Graph Drawings with Relative Edge Length Specifications*

Oswin Aichholzer† TU Graz
Michael Hoffmann‡ ETH Zürich
Marc van Kreveld Utrecht University
Günter Rote§ FU Berlin

Abstract

We study plane straight-line embeddings of graphs where certain edges are specified to be longer than other edges. We analyze which graphs are universal in the sense that they allow a plane embedding for any total, strict order on the edge lengths. In addition, we also briefly consider circular arc drawings with relative edge length specifications.

1 Introduction

There has been considerable research on embedding graphs with specified edge lengths [1, 4, 5, 6, 7]. To embed K_4 with edges of unit length one can take a regular tetrahedron in 3D, but there is no such embedding in the plane. Even if an embedding of a planar graph into the plane with certain specified edge lengths exists, edge crossings may be un-avoidable, for example, a K_4 with four edges of unit length and two non-adjacent edges of length $\sqrt{2}$. Cabello et al. [3] showed that it is NP-hard to decide whether a given 3-connected planar graph admits a plane straight-line embedding with unit edge lengths.

We propose a relaxed version of the problem in which edge lengths are not specified exactly. Instead, we just give a number of specifications of the type: edge e_i is longer than edge e_j. These specifications should form a partial order (with the antisymmetry and transitivity properties, but not reflexivity) on the edge lengths, or else the specifications can obviously not be met.

For trees, any edge length specification allows a plane embedding respecting these lengths. This is true if the length is specified exactly for each edge, and also in our case of relative length specifications. If we consider cycles, then exact edge lengths cannot always be respected (for instance, edge lengths $1, 1, 1, 7$ in a 4-cycle), but relative edge length specifications of our new type can. Forests are the most general class of graphs that allow any exact edge length specification.

For any planar graph, we call specifications of pairs of edges where in a drawing one edge must be strictly shorter than another edge a *relative edge length specification*, or *length specification* for short. For m edges we can specify up to $\binom{m}{2}$ such constraints, and if we have all and they are not inconsistent, we have specified a strict total order on the edge lengths. The question is whether there is a plane embedding for the given graph subject to these constraints. Provided that the edge lengths are all different, this problem is a relaxation of the embedding problem where all edge lengths are specified exactly.

Definition 1 *A graph G is said to be* (edge) length-universal *if any strict total order on the lengths of the edges of G allows a plane straight-line drawing of G that respects the relative edge lengths.*

This paper studies which graphs are length-universal and which are not. In addition to straight-line drawings, we also consider circular arc drawings in the last section.

2 Small graphs and observations

Example 1 K_4 is not length-universal.

To see this, consider two edges e, e' that do not share an endpoint and specify them both to be longer than all of the other four edges. Only one of the long edges, say e, can be on the outer triangle of a plane drawing; the other long edge e' must connect the interior vertex to the vertex v of the outer triangle that is not incident to e. It can easily be seen that e' does not fit inside the outer triangle if it is incident to v. △

The last argument can be phrased more generally as follows.

Observation 1 *(triangle-edge observation) If u, v, w is a triangle in a graph G, and ut is an edge specified to be longer than both uv and uw, then in any plane straight-line embedding of G, edge ut is outside the 3-cycle u, v, w.*

Given that (1-)trees are length-universal, it is natural to ask whether the same holds for *2-trees*. These are all

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graphs that can be recursively obtained from a triangle by adding a new vertex and connecting it by an edge to both endpoints of an existing edge.

Example 2 There exists a 2-tree on five vertices that is not length-universal.

To see this, consider the example depicted in Figure 1, where the numbers in increasing order show the length order from shortest to longest (not edge length itself). At least two 2-paths are on the same side of the shortest edge. Regardless of which pair of 2-paths we take, and which 2-path we put on the inside of the two, the triangle-edge observation will be violated. △

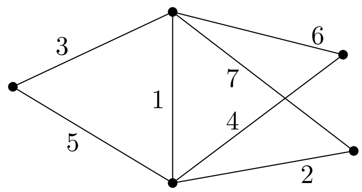

Figure 1: A 2-tree that does not admit a plane drawing for the specified length order.

