INTERVAL–VALUED FUZZY SUBALGEBRAS/IDEALS IN BCK–ALGEBRAS

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ABSTRACT. We define the notion of an interval-valued fuzzy subalgebra/$\circ$-subalgebra/ideal (briefly, an i-v fuzzy subalgebra/$\circ$-subalgebra/ideal) of a BCK-algebra. We study how the homomorphic images and inverse images of i-v fuzzy subalgebras become i-v fuzzy subalgebras. We give relations between i-v fuzzy subalgebras/$\circ$-subalgebras and i-v fuzzy ideals. We give a condition for an i-v fuzzy set in a BCK-algebra with condition (S) to be an i-v fuzzy ideal. We also state characterizations of an i-v fuzzy subalgebra/ideal.

1. Introduction

Fuzzy sets were initiated by Zadeh [8]. In [9], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. In [9], Zadeh also constructed a method of approximate inference using his i-v fuzzy sets. In [1], Biswas defined interval-valued fuzzy subgroups (i.e., i-v fuzzy subgroups) of Rosenfeld's nature, and investigated some elementary properties. In this paper we define the notion of an interval-valued fuzzy subalgebra/$\circ$-subalgebra/ideal (briefly, an i-v fuzzy subalgebra/$\circ$-subalgebra/ideal) of a BCK-algebra. We study how the homomorphic images and inverse images of i-v fuzzy subalgebras become i-v fuzzy subalgebras. We give relations between i-v fuzzy subalgebras/$\circ$-subalgebras and i-v fuzzy ideals. We give a condition for an i-v fuzzy set in a BCK-algebra with condition (S) to be an i-v fuzzy ideal. We also state characterizations of an i-v fuzzy subalgebra/ideal.

2. Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

An algebra $(\mathcal{K};*,0)$ of type $(2,0)$ is said to be a *BCK-algebra* if it satisfies the following conditions:

(I) $((x*y)*(x*z))*(z*y)=0$,
(II) $(x*(x*y))*y=0$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$

for all $x,y,z\in\mathcal{K}$. Define a binary relation $\leq$ on $\mathcal{K}$ by letting $x\leq y$ if and only if $x*y=0$. Then $(\mathcal{K};\leq)$ is a partially ordered set with the least element 0. A BCK-algebra $\mathcal{K}$ is said to be with *condition* (S) if, for all $x,y\in\mathcal{K}$, the set $\{z\in\mathcal{K}\mid z*x\leq y\}$ has a greatest

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element, written $x \circ y$. A mapping $f : \mathcal{K} \to \mathcal{K}'$ of BCK-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in \mathcal{K}$. A non-empty subset $\mathcal{M}$ of a BCK-algebra $\mathcal{K}$ is called a *subalgebra* of $\mathcal{K}$ if $x * y \in \mathcal{M}$ for all $x, y \in \mathcal{M}$. A non-empty subset $\mathcal{M}$ of a BCK-algebra $\mathcal{K}$ is called an *ideal* of $\mathcal{K}$ if

(I1) $0 \in \mathcal{M}$,

(I2) $x * y \in \mathcal{M}$ and $y \in \mathcal{M}$ imply $x \in \mathcal{M}$

for all $x, y \in \mathcal{K}$.

We now review some fuzzy logic concepts. Let $\mathcal{K}$ be a set. A *fuzzy set* in $\mathcal{K}$ is a function $\mu : \mathcal{K} \to [0, 1]$. Let f be a mapping from a set $\mathcal{K}$ into a set $\mathcal{K}'$. Let ν be a fuzzy set in $\mathcal{K}'$. Then the *inverse image* of ν, denoted by $f^{-1}[\nu]$, is the fuzzy set in $\mathcal{K}$ defined by $f^{-1}[\nu](x) = \nu(f(x))$ for all $x \in \mathcal{K}$. Conversely, let μ be a fuzzy set in $\mathcal{K}$. The *image* of μ, written as $f[\mu]$, is a fuzzy set in $\mathcal{K}'$ defined by

$$f[\mu](y) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in \mathcal{K}'$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

