Vertex Unfoldings of Orthogonal Polyhedra: Positive, Negative, and Inconclusive Results

Luis A. Garcia * Andres Gutierrez * Isaac Ruiz * Andrew Winslow *

Abstract

We obtain results for three questions regarding vertex unfoldings of orthogonal polyhedra. The (positive) first result is a simple proof that all genus-0 and genus-1 orthogonal polyhedra have grid vertex unfoldings. The (negative) second result is an orthogonal polyhedron that is not vertex-unfoldable. The (inconclusive) third result is a vertex unfolding of an orthogonal polyhedron that cannot be arranged orthogonally, evidence that deciding whether an orthogonal polyhedron has a vertex unfolding may not lie in NP.

1 Introduction

The study of unfolding polyhedral surfaces can be traced to as early as the 1500s when Albrecht Dürer considered unfoldings of convex polyhedra via cuts only along edges (called *edge unfoldings*).

To explore unfoldings of non-convex polyhedra, Biedl et al. [3] considered *orthogonal* polyhedra, in which all faces are perpendicular to one of the three axes, providing examples of edge ununfoldable orthogonal polyhedra as well as methods for unfolding some classes of orthogonal polyhedra. Subsequent work has further explored the boundary between unfoldable and ununfoldable orthogonal polyhedra by varying both the types of unfoldings permitted and classes of orthogonal polyhedra under consideration.

For instance, one line of work has considered broadening permitted unfoldings via *refinement*: adding a regular grid of (potential cut) edges to each face. Damian, Flatland, and O'Rourke [8] proved that exponential refinement was sufficient to unfold any orthogonal polyhedron; this refinement was later reduced to quadratic [7], and then linear [6].

To unfold the edge ununfoldable examples of Biedl et al. [3], it is sufficient to add *grid edges* formed by the intersection of orthogonal planes intersecting each vertex of the polyhedron. Such *grid edge unfoldings* have been found for several classes of orthogonal polyhedra [3, 5, 10, 13].

*Department of Computer Science, University of Texas Rio Grande Valley, Edinburg, TX. luis.a.garcia01@utrgv.edu, andres.a.gutierrez01@utrgv.edu, isaac.ruiz02@utrgv.edu, andrew.winslow@utrgv.edu

Extending grid unfoldings to allow cuts meeting at (but excluding) a vertex yields *grid vertex unfoldings*. Vertex unfoldings were first introduced for general polyhedra [11], and grid vertex unfoldings have been shown to exist for some classes of orthogonal polyhedra [12], including all genus-0 orthogonal polyhedra [9].

Here we obtain three new results on grid vertex unfoldings of orthogonal polyhedra:

- Every genus-0 and genus-1 orthogonal polyhedron has a grid vertex unfolding (extending a previous result of Damian, Flatland, and O'Rourke [9] to include genus 1 and providing an alternative, simpler proof).
- There exists an orthogonal polyhedron with faces homeomorphic to disks that does not have a vertex unfolding (complementing a vertex-ununfoldable topologically convex polyhedron of Abel, Demaine, and Demaine [2] and vertex-ununfoldable orthogonal polyhedra with faces not homeomorphic to disks by Biedl et al. [3]).
- There exists a "maximally cut" vertex unfolding of an orthogonal polyhedron that cannot be made orthogonal, raising the question of whether deciding if an orthogonal polyhedron has a vertex unfolding is in NP (in contrast with the trivial containment in NP of deciding whether an orthogonal polyhedron has an *edge* unfolding [1]).

2 Definitions

This work considers orthogonal polyhedra, i.e. polyhedra where all edges are parallel to the x-, y-, or z-axis. A *gridded* polyhedron has edges everywhere that an xy-, xz-, or yz-plane intersects a vertex of the polyhedron, resulting in faces that are edge-adjacent rectangles. A *polycube* is a special case of gridded orthogonal polyhedra whose faces are unit squares.

An *unfolding* is a connected planar arrangement of the surface of a polyhedron by the addition of cuts. If the cuts are restricted to the polyhedron's edges, then the resulting unfolding is an *edge unfolding* if the surface remains strongly connected, and a *vertex unfolding* otherwise. A surface that permits no additional edges or vertices to be cut without disconnecting the surface is *maximally cut*.