We can make another observation about triangles, or in fact, any cycles, in graphs:

Observation 2 *(cycle observation) In a triangle, the longest edge is less than twice as long as second-longest edge. In other words, their length ratio is in the open interval* $(1, 2)$*. For* k*-cycles, the length ratio between the longest and the second-longest edge is in the open interval* $(1, k-1)$*.*

Lemma 1 *(area-length lemma) Consider a triangle* C *in* $\mathbb{R}^2$ *whose shortest side has unit length. Then* C *has (i) area* $\geq \frac{1}{4}\sqrt{3}$*, or (ii) an edge of length* $> \frac{3}{2}$*.*

Proof. Consider $C = pqr$ such that pq has unit length. As pq is the shortest side of C, the third point r lies outside the union of the two unit disks centered at p and q. If (ii) does not hold, then r lies in the intersection of the two disks of radius $3/2$ centered at p and q. In the resulting region of possible placements for r (see Figure 2) the point closest to the line pq is the one in unit distance to both p and q, leading to an equilateral triangle pqr. □

The examples studied so far might suggest that triangles in the graph are the main obstruction for length universality. However, this is not the case.

Example 3 There exists a bipartite graph on six vertices that is not length-universal.

Let the graph G be $K_{3,3}$ minus an edge $\{u, v\}$ and specify that the four edges incident to u and v must be longer than the remaining four edges (Figure 3). In any

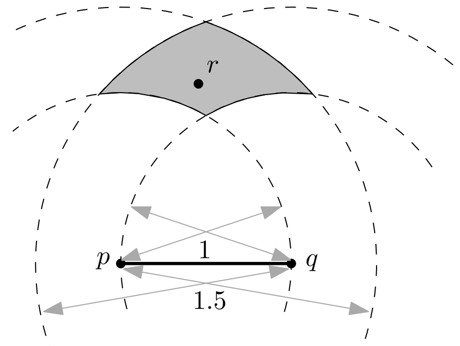

Figure 2: Any triangle has some positive area or some difference in edge lengths.

plane straight-line drawing of G, the four short edges form a simple polygon P such that exactly one of u or v, say v, is inside P. Let $P = axby$ such that a and b are the neighbors of v. At least one of a or b, say a, is a convex vertex of P. Given that av is longer than both ax and ay, it follows that v lies outside the triangle axy. This implies that b is a convex vertex of P as well. Similar as for a, we conclude that v lies outside the triangle bxy. As $P = axy \cup bxy$, this yields a contradiction to our assumption $v \in P$. △

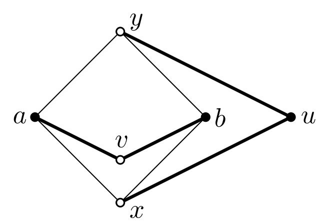

Figure 3: If the fat edges must be longer than the thin edges, then no plane embedding exists.

3 Outerplanar graphs

In the negative examples discussed in the previous section, obstructions mostly come from nested cycles with incompatible length specifications. It is, therefore, natural to ask: What happens if nested cycles are forbidden? We arrive at the class of outerplanar graphs, which also may be considered as a generalization of trees (the only length-universal family mentioned thus far).

Example 4 There are outerplanar graphs with maximum vertex degree 3 that are not length-universal.

To show this, we construct a graph $G = G_1 \cup G_2 \cup G_3$ that consists of three (connected) components that are described in the following paragraphs.

Consider a tree with vertex degrees 1 or 3, such that all leaves have the same distance from some central vertex. Replace every vertex by a triangle (Figure 4); this is G_1. If the original tree had graph-diameter k, then G_1 has graph-diameter $2k+1$ and consists of $3 \cdot 2^{k/2} - 2$ triangles. For one triangle in G_1 with edges e_1, e_2, and e_3, we specify $|e_1| < |e_2| < |e_3|$. For all other edges of G_1 we specify their length to be $> |e_2|$ and $< |e_3|$. Normalizing $|e_2| = 1$ we conclude from Obervation 2 that all edge lengths other than $|e_1|$ are in the interval $(1, 2)$. Therefore, the Euclidean diameter of G_1 is $\Theta(k)$, and so the embedding of G_1 fits into some square of area $O(k^2)$. For k large enough, many triangles have area less than $\frac{1}{4}\sqrt{3}$. Together with Lemma 1 it follows that $3/2 < |e_3|/|e_2| < |e_3|/|e_1|$.