A fuzzy set μ in a BCK-algebra $\mathcal{K}$ is called a *fuzzy subalgebra* of $\mathcal{K}$ if

$$\mu(x * y) \geq \min\{\mu(x * y), \mu(y)\}$$

for all $x, y \in \mathcal{K}$. A fuzzy set μ in a BCK-algebra $\mathcal{K}$ is called a *fuzzy ideal* of $\mathcal{K}$ if

(FI1) $\mu(0) \geq \mu(x)$,

(FI2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

for all $x, y \in \mathcal{K}$. Note that a fuzzy set μ in a BCK-algebra $\mathcal{K}$ is a fuzzy ideal of $\mathcal{K}$ if and only if the non-empty level set $U(\mu; \alpha) := \{x \in \mathcal{K} \mid \mu(x) \geq \alpha\}$ is an ideal of $\mathcal{K}$ for every $\alpha \in [0, 1]$.

An *interval-valued fuzzy set* (briefly, *i-v fuzzy set*) A defined on $\mathcal{K}$ is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\}, \forall x \in \mathcal{K} \ \text{(briefly, denoted by } A = [\mu_A^L, \mu_A^U]),$$

where μ_A^L and μ_A^U are two fuzzy sets in $\mathcal{K}$ such that $\mu_A^L(x) \leq \mu_A^U(x)$ for all $x \in \mathcal{K}$.

Let $\bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)], \forall x \in \mathcal{K}$ and let $D[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$. If $\mu_A^L(x) = \mu_A^U(x) = c$ (say) where $0 \leq c \leq 1$, then we have $\bar{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0, 1]$. Thus $\bar{\mu}_A(x) \in D[0, 1], \forall x \in \mathcal{K}$, and therefore the i-v fuzzy set A is given by

$$A = \{(x, \bar{\mu}_A(x))\}, \forall x \in \mathcal{K}, \ \text{where } \bar{\mu}_A : \mathcal{K} \to D[0, 1].$$

Now let us define what is known as *refined minimum* (briefly, rmin) of two elements in $D[0, 1]$. We also define the symbols "$\geq$", "$\leq$", "$=$" in case of two elements in $D[0, 1]$. Consider two elements $D_1 := [a_1, b_1]$, $D_2 := [a_2, b_2] \in D[0, 1]$. Then

$$\begin{aligned} &\text{rmin}(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]; \\ &D_1 \geq D_2 \text{ if and only if } a_1 \geq a_2 \text{ and } b_1 \geq b_2; \end{aligned}$$

and similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

3. Interval-valued fuzzy subalgebras

In what follows, let $\mathcal{K}$ denote a BCK-algebra unless otherwise specified. We begin with the following two propositions.

Proposition 3.1. *Let f be a homomorphism from a BCK-algebra $\mathcal{K}$ into a BCK-algebra $\mathcal{K}'$. If ν is a fuzzy subalgebra of $\mathcal{K}'$, then the inverse image $f^{-1}[\nu]$ of ν is a fuzzy subalgebra of $\mathcal{K}$.*

Proof. For any $x, y \in \mathcal{K}$, we have

$$\begin{aligned} f^{-1}[\nu](x*y) &= \nu(f(x*y)) = \nu(f(x)*f(y)) \\ &\geq \min\{\nu(f(x)), \nu(f(y))\} \\ &= \min\{f^{-1}[\nu](x), f^{-1}[\nu](y)\}. \end{aligned}$$

Hence $f^{-1}[\nu]$ is a fuzzy subalgebra of $\mathcal{K}$. □

Proposition 3.2. *Let $f : \mathcal{K} \to \mathcal{K}'$ be a homomorphism between BCK-algebras $\mathcal{K}$ and $\mathcal{K}'$. For every fuzzy subalgebra μ of $\mathcal{K}$, the image $f[\mu]$ of μ is a fuzzy subalgebra of $\mathcal{K}'$.*