Unfoldings that are (possibly weakly connected) orthogonal polygons are also called *orthogonal*. In the case of vertex unfoldings, point connectivity on the surface yields a "hinge" or "joint" allowing portions of the surface to rotate relative to each other. Thus vertex unfoldings are weakly connected polygons that may either be orthogonal or not.

The flexibility around vertices allows for faces in a vertex unfolding to (potentially) not appear in the same clockwise order around a common vertex as they do on the original surface; here we allow such "rearrangement" of vertex-adjacent faces (see [11] for further discussion).

3 A Simple Proof that Genus-≤ 1 Orthogonal Polyhedra have Grid Vertex Unfoldings

Here we prove the existence of vertex unfoldings for low-genus polycubes using an approach reminiscent of a proof of a similar result [11] for polyhedra with (possibly intersecting) triangular faces. This implies a corollary (Corollary 2) for grid vertex unfoldings that extends the previous result by Damian, Flatland, and O'Rourke [9] on grid vertex unfoldings to include genus-1 polyhedra.

Theorem 1 *Every genus-0 and genus-1 polycube has a vertex unfolding.*

Proof. First, we prove that the face dual graph (obtained by creating a vertex for every face of the polycube and connecting pairs of edge-adjacent faces) of every such polycube has a Hamiltonian path. Afterwards, we show how to use a Hamiltonian path through the faces to obtain a vertex unfolding.

Bodini and Lefranc [4] prove that polycube face dual graphs are 4-connected. Intuitively, this is due to the four directions that must be traversed to obtain a disconnecting cut of the surface of a polycube (i.e. non-contractable cycle on the surface). As they observe, Tutte [15] proves that all genus-0 4-connected graphs are Hamiltonian, implying that all face dual graphs of genus-0 polycubes are Hamiltonian. Recently, Thomas, Yu, and Zang [14] proved that all genus-1 4-connected graphs have a Hamiltonian path.

The vertex unfolding consists of a path of (vertex) connected faces appearing in the same order as on the Hamiltonian path previously proved to exist. These faces are arranged left to right, and remain either edge-connected (if the previous face is left of the next face) or cut to become vertex-connected and rotated by $\pm 90°$ (if the previous face is above or below the next face). These three cases are seen in Figure 1.

Only these three cases need be considered due to maintaining the invariant that before the edge connecting the previous (gray) face to the next (green) face is cut, the next face is above, below, or to the right of the previous face. $\square$

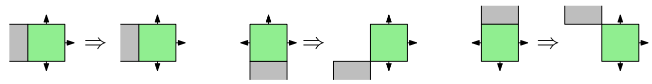

Figure 1: The three cases for vertex unfolding a sequence of consecutive faces along a Hamiltonian path of the face dual graph. The invariant maintained is that each subsequent face (in green) begins attached to the above, below, or right of the previous face (indicated as arrows). In the right two cases, the face is rotated $\pm 90°$ to maintain the invariant.

Observe that the resulting unfolding is also orthgonal. Moreover, because the resulting unfolding can be partitioned into vertical strips each containing one face of the polycube, a similar result holds if the faces are edge-adjacent rectangles, as they are in all "gridded" orthogonal polyhedra:

Corollary 2 *Every genus-0 or genus-1 orthogonal polyhedron has an orthogonal grid vertex unfolding.*

4 A Vertex-Ununfoldable Orthogonal Polyhedron

Theorem 3 *There is an orthogonal polyhedron with simple faces that cannot be vertex-unfolded.*

Proof. The vertex-ununfoldable polyhedron consists of a box with a thin "ridge" through two adjacent faces of the box (see Figure 2).

