon > 0$, to be a slight perturbation of the antipodal point. We proceed to cases as before.

Suppose that the total number of facets visited is 3. If the facet visited between the bottom and the top facets is the front or the back facet, the minimal path length is $\sqrt{2^2+\epsilon^2} > 2$. If it is the left or the right facet, the minimal path length is $\sqrt{(2\pm\epsilon)^2+(1-2x)^2}$, which is greater than 2 for small enough ϵ.

Finally, suppose that the total number of facets visited is at least 4. Previous analysis shows that, when Q' is exactly the antipodal point of P', the minimal path length is greater than 2. Thus, as long as ϵ is small enough, the minimal path length is still greater than 2. Combining all the cases, we see that $d_V(P', Q') > 2$ for $Q' = (1-x, \frac{1}{2}+\epsilon, 1)$, where $\epsilon > 0$ is small enough. $\square$

The next lemma shows that Lemma 2 is true even if the condition $P' \in \partial U$ is replaced by $P' \in V$.

Lemma 3 *For any $P' \in V$, there exists $Q' \in \partial U$ such that $d_V(P', Q') \geq 2$. If P' is not the center of any of the facets, then Q' can be chosen so that the inequality is strict.*

Proof. The lemma is true if $P' \in \partial U$ by Lemma 2. Now suppose $P' \notin \partial U$. Let $P'_1 = \pi(P') \in \partial U$, where π is as defined in the proof of Lemma 1. Let Q' be the antipodal point of P'_1. We claim that $d_V(P', Q') > 2$.

Let p be any path in V between P' and Q'. Then $\pi(p)$ is a path in ∂U between P'_1 and Q'. The proof of Lemma 2 implies that $l(\pi(p)) \geq 2$. Because p is not entirely in ∂U, the same argument as in the proof of Lemma 1 shows that $l(p) > l(\pi(p)) \geq 2$. Since p is arbitrary, we have $d_V(P', Q') > 2$. $\square$

We now proceed to prove our main result.

Theorem 4 *Any wrapping of the unit cube by a square of side length $2\sqrt{2}$ is equivalent to the standard wrapping, up to the symmetries of the unit cube and the placement of the junk material.*

Proof. Let $f : S \to V$ represent the given wrapping, then f is continuous and satisfies (1), (2), (3) and (4). The proof is divided into six steps, as we gradually match f with the standard wrapping.

Step 1: *Let $O = (\sqrt{2}, \sqrt{2})$ be the center of S, then $f(O)$ is the center of some facet of U.*

To establish **Step 1**, suppose for the sake of contradiction that $f(O)$ is not the center of any of the facets. By Lemma 3, there exists $Q' \in \partial U$ such that $d_V(f(O), Q') > 2$. By (1), there exists $Q \in S$ such that $f(Q) = Q'$. By (2), we have

$$d(O, Q) \geq d_V(f(O), Q') > 2,$$

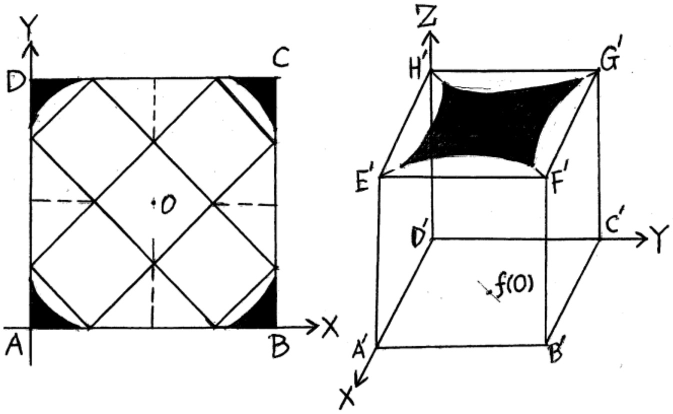

Figure 6: $\Phi_S(O,t)$ (left) and $\Phi_V(f(O),t)$ (right), for some $\frac{\sqrt{10}}{2} \le t \le 2$

which is impossible.

Without loss of generality, we assume that $f(O) = (\frac{1}{2}, \frac{1}{2}, 0)$, the center of the bottom facet. Let $A = (0,0)$, $B = (2\sqrt{2}, 0)$, $C = (2\sqrt{2}, 2\sqrt{2})$, and $D = (0, 2\sqrt{2})$ be the four corners of S, and $O' = (\frac{1}{2}, \frac{1}{2}, 1)$ the center of the top facet of U.