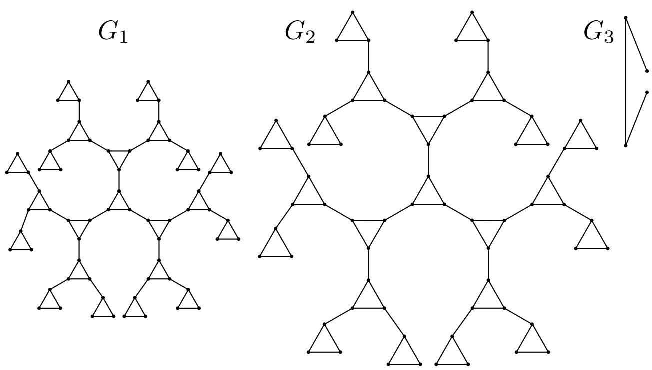

Figure 4: An outerplanar graph with maximum vertex degree three that is not length-universal.

As a second step, take an identical copy G_2 of G_1 along with all edge length specifications. Let e'_i denote the edge in G_2 corresponding to e_i in G_1, for $i \in \{1, 2, 3\}$. Now we specify $|e_3| < |e'_1|$, implying that $|e'_3|/|e_1| > 9/4$.

Finally, we create a triangle G_3 with edges e''_1, e''_2, and e''_3. We specify $|e''_1|, |e''_2| < |e_1|$ and $|e''_3| > |e'_3|$. This enforces $|e''_3| > 2.25 \max\{|e''_1|, |e''_2|\}$, in contradiction to Observation 2 applied to G_3. △

Looking at the above construction, one might suspect the many triangles of G to be the culprit. But the construction can be adapted to yield triangle-free outerplanar graphs.

We formulate the central argument of the construction more generally as follows, without proof.

Lemma 2 *Let G_1 be a graph which cannot be embedded with all edge lengths in the range $[1, 1+\varepsilon]$.*

Choose integers $g \geq 3$ and r such that $(1+\varepsilon)^r > g-1$. Then a graph containing r edge-disjoint copies of G_1 and an additional edge-disjoint g-cycle is not length-universal.

In the previous example, we have $\varepsilon = 1/2$, $r = 2$, and $g = 3$. The construction can be modified to use odd cycles of any length instead of triangles. We will state and prove it specifically for pentagons. We need the following geometric lemma, similar to Lemma 1 for triangles.

Lemma 3 *Consider a (simple) pentagon C in $\mathbb{R}^2$ whose shortest side has unit length. Then C has (i) area $\geq \sqrt{8}/9 > 0.31$, or (ii) an edge of length $> 4/3$.*

Proof. Consider a simple pentagon where all edges have lengths in $[1, 4/3]$. Böröczky et al. [2] have analyzed the infimums of the areas of simple polygons with specified number of edges and edge lengths. The infimum corresponds to a degenerate situation in which the edges are partitioned into three classes, and the edges are aligned (in the same or in opposite directions) so that the result looks like a triangular area with zero-area extension of the edges. (Of course, we cannot have aligned or overlapping edges in a simple polygon, but can come arbitrarily close to such a degenerate solution.) As a consequence, at two of the five vertices of a close-to-minimum area pentagon, the angle is near 0 or near π. The edge lengths do not allow two pairs of edges to be aligned, because the resulting triangle would have edge lengths in the ranges $[0, 1/3]$, $[0, 1/3]$, and $[1, 4/3]$, which contradicts the triangle inequality. The only remaining option is to align a triple of edges. The resulting class of triangles has side lengths in the intervals $[2/3, 5/3] \cup [3, 4]$, $[1, 4/3]$ and $[1, 4/3]$. The interval $[3, 4]$ cannot be used, by the triangle inequality. The smallest area is realized with edge lengths 1, 1, and $1 + 1 - 4/3 = 2/3$, giving an area of $\sqrt{8}/9$. □

Example 5 There are triangle-free outerplanar graphs of maximum degree three that are not length-universal.

We use a construction analogous to Example 4. When building G_1, we use 5-cycles instead of triangles, and connect them symmetrically in such a way that on each 5-cycle, the two vertices that connect to the children are neighbors.