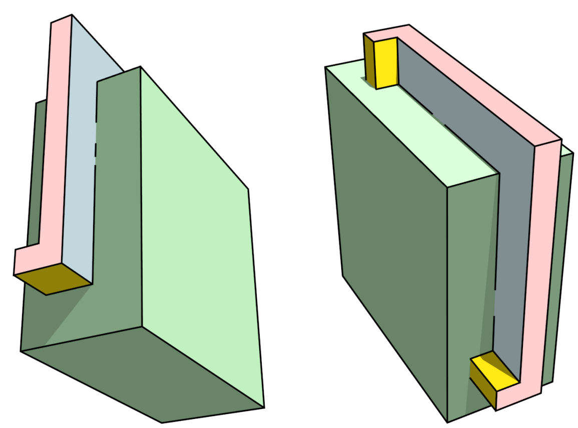

Figure 2: An orthogonal polyhedron that cannot be vertex unfolded.

We prove that the polyhedron is ununfoldable by considering only a portion of the surface consisting of two large *base faces* containing the ridge (green in Figure 2), and the *ell*, *putt*, and *bungie faces* on the ridge (blue, pink, and yellow, respectively, in Figure 2).

In the unfolding, at least one of the two (symmetric) putt faces must be connected to a base face via a sequence of faces that excludes the other putt face. For the remainder of the proof, we consider only this sequence of faces, proving that any such sequence cannot be arranged without overlap.

Since the boundary of the putt shares no boundary with the base face on the surface, the sequence must contain either a bungie or ell face. Regardless of the sequence of faces connecting the putt to the base face, the end of the face sequence is either:

1. ell $\rightarrow$ base, or
2. putt $\rightarrow$ bungie $\rightarrow$ base, or
3. ell $\rightarrow$ bungie $\rightarrow$ base.

Case 1: ell $\rightarrow$ base. The first case implies overlap between the ell face and base face (see Figure 3), as these two faces are connected by either the unique edge they share on the surface or a vertex of this edge.

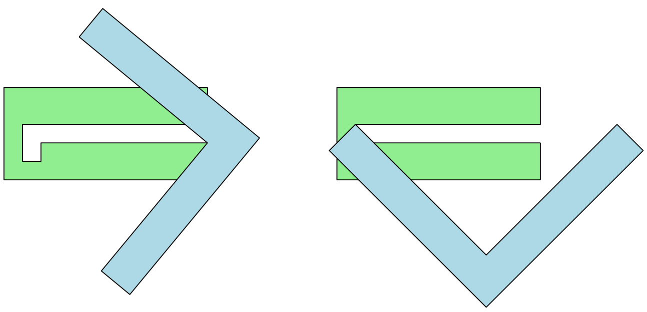

Figure 3: Any attached ell and base face must overlap.

For the other two cases, we simplify the analysis by considering the bungie faces as a zero-width curve of length between 0 and the maximum distance between two locations on a sequence of connected bungie faces (see Figure 4) of $\approx 7.30 < 8$. That is, we suppose they behave as an elastic "bungie cord".

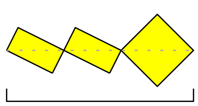

Figure 4: The maximum length of a connected sequence of bungie faces is $\sqrt{5} + \sqrt{5} + \sqrt{8} \approx 7.30$.

Case 2: putt $\rightarrow$ bungies $\rightarrow$ base face. In the unfolding, a bungie face connects to the putt at boundary location(s) limited to those drawn in red in Figure 5. Any non-overlapping arrangement of the putt and base faces either has the entire notch filled by the putt (leaving no available location for any bungie face in the unfolding) or has no portion of the boundary of the putt to which the bungie faces can attach more than distance 1 from the entrance of the notch. In the latter case, the minimum distance between the bungie faces' connection to the base face and putt's boundary is at least 8, exceeding the maximum distance that can be spanned by the bungie faces.

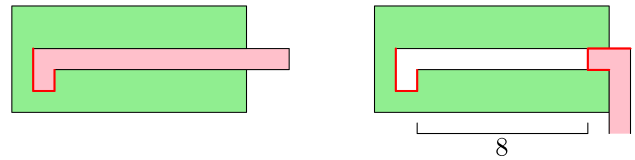

Figure 5: The two potential putt placements for Case 2. The red boundary portions denote where bungie faces must connect (to both the putt and base faces).