Step 2: *f(A) = f(B) = f(C) = f(D) = O'.*

To establish **Step 2**, we first define

$$\begin{aligned} \Phi_S(P,t) &= \{Q \in S \mid d(P,Q) \ge t\} \\ \Phi_V(P',t) &= \{Q' \in \partial U \mid d_V(P',Q') \ge t\} \end{aligned} \tag{5}$$

for any $P \in S$, $P' \in V$, and $t \ge 0$.

The key observation is that for

$$\sqrt{(\frac{3}{2})^2 + (\frac{1}{2})^2} = \frac{\sqrt{10}}{2} \le t \le 2,$$

we have

$$a(\Phi_S(O,t)) = a(\Phi_V(f(O),t)). \tag{6}$$

The two regions are shaded in Figure 6. To see (6), notice that for a point P' in the triangular region $O'E'F'$ in the top facet, the shortest surface path from $f(O) = (\frac{1}{2}, \frac{1}{2}, 0)$ to P' visits the facets in the sequence of bottom, front, and top. This can be established through the general scheme described prior to Lemma 2. We omit the details. Thus, if we form the 2-dimensional region Γ as in the scheme by concatenating the bottom, front, and top facets in a plane, we see that the part of triangle $O'E'F'$ in ∂U that belongs to $\Phi_V(f(O),t)$ corresponds exactly to the part of $O'E'F'$ in Γ that lies outside of the circle of center $f(O)$ and radius t (see Figure 7). The situations for the other three triangles $O'F'G'$, $O'G'H'$, and $O'H'E'$ are similar.

On the other hand, the part of S that belongs to $\Phi_S(O,t)$ is exactly the region that lies outside of the circle of center O and radius t. For t in the range specified above, that region consists of four disjoint parts that lie inside, respectively, the four triangles at the corners of S that are combined to cover the top facet of U in the standard wrapping. It is easy to see that each of the four disjoint parts that belongs to $\Phi_S(O,t)$ is congruent to the part of each of $O'E'F'$, $O'F'G'$, $O'G'H'$, and $O'H'E'$ that belongs to $\Phi_V(f(O),t)$. Furthermore, the non-top facets of U don't intersect $\Phi_V(f(O),t)$ at all (for example, any point on the front facet can be connected to $f(O)$ via a surface path of length at most $\sqrt{(\frac{3}{2})^2 + (\frac{1}{2})^2} = \frac{\sqrt{10}}{2}$ that visits only the front and the bottom facets). Thus, (6) is established.

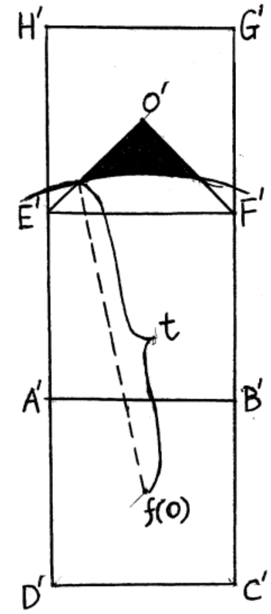

Figure 7: The part of triangle $O'E'F'$ that belongs to $\Phi_V(f(O),t)$

From (2) and (5), we see that $f(P) \in \Phi_V(f(O),t)$ implies that $P \in \Phi_S(O,t)$. Combining this with (1), (3) and (6), we see that the converse is also true for $\frac{\sqrt{10}}{2} \le t \le 2$. In other words, $\Phi_V(f(O),t)$ can only be covered by material from $\Phi_S(O,t)$, and $\Phi_S(O,t)$ has just enough material to do the job. So no waste can occur.

As t approaches 2, $\Phi_S(O,t)$ shrinks to the four corners of S, and $\Phi_V(f(O),t)$ shrinks to the point $(\frac{1}{2}, \frac{1}{2}, 1) \in \partial U$. So **Step 2** is established.

We let p_1, p_2, p_3, and p_4 be the paths on ∂U covered by OA, OB, OC, and OD in the standard wrapping.