Specifying the length order for the edges of one 5-cycle, Observation 2 guarantees that the length ratio between the two longest edges is in $(1, 4)$. As above, we specify all other edges to have length in between these two. It follows that both graph and Euclidean diameter of G_1 are in $\Theta(k)$. The area bound in combination with Lemma 3 enforces an edge length ratio $> 4/3$ between the shortest and the longest edge if k is large enough. Now we apply Lemma 2 with $1 + \varepsilon = 4/3$, $g = 4$, and $r = 4$. △

Lemma 2 gives us a method for constructing graphs which are not length-universal, by starting from a graph G_1 whose edges cannot be embedded with equal lengths, not even approximately. This construction has a positive counterpart, a principle for showing that a graph is length-universal, which we well see in action in several places: if we can find a drawing in which all edge lengths are equal, we will usually have enough freedom to perturb the drawing to achieve any desired relative ordering of edge lengths.

In a desperate attempt to further restrict the class of graphs, let us consider maximal outerplanar graphs, that is, graphs that can be realized as triangulated simple polygons. Of course, all examples discussed so far can be extended to maximal planar graphs by adding edges (which cannot make embedding easier), but then the vertex degree goes up. Hence our real interest is in maximal outerplanar graphs of low vertex degree.

Example 6 There is a maximal outerplanar graph of maximum vertex degree five that is not length-universal.

Consider again the construction from Example 4. We claim that we can add edges to G_1 to convert it into a maximal outerplanar graph in such a way that every vertex is incident to at most two new edges (and so has maximum degree five). Roughly speaking, we walk around the outer boundary of the tree, skipping every other vertex, and remove duplicate edges. More precisely, consider G_1 as a rooted and embedded tree on the triangles. Each non-root triangle has one *top vertex* that is connected to the parent triangle and two *children vertices*—a left one and a right one—that are connected to the children or not connected at all, for the leaf triangles. For each non-root triangle t add two edges connecting it to its parent triangle p. If t is a left child of p, then add an edge from the left child vertex of t to the left child vertex of p and an edge from the top vertex of t to the right child vertex of p. If t is a right child of p, then add an edge from the left child vertex of t to the right child vertex of p and an edge from the top vertex of t to the top vertex of p. It is easy to check that no vertex gets more than two new edges this way. Finally, the graphs G_1, G_2, and G_3 can easily be connected by adding edges to connect leaf triangles (Figure 5). Note that the right child vertex of each leaf triangle has degree two only and its left child vertex has degree three. The resulting graph is maximal outerplanar with maximum vertex degree five. $\triangle$

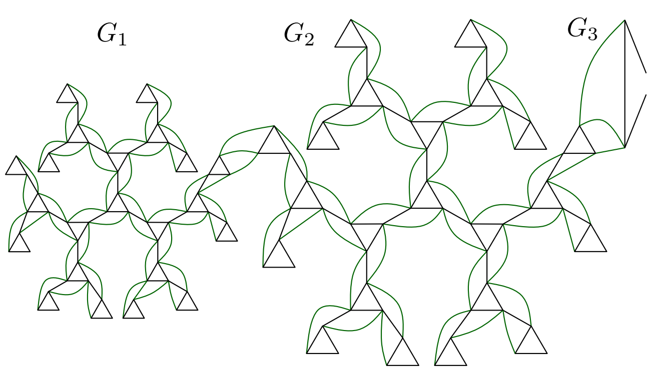

Figure 5: A maximal outerplanar graph with maximum vertex degree five that is not length-universal.

After such a large collection of negative examples, it is about time to finally present some positive result—if only to illustrate that the class of edge length-universal graphs goes beyond trees and cycles.

Lemma 4 *Every maximal outerplanar graph of maximum vertex degree four is length-universal.*

Proof. Such a graph is always a triangle strip with $n \geq 3$ vertices and $m = 2n - 3$ edges. It can be embedded on the triangular grid with unit edge lengths (Figure 6). Whatever the edge length specifications are, we can always change the lengths of the edges by very small amounts. For instance, if $e_0, \ldots, e_{m-1}$ is a total order of the edges by increasing length that conforms with the specifications, then we set $|e_i| = 1 + \delta i/m$, for $i \in \{0, \ldots, m-1\}$ and some $\delta > 0$ to be determined.