Case 3: ell $\rightarrow$ bungies $\rightarrow$ base face. This case is proved similarly to Case 2. Since ell faces are 2 units wide, any optimal non-overlapping arrangement of the ell and base faces forms a right triangle consisting of a portion of the ell face, with the 90° vertex on the ell and two remaining vertices at the entrance of the notch (see Figure 6).

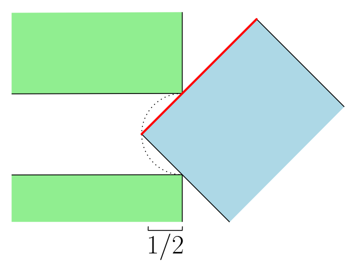

Figure 6: Arranging an ell face as deep in the notch as possible.

By Thales's theorem, the 90° vertex lies on a circle whose diameter is the notch entrance. Then since this circle has radius 1/2, the 90° vertex (and all other locations on the ell) has distance at most 1/2 from the entrance of the notch. So as in Case 2, the minimum distance between the bungie faces' connection to the base face and putt's boundary exceeds the maximum distance that can be spanned by the bungie faces. $\square$

5 Evidence that Vertex Unfolding Orthogonal Polyhedra is not in NP

Finally, we consider the complexity of deciding whether a polycube has a vertex unfolding. Specifically, whether the problem lies in NP.

As Abel and Demaine [1] observe, the same problem limited to edge unfoldings is easily seen to be in NP. One proof uses the following simple algorithm: nondeterministically select a set of maximal set of polycube faces to cut that leaves the face dual graph connected (i.e. a set of cut edges that yields a tree-shaped face dual graph). Then check whether the resulting surface is indeed an unfolding, i.e. has no overlaps.

This NP algorithm for edge unfolding relies in part on the uniqueness of the induced unfolding. However, in vertex unfoldings, faces may be connected by a single vertex, allowing infinitely many angles at which these two faces may be arranged. Thus the same algorithmic approach for vertex unfoldings requires efficiently determining whether a maximally set of cut edges yields a surface that can be arranged into a vertex unfolding (by careful selection of adjacent face angles).

One natural approach to resolving this issue is to prove that any maximally cut polycube surface has a vertex unfolding only if there is such an unfolding that is orthogonal, i.e. where all adjacent face angles from the set $\{0°, 90°, 180°, 270°\}$. Here, we prove by example that this is not the case.

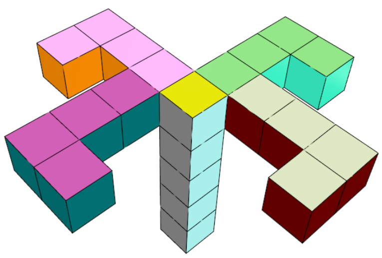

Figure 7: A polycube with a maximally cut vertex unfolding that cannot be arranged orthogonally.

Theorem 4 *There is a maximally cut vertex unfolding of a polycube with no orthogonal arrangement.*

Proof. The maximally cut unfolding with no orthogonal arrangement is seen in Figure 8. The four L-shaped regions adjacent to face c (colored fushia, eggshell, light green, and pink in Figure 8) are *claws*. The two regions attached by a single vertex to face c consist of face sets $A = \{a_1, \ldots, a_5\}$ and $B = \{b_1, \ldots, b_5\}$

For the remainder of the proof, we assume face c is orthogonal. In any orthogonal arrangement of the unfolding, each of six face adjacent to c lies in one of the eight orthogonal locations adjacent to c (see Figure 9). Moreover, faces a_1 and b_1 must lie in locations 2, 3, or 4.[1]

Claw arrangements up to symmetry. Observe that the claws come in symmetric pairs (colored fushia/pink and eggshell/green in Fig. 8) and each pair is also symmetric. Without loss of generality, assume that the claws appear in the relative order around c seen in Figure 8 (i.e. in clockwise order, fushia, pink, light green, eggshell).

[1]Recall that we allow vertex-adjacent faces to appear in a different clockwise ordering around the vertex than they appear on the surface; the proof holds even when such unfolding are permitted.