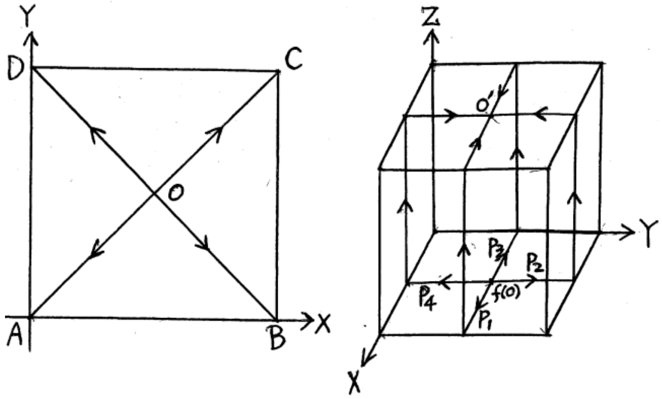

Figure 8: Segments OA, OB, OC, and OD (left), and paths p_1, p_2, p_3, and p_4 (right)

In other words,

$$p_1 : (\frac{1}{2}, \frac{1}{2}, 0) \to (1, \frac{1}{2}, 0) \to (1, \frac{1}{2}, 1) \to (\frac{1}{2}, \frac{1}{2}, 1)$$
$$p_2 : (\frac{1}{2}, \frac{1}{2}, 0) \to (\frac{1}{2}, 1, 0) \to (\frac{1}{2}, 1, 1) \to (\frac{1}{2}, \frac{1}{2}, 1)$$
$$p_3 : (\frac{1}{2}, \frac{1}{2}, 0) \to (0, \frac{1}{2}, 0) \to (0, \frac{1}{2}, 1) \to (\frac{1}{2}, \frac{1}{2}, 1)$$
$$p_4 : (\frac{1}{2}, \frac{1}{2}, 0) \to (\frac{1}{2}, 0, 0) \to (\frac{1}{2}, 0, 1) \to (\frac{1}{2}, \frac{1}{2}, 1),$$

where arrows mean heading straight towards. We include a time parameter s, $0 \le s \le 1$, and imagine walking on each of the four paths in constant speed from the start to the end as s goes from 0 to 1. Let $p_i(s)$ be our position at time s when we are tracing p_i. For example, we have $p_1(\frac{1}{4}) = (1, \frac{1}{2}, 0)$.

Step 3: *Up to some symmetries of the unit cube, f maps the segments OA, OB, OC, and OD to p_1, p_2, p_3, and p_4, respectively (see Figure 8). Moreover, the mappings are uniform in that if P on OA is such that $d(O,P) = s \cdot d(O,A)$, then $f(P) = p_1(s)$. Similar results hold for points on OB, OC, and OD.*

To establish **Step 3**, we first claim that p_1, p_2, p_3, and p_4 are the only four paths of length at most 2 through V connecting the antipodal pair $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 1)$ (their lengths are exactly 2, the smallest possible by Lemma 2). To see that, notice that Lemma 1 suggests that we only need to consider paths in ∂U. Given such a path of length 2, consider the sequence of visited facets as in the general scheme. The proof of Lemma 2 shows that the sequence contains no more than three facets (otherwise length is greater than 2). Suppose the sequence is bottom, front, and top, then in the resulting 2-dimensional region Γ, the points corresponding to $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 1)$ are exactly Euclidean distance 2 apart (see Figure 3 with $x = y = \frac{1}{2}$). Therefore, only the straight segment between them has length 2 in Γ, which corresponds to the path p_1 on ∂U.

Notice that $f(OA)$, $f(OB)$, $f(OC)$, and $f(OD)$ are paths in V from $(\frac{1}{2}, \frac{1}{2}, 0)$ to $(\frac{1}{2}, \frac{1}{2}, 1)$. By (4), their lengths are at most 2, because OA, OB, OC, and OD have length 2. Thus, each of $f(OA)$, $f(OB)$, $f(OC)$, and $f(OD)$ must coincide with some p_i.

Suppose OA is mapped to some p_i. For any $0 \le s \le 1$, let $P \in OA$ be such that $d(O,P) = s \cdot d(O,A)$. Let $0 \le s' \le 1$ be the unique value such that $f(P) = p_i(s')$. If $s < s'$, then OP is shorter than the part of p_i from $f(O) = (\frac{1}{2}, \frac{1}{2}, 0)$ to $f(P)$, contradicting (4). If $s > s'$, then PA is shorter than the part of p_i from $f(P)$ to $f(A) = (\frac{1}{2}, \frac{1}{2}, 1)$, again contradicting (4). As a result, $s = s'$. Since s is arbitrary, the mapping from OA to p_i is uniform, as required. Similar analysis holds for OB, OC, and OD.