Then each triangle has edge lengths in the interval $[1, 1+\delta]$ and the angles in such a triangle are in the interval

$$\left[2\arcsin\left(\frac{1}{2(1+\delta)}\right), 2\arcsin\left(\frac{1+\delta}{2}\right)\right].$$

Both of these expressions are continuous functions of δ on $[0,1]$. Therefore, for $\varepsilon = \pi/(6n) > 0$ we find $\zeta > 0$ such that all triangle angles are in $[\pi/3 - \varepsilon, \pi/3 + \varepsilon]$, for $\delta \in [0, \zeta]$. This implies that all edges that are parallel in a unit edge-length embedding deviate in angle by at most $\pi/6$ with respect to each other. $\square$

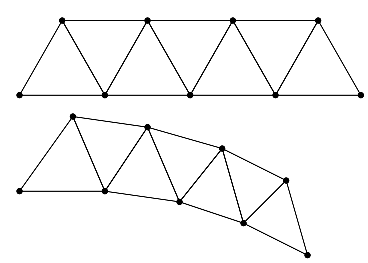

Figure 6: Maximal outerplanar graphs with maximum vertex degree four are length-universal.

4 The graphs $K_{2,m}$

We have seen that some small 2-tree is not length-universal, and also some small bipartite graph is not length-universal. However, it appears that a slight variation on these examples yields a class of graphs that is length-universal.

Theorem 5 *Every graph $K_{2,m}$ is length-universal.*

Proof. Let us call the small rectangle $[-\varepsilon, \varepsilon] \times [0, \varepsilon]$ above the origin an ε-box. We prove the following statement by induction on m: For every $\varepsilon > 0$, the graph

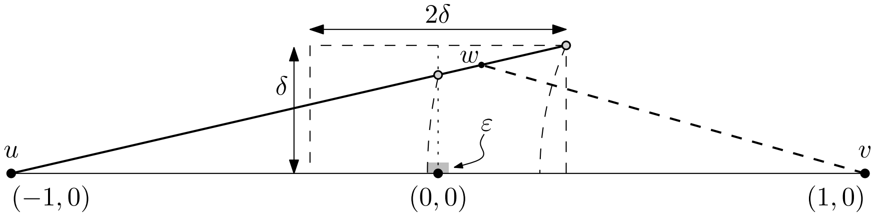

Figure 7: Drawing a $K_{2,m}$ with edge length specifications.

$K_{2,m}$ with arbitrary length specifications can be drawn with the two degree-m vertices u and v on $(-1, 0)$ and $(1, 0)$ and all other vertices in the ε-box.

As induction base, we can take the trivial case $m = 0$. Now, let $m \geq 1$. Assume without loss of generality that the longest edge is incident to u. Let us denote it by uw, and consider the graph $G' = K_{2,m} - \{w\}$. We rename ε to δ for the induction claim, but we will continue to use ε for the induction hypothesis. We show that, for any $0 < \delta < 0.5$, there is an $\varepsilon > 0$ such that if G' is embedded, by the induction hypothesis, with all vertices $\neq u, v$ in an ε-box, the vertex w can be placed in a δ-box so that the two newly placed edges have the correct relative lengths with respect to the edges of G', see Figure 7. We place w on the line segment between u and the upper right corner of the δ-box, in the half-plane with positive x-coordinates. In other words, it will lie between the points $(0, \delta/(1 + \delta))$ and (δ, δ). If ε is small enough with respect to δ, this ensures that uw becomes the longest edge, no matter where on the segment we put it. The placement of w allows vw (the other new edge) to assume any length between $\sqrt{(1-\delta)^2 + \delta^2}$ and $\sqrt{1 + (\delta/(1 + \delta))^2}$. All previously placed vertices of G' are in an ε-box, so the edge lengths of G' are between $1 - \varepsilon$ and $\sqrt{(1+\varepsilon)^2 + \varepsilon^2}$. If ε is sufficiently small compared to δ, for example $\varepsilon = \delta^2/8$, then $(1-\varepsilon, \sqrt{(1+\varepsilon)^2 + \varepsilon^2}) \subset (\sqrt{(1-\delta)^2 + \delta^2}, \sqrt{1 + (\delta/(1 + \delta))^2})$, showing that the length vw can be chosen to satisfy any length specifications with respect to the edges of G'.

Since the two new edges of P_i avoid the ε-box in which the vertices of $G' - \{u, v\}$ lie, there are no new intersections between edges, and the resulting drawing is plane. $\square$

Note that adding any edge except uv to $K_{2,m}$ gives the graph of Figure 3 as a subgraph. Thus, this slightly modified version of $K_{2,m}$ is already not length-universal.

5 Relative edge lengths and circular arcs

We consider drawing a graph with relative edge length specifications using circular arcs. Not unexpectedly, this version allows more graphs to be drawn with the edge length specifications, and the triangle and cycle observations no longer hold.