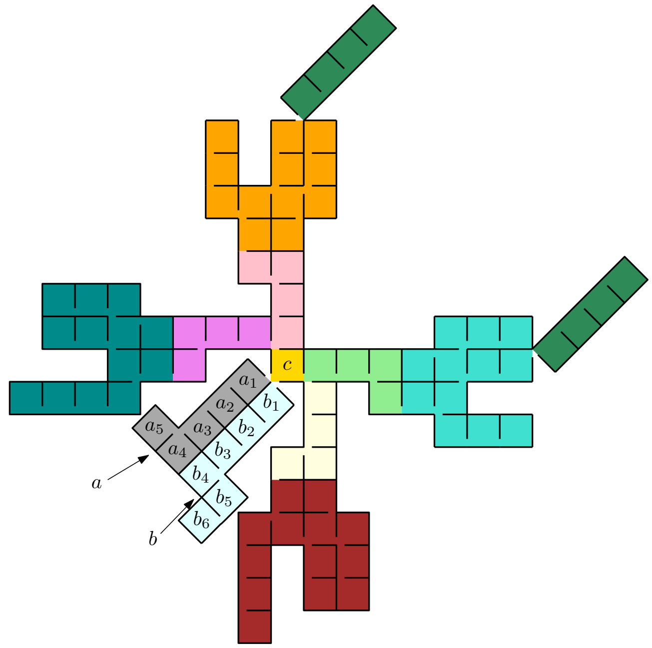

Figure 8: A maximally cut vertex unfolding of the polycube in Figure 7 that cannot be arranged orthogonally. The solid black lines represent cuts.

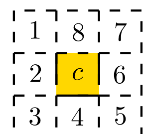

Figure 9: The eight possible locations for faces adjacent to c in orthogonal unfolding.

Then the fushia and pink claws must have faces in locations 1 and 8, respectively, since otherwise overlap occurs between a_1, b_1, c and two fushia faces sharing a common vertex (see Figure 10). By symmetry, the eggshell and light green claws must lie in locations 5 and 6.

Next, consider the arrangement of the fushia claw faces. Figure 11 enumerates the five possible arrangements. The remainder of the proof is dedicated to proving that each arrangement leads to overlap.

Arrangements 4 and 5. Both arrangements cause overlap due to more than four faces sharing a common vertex location. For arrangement 4, these faces consist of two fushia faces, c, a_1, and b_1. For arrangement 5, these faces consist of four fushia faces, one pink face, and c.

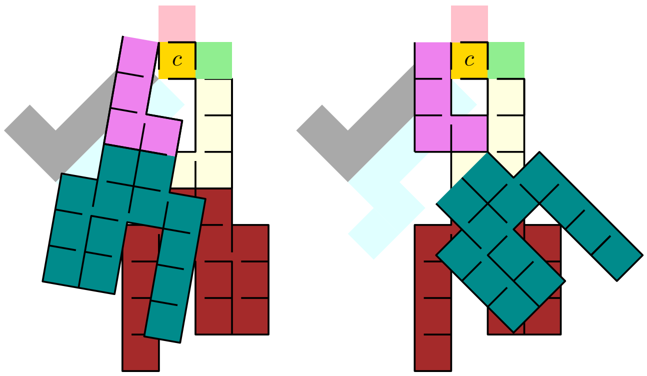

Figure 10: Attemping to place the first face of the fushia claw into location 2 causes overlap.

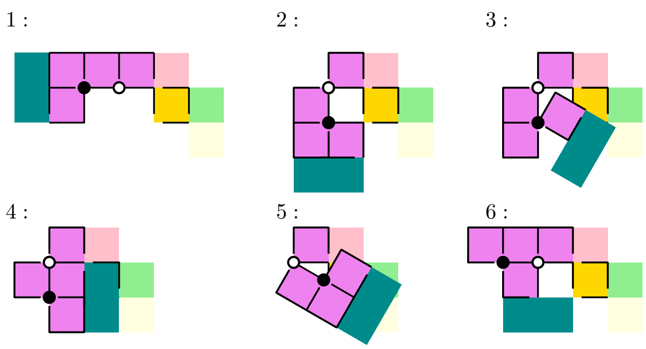

Figure 11: The five possible arrangements of the fushia claw faces.