No two of OA, OB, OC, and OD can be mapped to the same p_i. Otherwise, for any $\frac{\sqrt{10}}{2} \le t < 2$, a positive measure amount of material in $\Phi_S(O,t)$ is wasted due to overlapping, so $\Phi_S(O,t)$ can't cover $\Phi_V(f(O),t)$ completely, contradicting the analysis in **Step 2**.

Without loss of generality, suppose that OA is mapped to p_1. If OB is mapped to p_3, then consider $P \in OA$ and $Q \in OB$ that are both ϵ away from O, for some small ϵ. By (2), we have

$$\sqrt{2}\epsilon = d(P,Q) \ge d_V(P',Q') = 2\epsilon,$$

a contradiction. Therefore, OB is mapped to p_2 or p_4. Without loss of generality, suppose OB is mapped to p_2. Similar analysis then shows that OC can't be mapped to p_4, so it must be mapped to p_3, and so OD must be mapped to p_4.

Step 4: *The action of f on $\Phi_S(O, \frac{\sqrt{10}}{2})$ matches the standard consturction.*

To establish **Step 4**, we first define

$$\begin{aligned} \phi_S(P,t) &= \{Q \in S \mid d(P,Q) = t\} \\ \phi_V(P',t) &= \{Q' \in \partial U \mid d_V(P',Q') = t\} \end{aligned} \tag{7}$$

for any $P \in S$, $P' \in V$, and $t \ge 0$. Intuitively, $\phi_S(P,t)$ and $\phi_V(P',t)$ can be thought of as the derivatives of, respectively, $\Phi_S(P,t)$ and $\Phi_V(P',t)$ with respect to t.

For any $\frac{\sqrt{10}}{2} \le t < 2$, $\phi_S(O,t)$ is the portion of the boundary of the circle of center O and radius t that lies in S. It consists of four congruent and disjoint arcs (see Figure 9). Denote the arc that is closest to A, B, C, and D by $\gamma_A(t)$, $\gamma_B(t)$, $\gamma_C(t)$, and $\gamma_D(t)$, respectively. On the other hand, $\phi_V(f(O),t)$ is the boundary of $\Phi_V(f(O),t)$ on the top facet of U; it is composed of four arcs of the same shape, but connected (see Figure 9). Denote the arc that lies in triangles $O'E'F'$, $O'F'G'$, $O'G'H'$, and $O'H'E'$ by $\gamma'_A(t)$, $\gamma'_B(t)$, $\gamma'_C(t)$, and $\gamma'_D(t)$, respectively. All eight arcs just defined are congruent. So $\phi_S(O,t)$ and $\phi_V(f(O),t)$ have the same total length.

By the discussion in **Step 2**, for any $\frac{\sqrt{10}}{2} \le t < 2$, $P \in \Phi_S(O,t)$ if and only if $f(P) \in \Phi_V(f(O),t)$. By

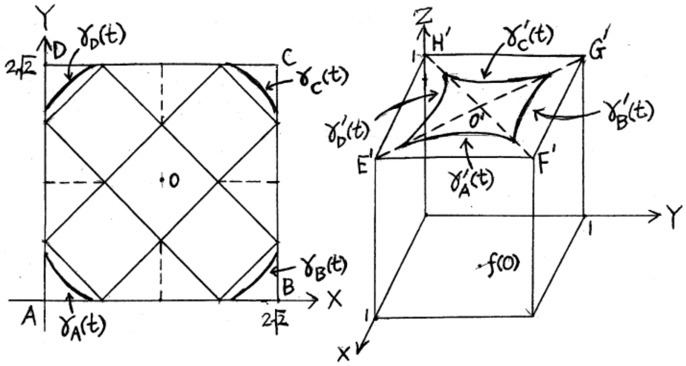

Figure 9: Arcs $\gamma_A(t)$, $\gamma_B(t)$, $\gamma_C(t)$, and $\gamma_D(t)$ composing $\phi_S(O,t)$ (left), and arcs $\gamma'_A(t)$, $\gamma'_B(t)$, $\gamma'_C(t)$, and $\gamma'_D(t)$ composing $\phi_V(f(O),t)$ (right), for some $\frac{\sqrt{10}}{2} \leq t < 2$

varying t, we see that, for any $\frac{\sqrt{10}}{2} \leq t < 2$, $P \in \phi_S(O,t)$ if and only if $f(P) \in \phi_V(f(O),t)$.