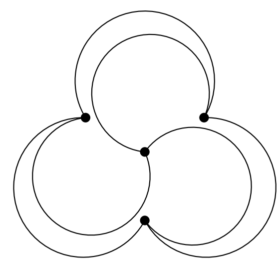

Figure 8: Drawing K_4 using circular arcs of the same length.

Example 7 We show that K_4 is length-universal for circular-arc drawings.

In Figure 8, all six arc lengths are equal. By perturbing the arcs, one can achieve any desired order of edge lengths. It is perhaps surprising that the inner edges can be longer than the outer arcs. In this drawing, they can become longer than the outer arcs by a factor of up to 5/4. $\triangle$

When constructing drawings with circular arcs as edges, it is useful to study 3-cycles which will be drawn as circular-arc triangles, called *criangles* from now on. The longest arc in a criangle can be arbitrarily much longer than the other arcs, and the second-longest arc can be arbitrarily much longer than the shortest arc at the same time. However, it is easy to see that if the longest arc is concave, then its length is at most twice the length of the second-longest arc. This allows us to show that wheel graphs are not length-universal for circular-arc drawings.

Lemma 6 *Given a criangle with a shortest edge of length* 1 *and longest edge of length at most* 2*, at least one of the following statements holds:*

(i) the tangent angle at the vertex incident to the shorter two edges is at least α, for some constant $\alpha > 0$;

(ii) the criangle contains an area of at least δ inside, for some constant $\delta > 0$.

Proof. Let the arcs be a, b, c and assume $1 \leq ||a|| \leq ||b|| \leq ||c|| \leq 2$. Let the vertices be u, v, w where u is opposite a, v is opposite b, and w is opposite c. See Figure 9(a).

We observe that the curvatures of a, b, c are less than 2π, since no arc can be a full circle and the arcs have length at least 1. In other words, they are parts of a circle with radius at least $1/(2\pi)$.

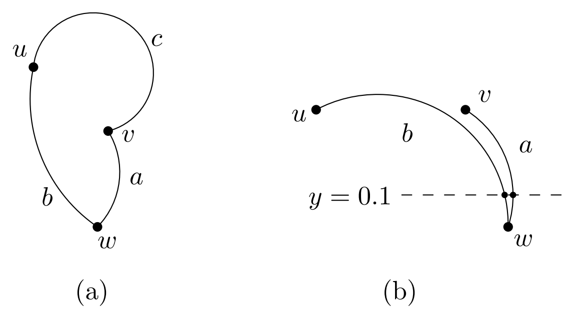

Figure 9: Notation for the proof of Lemma 6.

Suppose *(i)* does not hold. We can assume after translation and rotation that vertex w lies at the origin, and the positive y-axis is the bisector of the two tangents of a and b at w. After reflection we can assume that arc a has a positive tangent at w and arc b has a negative tangent at w, unless the tangents are the same (and equal to the y-axis). In this case we can assume after reflection that locally at w, arc a is right of arc b.

Consider the horizontal line $y = 0.1$ and where it cuts a and b. Due to the curvature bound it must cut both; if there are more intersections we take the ones closest to w on the arcs. See Figure 9(b). We distinguish two cases.

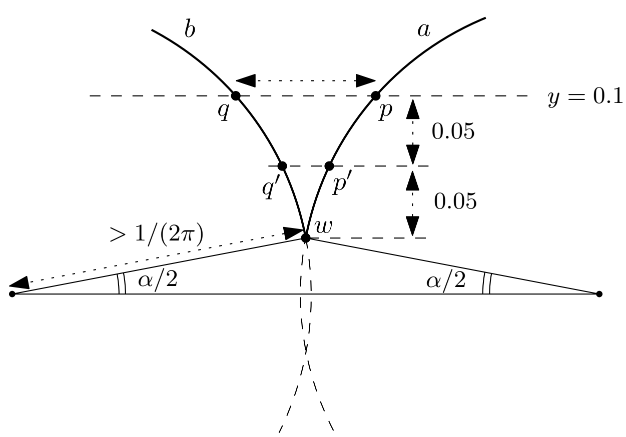

Figure 10: Case 1 in the proof.