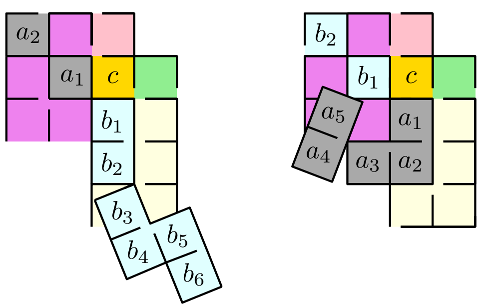

Figure 12: Two possible configurations if crossings are allowed.

Arrangement 3. The faces of A or B may be arranged so that the first face (a_1 or b_1) lies in location 2 as seen in Figure 12. As shown the figure, either option causes overlap.

Arrangements 1 and 2. Here we consider placing A and B. Either a_1 or b_1 must be placed in location 2 or 4. By symmetry (and ignoring the existence of b_6), it suffices to consider two cases:

1. a_1 is placed in location 2.
2. b_1 is placed in location 2.

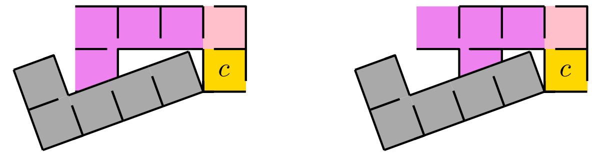

Figure 13: Attempting to unfold A and B by placing a_1 in location 2.

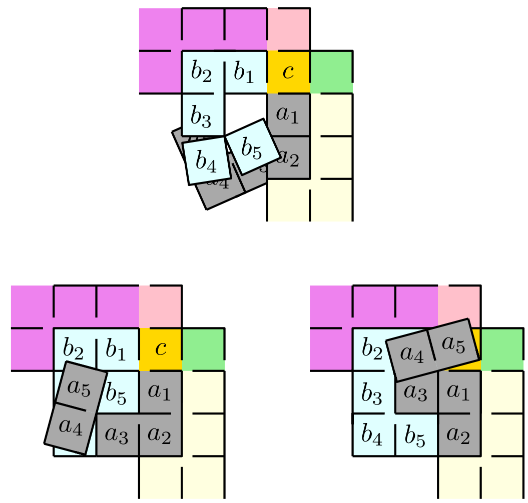

Figure 14: Top: required arrangement of faces b_1, b_2, b_3, a_1, a_2. Bottom: the two options for arranging b_4, b_5, and a_3 that further avoid overlap (in both cases, overlap involving a_4 and a_5 still occurs).

We consider the cases in order. In the case that a_1 is in location 2, A overlaps with the fushia claw, since due to the cuts, a_2 is left of a_1 or overlaps the fushia claw, and likewise a_3 is either left or above a_2.

Next, consider the case that b_1 is placed in location 2. In this case, there are unique placements of faces b_2, b_3, a_1, and a_2 that avoid overlap (the top portion of Figure 14) and only two placements of b_4, b_5 and a_3 that also avoid overlap (the bottom portion of Figure 14). In both cases, a_4 and a_5 overlap with other faces due to a vertex incident to more than four faces. □

6 Open Problems

Each of our results leads directly to a natural open problem in the same direction. Since all genus-0 and genus-1 orthogonal polyhedra have grid vertex unfoldings, what about genus-2?

Open Problem 1 *Does every genus-2 orthogonal polyhedron have a grid vertex unfolding?*

Since there is an orthogonal polyhedron with simple faces and no vertex unfolding, does the same hold for simple faces that are also edge-incident rectangles?