For a fixed $\frac{\sqrt{10}}{2} \leq t < 2$, let $P_A = \phi_S(O,t) \cap OA$, and $P'_A = \phi_V(f(O),t) \cap p_1$ (p_1 is as in **Step 3**). By **Step 3**, we have $f(P_A) = P'_A$. Now P_A is the midpoint of $\gamma_A(t)$, and P'_A is the midpoint of $\gamma'_A(t)$, where $\gamma_A(t)$ and $\gamma'_A(t)$ are congruent. By (4) and the fact that $f(\phi_S(O,t)) \subset \phi_V(f(O),t)$, we deduce that $f(\gamma_A(t)) \subset \gamma'_A(t)$, because $\gamma_A(t)$ doesn't have enough length to "escape" outside of $\gamma'_A(t)$ while staying within $\phi_V(f(O),t)$.

Similarly, it must be that $f(\gamma_B(t)) \subset \gamma'_B(t)$, $f(\gamma_C(t)) \subset \gamma'_C(t)$, and $f(\gamma_D(t)) \subset \gamma'_D(t)$. However, we noted before that $\phi_V(f(O),t)$ must be completely covered by $\phi_S(O,t)$, that is, $\phi_V(f(O),t) \subset f(\phi_S(O,t))$, or, equivalently,

$$\gamma'_A(t) \cup \gamma'_B(t) \cup \gamma'_C(t) \cup \gamma'_D(t) \subset f(\gamma_A(t) \cup \gamma_B(t) \cup \gamma_C(t) \cup \gamma_D(t)).$$

Therefore, we have $f(\gamma_A(t)) = \gamma'_A(t)$, $f(\gamma_B(t)) = \gamma'_B(t)$, $f(\gamma_C(t)) = \gamma'_C(t)$, and $f(\gamma_D(t)) = \gamma'_D(t)$. Since the midpoint of $\gamma_A(t)$ is mapped to the midpoint of $\gamma'_A(t)$, by (4), the only points on $\gamma_A(t)$ that can possibly be mapped to either end points of $\gamma'_A(t)$ are the two end points of $\gamma_A(t)$. Therefore, the two end points of $\gamma_A(t)$ are mapped to the two end points of $\gamma'_A(t)$. Using a similar argument as in **Step 3**, we see that the mapping of $\gamma_A(t)$ to $\gamma'_A(t)$ is uniform. That is, if we walk from one end point of $\gamma_A(t)$ to another in constant speed, our image under f traces $\gamma'_A(t)$ in constant speed.

There is a catch. There are still two ways to map $\gamma_A(t)$ to $\gamma'_A(t)$ based on which end point gets mapped to which. Unfortunately, given the assumptions we have made so far, we can't eliminate this uncertainty using "up to symmetries" anymore. One of the two orientations is impossible. Specifically, the end point of $\gamma_A(t)$ on AB must be mapped to the end point of $\gamma'_A(t)$ on $O'F'$. Now we prove that the other orientation is impossible.

Suppose that, for some t, the end point of $\gamma_A(t)$ on AB is mapped to the end point of $\gamma'_A(t)$ on $O'E'$. Because orientation clearly depends continuously on t, the orientation must be "wrong" for all $\frac{\sqrt{10}}{2} \leq t < 2$. In particular, by taking $t = \frac{\sqrt{10}}{2}$, we see that

$$f((\frac{\sqrt{2}}{2}, 0)) = E' = (1,0,1).$$

Let $Q = (\frac{5\sqrt{2}}{4}, \frac{3\sqrt{2}}{4})$ be the point on OB such that $d(O,Q) = \frac{1}{4}d(O,B)$. By **Step 3**, we have

$$f(Q) = p_2(\frac{1}{4}) = (\frac{1}{2}, 1, 0).$$

So by (2),

$$\begin{aligned} &d_V((1,0,1),(\frac{1}{2},1,0)) \\ \leq &d((\frac{\sqrt{2}}{2},0),(\frac{5\sqrt{2}}{4},\frac{3\sqrt{2}}{4})) = \frac{3}{2}. \end{aligned} \tag{8}$$

However, this is incorrect. Since $(\frac{1}{2},1,0)$ is $\frac{1}{2}$ away from $C' = (0,1,0)$, (8) would imply that $d_V(E',C') \leq 2$. This contradicts the proof of Lemma 2, as C' is the antipodal point of E' without being the center of any facet. (We could also have directly showed that $d_V((1,0,1),(\frac{1}{2},1,0)) > \frac{3}{2}$ using the general scheme described prior to **Step 2**.)