Case 1: If the distance between these intersection points is at least γ for some constant $\gamma > 0$, then we argue as follows. See Figure 10. Let p be the intersection point of $y = 0.1$ and a and let q be the intersection point of $y = 0.1$ and b. Due to the curvature bounds on a, b, and c, the distance between p and q is small enough so that c cannot intersect pq if α is a small enough constant. Now we argue that the part of the criangle below pq has at least some positive constant area. Consider the intersection points of a and b with the line $y = 0.05$, and call them p' and q', respectively. The distance between p' and q' is at least some constant depending on α and γ. This is true for any horizontal line between $y = 0.05$ and $y = 0.1$, which implies that the area of the criangle between $y = 0.05$ and $y = 0.1$ is at least the minimum of these distances times 0.05, which is a positive constant.

Case 2: If the distance between these intersection points is less than γ, then we use the fact that arcs a and b have nearly the same curvature: the circles on which the arcs a and b lie are fully determined by the fact that they contain w, by their tangents at w, and by the intersections we chose on $y = 0.1$.

Let v' be the point on b closest to v. The distance ε between v and v' is small if γ is small. More precisely, for any constant $\varepsilon > 0$ there exists a constant $\gamma > 0$ such that the distance between v and v' is at most ε.

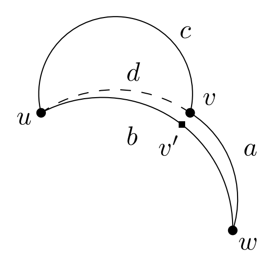

Figure 11: Case 2 in the proof.

Let d be the shortest circular arc inside the criangle between u and v, see Figure 11. The length of d is less than the sum of the distance from v to v' and the distance from v' along b to u. In a formula, $||d|| \leq ||b|| - ||a|| + \eta$ for some $\eta = \eta(\alpha, \varepsilon) = \eta(\alpha, \gamma) > 0$ constant. For any constant $\eta > 0$ there exist constants $\alpha > 0$ and $\gamma > 0$ such that the inequality is true.

We argue next that c and d are significantly different in length. $||c|| - ||d|| \geq ||b|| - (||b|| - ||a|| + \eta) = ||a|| - \eta \geq 1 - \eta$. Moreover, for any constant $\eta > 0$ there exist constants $\alpha > 0$ and $\gamma > 0$ such that the inequality is true. We conclude that there exist $\alpha > 0$ and $\gamma > 0$ such that c is at least some significant amount, say, 0.5, longer than d. The area in between is a lens bounded by a circular arc of length at most 2 and another circular arc that has length at most 1.5. This lens is fully inside the criangle and it has at least some positive constant area. This concludes the second case. $\square$

Example 8 Wheel graphs are not length-universal for circular-arc drawings, for sufficiently large wheels.

To see this, take a spoke c of the wheel and let the incident criangles have arcs a, b and d, e. Now specify the lengths to be such that c is longest and b is second-longest among a, b, c, d, e. For all other arcs of the wheel, specify their lengths to be greater than $||b||$ and smaller than $||c||$. If we normalize $||b|| = 1$, then all arcs of the wheel, except for a, b, c, d, e, have lengths strictly between 1 and 2, because c is concave in criangle abc or in cde if the wheel has its usual embedding. To deal with embeddings where abc is inside cde or vice versa, we take another spoke c' and incident criangles $a'b'c'$ and $c'd'e'$ and specify relative lengths of these five arcs in the same way. Now we observe that in one of the four criangles, c or c' must be concave in any embedding, and hence we can force all other arcs of the wheel to have arc lengths between 1 and 2 after normalization by specifying their lengths to be between $||b'||$ and $||c'||$ as well.

Given any criangle of the wheel with arc lengths between 1 and 2, we specify its non-spoke arc to be longer than its spokes. Then the criangle's angle at the central vertex is at least α, or its area is at least δ by Lemma 6. Furthermore, such a criangle lies fully inside a circle with radius 3 around the central vertex of the wheel. This means that if we have more than $2\pi/\alpha + 9\pi/\delta + 4$ criangles in the wheel, then the drawing cannot be plane. $\triangle$

6 Conclusions

We initiated research on drawing graphs where relative edge lengths are pre-specified. This is a generalization of drawing graphs where edge lengths are specified precisely. Several graph classes are shown not to be universal in the sense that they do not allow a plane drawing for all relative edge length specifications. Two rather restricted classes of graphs are shown to be length-universal. It remains to give a full characterization of graphs that are length-universal.

We also considered circular-arc drawings briefly, and gave one positive and one negative result. Here a full characterization seems further away than for straight-line drawings.

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