Open Problem 2 *Does every orthogonal polyhedron (of any genus) have a grid vertex unfolding?*

Finally, we provided an example of a maximally cut polycube with an unfolding, but no orthogonal unfolding,[2] demonstrating that the orthogonal unfoldings of maximally cut grided orthogonal polyhedra do not characterize the (unrestricted) unfoldings of maximally cut gridded orthogonal polyhedra (eliminating one particularly simple proof that deciding whether an orthogonal polyhedron has a vertex unfolding is in NP). Thus the following two related problems remain open:

Open Problem 3 *Is deciding if an orthgonal polyhedron is grid vertex-unfoldable in* NP*? Is the problem* NP*-hard?*

Additionally, the relationship between orthogonal grid vertex-unfoldings and (unrestricted) grid vertex-unfoldings also remain open:

Open Problem 4 *Does there exist a grid vertex-unfoldable orthogonal polyheron with no orthogonal grid vertex-unfolding?*

To our knowledge, the same question also remains open for non-grid vertex-unfoldings:

Open Problem 5 *Does there exist a vertex-unfoldable orthogonal polyhedron no orthogonal vertex-unfolding?*

Acknowledgments

We would like to thank Erik Demaine for discussion of known results related to vertex unfoldings, and anonymous reviewers whose suggestions improved the readability and correctness of the paper.

References

[1] Z. Abel and E. D. Demaine. Edge-unfolding orthogonal polyhedra is strongly NP-complete. In *Proceedings of the 23rd Canadian Conference on Computational Geometry (CCCG)*, 2011.

[2] Z. Abel, E. D. Demaine, and M. L. Demaine. A topologically convex vertex-ununfoldable polyhedron. In *Proceedings of the 23rd Canadian Conference on Computational Geometry (CCCG)*, 2011.

[3] T. Biedl, E. Demaine, M. Demaine, A. Lubiw, M. Overmars, J. O'Rourke, S. Robbins, and S. Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proceedings of the 10th Canadian Conference on Computational Geometry (CCCG)*, pages 70–71, 1998.

[2] Note that the polycube from which the example is obtained is indeed vertex (and also edge) unfoldable, and so is not a potential counterexample to Open Problem 2.

[4] O. Bodini and S. Lefranc. How to tile by dominoes the boundary of a polycube. In *Proceedings of the 13th International Conference on Discrete Geometry for Computer Imagery (DGCI)*, volume 4245 of *LNCS*, pages 630–638. Springer, 2006.

[5] E. W. Chambers, K. A. Sykes, and C. M. Traub. Unfolding rectangle-faced orthostacks. In *Proceedings of the 24th Canadian Conference on Computational Geometry (CCCG)*, pages 23–28, 2012.

[6] Y.-J. Chang and H.-C. Yen. Improved algorithms for grid-unfolding orthogonal polyhedra. *International Journal of Computational Geometry & Applications*, 27(1):33–56, 2017.

[7] M. Damian, E. D. Demaine, and R. Flatland. Unfolding orthogonal polyhedra with quadratic refinement: The delta-unfolding algorithm. *Graphs and Combinatorics*, 30(1):25–40, 2014.

[8] M. Damian, R. Flatland, and J. O'Rourke. Epsilon-unfolding orthogonal polyhedra. *Graphs and Combinatorics*, 23:179–194, 2007.

[9] M. Damian, R. Flatland, and J. O'Rourke. Grid vertex-unfolding orthogonal polyhedra. *Discrete & Computational Geometry*, 39(1–3), 2008.

[10] M. Damian and H. Meijer. Grid edge-unfolding orthostacks with orthogonally convex slabs. In *Proceedings of the 14th Annual Fall Workshop on Computational Geometry*, pages 20–21, 2004.

[11] E. D. Demaine, D. Eppstein, J. Erickson, G.W. Hart, and J. O'Rourke. Vertex-unfoldings of simplicial manifolds. In *Proceedings of the 18th Symposium on Computational Geometry (SoCG)*, pages 237–243. ACM Press, 2002.

[12] E. D. Demaine, J. Iacono, and S. Langerman. Grid vertex-unfolding orthostacks. *International Journal of Computational Geometry & Applications*, 20(3):245–254, 2010.

[13] J. O'Rourke. Unfolding orthogonal terrains. https://arxiv.org/abs/0707.0610, 2007.

[14] R. Thomas, X. Yu, and W. Zang. Hamilton paths in toroidal graphs. *Journal of Combinatorial Theory, Series B*, 94:214–236, 2005.

[15] W. T. Tutte. A theorem on planar graphs. *Transactions of the American Mathematical Society*, 82:99–116, 1956.