Thus, we have eliminated the possibility of the "wrong" orientation. Similar argument works for $\gamma_B(t)$, $\gamma_C(t)$, and $\gamma_D(t)$. As a result, the action of f on $\Phi_S(O, \frac{\sqrt{10}}{2})$ is completely matched with the standard wrapping.

For the next step, we define

$$\Psi(t) = \Phi_S(A,t) \cap \Phi_S(B,t) \cap \Phi_S(C,t) \cap \Phi(D,t).$$

We are mainly interested in $\Psi(t)$ for $\frac{\sqrt{10}}{2} \leq t \leq 2$. The shaded region in Figure 10 (left) shows $\Psi(t)$ for some t in this range.

Step 5: *The action of f on $\Psi(\frac{\sqrt{10}}{2})$ matches the standard wrapping.*

Consider $\Phi_V(O',t)$, shown by the shaded region in Figure 10 (right). It is the antipodal set of $\Phi_V(f(O),t)$, which we described in **Step 2**. For $\frac{\sqrt{10}}{2} \leq t \leq 2$, $\Phi_V(O',t)$ lies entirely in the bottom facet. Now, $\Psi(t)$ is the part of S outside of the four circles of radius t and centers A, B, C, and D. Simple calculation shows that $\Psi(t)$ is in fact congruent to $\Phi_V(O',t)$, for t in the above range.

Let $P \in S$ be such that $f(P) \in \Phi_V(O',t)$. Because $f(A) = O'$, we have by (2) and (5)

$$d(P,A) \geq d_V(f(P),O') \geq t.$$

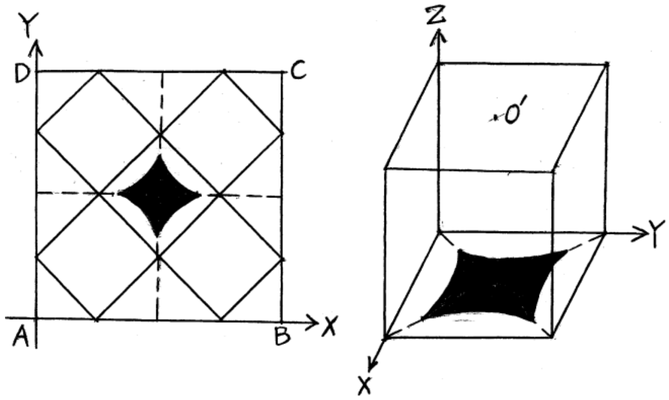

Figure 10: $\Psi(t)$ (left) and $\Phi_V(O',t)$ (right), for some $\frac{\sqrt{10}}{2} \leq t \leq 2$

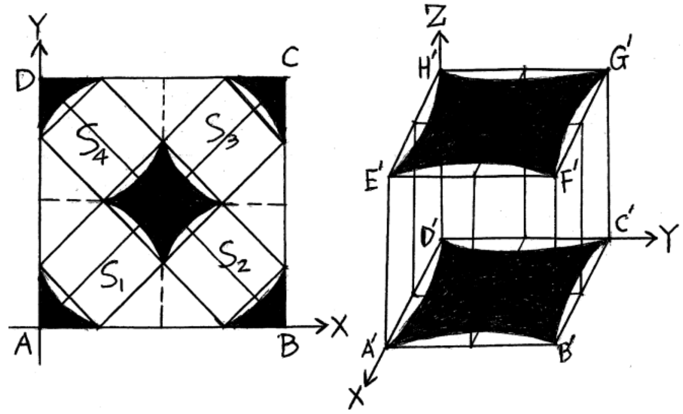

Figure 11: At the beginning of **Step 6**, the shaded region on the left has been matched with the shaded region on the right.

Thus, $P \in \Phi_S(A,t)$. Similarly, P belongs to $\Phi_S(B,t)$, $\Phi_S(C,t)$, and $\Phi_S(D,t)$. Thus, $P \in \Psi(t)$.

The last paragraph shows that $f(P) \in \Phi_V(O',t)$ implies $P \in \Psi(t)$. By (1), (3), and the fact that $\Phi_V(O',t)$ and $\Psi(t)$ are congruent, we see that the converse is also true for $\frac{\sqrt{10}}{2} \leq t \leq 2$ (a similar argument was used in **Step 2** for $\Phi_S(O,t)$ and $\Phi_V(f(O),t)$). From here on, we can use essentially the same arguments as in **Step 2** and **Step 4** to show that the way f maps $\Psi(\frac{\sqrt{10}}{2})$ to $\Phi_V(O', \frac{\sqrt{10}}{2})$ matches the standard wrapping. Almost all changes are notational, so we omit the details.

Step 6: *The action of f on S matches the standard wrapping, up to the placement of the junk material.*

Figure 11 shows what we've matched so far. The four shaded regions near the corners of S, namely $\Phi_S(O, \frac{\sqrt{10}}{2})$, are mapped to cover the shaded region on the top facet of U, namely $\Phi_V(f(O), \frac{\sqrt{10}}{2})$. The shaded region in the middle of S, namely $\Psi(\frac{\sqrt{10}}{2})$, is mapped to cover the shaded region on the bottom facet of U, namely $\Phi_V(O', \frac{\sqrt{10}}{2})$. Lastly, the segments OA, OB, OC, and OD are mapped to cover the paths p_1, p_2, p_3, and p_4 as defined in **Step 3**. All those partial mappings match the standard wrapping.

We match the rest using the following argument. If $P,Q \in S$ are such that 1) $f(P)$ and $f(Q)$ are on the same facet of U, and 2) $d(P,Q) = d(f(P), f(Q)) = a$, then f must map the segment PQ uniformly to the segment connecting $f(P)$ and $f(Q)$. To see why it must be so, notice that 1) By (4), $f(PQ)$ is a path of length at most a in V from $f(P)$ to $f(Q)$, and 2) The only path in V of length at most a from $f(P)$ to $f(Q)$ is the segment between them.

Pick $P = (\frac{\sqrt{2}}{2}, \sqrt{2})$ and $Q = (0, \frac{\sqrt{2}}{2})$. We know that $f(P) = A' = (1,0,0)$ and $f(Q) = E' = (1,0,1)$. Since A' and E' are both on the front facet with $d(A',E') = 1$, and since $d(P,Q) = 1$, our argument shows that PQ is mapped uniformly to $A'E'$. Similarly, we see that the segment between $(\sqrt{2}, \frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2}, 0)$ is mapped uniformly to $B'F'$.

By picking P to be arbitrary points on the segment between $(\frac{\sqrt{2}}{2}, \sqrt{2})$ and $(0, \frac{\sqrt{2}}{2})$, and Q to be arbitrary points on the segment between $(\sqrt{2}, \frac{\sqrt{2}}{2})$ and $(\frac{\sqrt{2}}{2}, 0)$, we see that the unit square S_1 (see Figure 11) is mapped to the front facet of U in a way that matches the standard wrapping. Similarly, we deduce that the unit squares S_2, S_3, and S_4 (see Figure 11) are mapped to the right, back, and left facets of U, respectively, in ways that match the standard wrapping. At this point, all except for the eight tiny gaps and the junk material have been matched. It is easy to see that our argument can match those gaps as well: for any point inside one of the gaps, just pick P and Q to be the intersections of any line through that point with the boundary of that gap. The argument goes through since the boundary of each of those gaps is already matched, and its image is on a single facet, congruent to the boundary itself.

The proof of the theorem is complete.

$\square$

3 Conclusion

Although it is likely that our method of deduction would assist in solving similar uniqueness problems, its application is so far limited by the fact that very few instances of optimal wrapping of 3D objects are solved. For example, it remains unknown how big a square is needed to wrap a regular tetrahedron, or general rectangular boxes.

A possible direction is to consider wrapping a unit cube with a square of side length $2\sqrt{2} + \epsilon$, for some small $\epsilon > 0$. The proof for Lemma 2 actually implies that, for any $\delta > 0$, there exists $\epsilon > 0$ such that in any wrapping of the unit cube with a square of side length at most $2\sqrt{2}+\epsilon$, the center of the square must be within δ from the center of some facet. Could similar statements be proved for other points on the square for slightly suboptimal wrappings? It seems highly likely.

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