Proof. We first prove that

$$(*1) \qquad f^{-1}(y_1) * f^{-1}(y_2) \subseteq f^{-1}(y_1 * y_2)$$

for all $y_1, y_2 \in \mathcal{K}'$. For, if $x \in f^{-1}(y_1) * f^{-1}(y_2)$, then $x = x_1 * x_2$ for some $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Since f is a homomorphism, it follows that $f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$ so that $x \in f^{-1}(y_1 * y_2)$. Hence $(*1)$ holds. Now let $y_1, y_2 \in \mathcal{K}'$ be arbitrarily given. Assume that $y_1 * y_2 \notin \mathrm{Im}(f)$. Then $f[\mu](y_1 * y_2) = 0$. But if $y_1 * y_2 \notin \mathrm{Im}(f)$, i.e., $f^{-1}(y_1 * y_2) = \emptyset$, then $f^{-1}(y_1) = \emptyset$ or $f^{-1}(y_2) = \emptyset$ by $(*1)$. Thus $f[\mu](y_1) = 0$ or $f[\mu](y_2) = 0$, and so

$$f[\mu](y_1 * y_2) = 0 = \min\{f[\mu](y_1), f[\mu](y_2)\}.$$

Suppose that $f^{-1}(y_1 * y_2) \neq \emptyset$. Then we should consider the two cases:

$$\text{(i)} \quad f^{-1}(y_1) = \emptyset \ \text{ or } \ f^{-1}(y_2) = \emptyset,$$
$$\text{(ii)} \quad f^{-1}(y_1) \neq \emptyset \ \text{ and } \ f^{-1}(y_2) \neq \emptyset.$$

For the case (i), we have $f[\mu](y_1) = 0$ or $f[\mu](y_2) = 0$, and so

$$f[\mu](y_1 * y_2) \geq 0 = \min\{f[\mu](y_1), f[\mu](y_2)\}.$$

Case (ii) implies from $(*1)$ that

$$\begin{aligned} f[\mu](y_1 * y_2) &= \sup_{z \in f^{-1}(y_1 * y_2)} \mu(z) \geq \sup_{z \in f^{-1}(y_1) * f^{-1}(y_2)} \mu(z) \\ &= \sup_{x_1 \in f^{-1}(y_1),\, x_2 \in f^{-1}(y_2)} \mu(x_1 * x_2). \end{aligned}$$

Since μ is a fuzzy subalgebra of $\mathcal{K}$, it follows from the definition of a fuzzy subalgebra that

$$\begin{aligned} f[\mu](y_1 * y_2) &\geq \sup_{x_1 \in f^{-1}(y_1),\, x_2 \in f^{-1}(y_2)} \min\{\mu(x_1), \mu(x_2)\} \\ &= \sup_{x_1 \in f^{-1}(y_1)} (\min\{\sup_{x_2 \in f^{-1}(y_2)} \mu(x_1), \mu(x_2)\}) \\ &= \sup_{x_1 \in f^{-1}(y_1)} (\min\{\mu(x_1), \sup_{x_2 \in f^{-1}(y_2)} \mu(x_2)\}) \\ &= \sup_{x_1 \in f^{-1}(y_1)} (\min\{\mu(x_1), f[\mu](y_2)\}) \\ &= \min\{\sup_{x_1 \in f^{-1}(y_1)} \mu(x_1), f[\mu](y_2)\} \\ &= \min\{f[\mu](y_1), f[\mu](y_2)\}. \end{aligned}$$

Hence $f[\mu](y_1 * y_2) \geq \min\{f[\mu](y_1), f[\mu](y_2)\}$ for all $y_1, y_2 \in \mathcal{K}'$. This completes the proof. □

Definition 3.3. An i-v fuzzy set A in $\mathcal{K}$ is called an *interval-valued fuzzy subalgebra* (briefly, *i-v fuzzy subalgebra*) of $\mathcal{K}$ if

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$$

for all $x, y \in \mathcal{K}$.

Example 3.4. Let $\mathcal{K} = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let an i-v fuzzy set A defined on $\mathcal{K}$ be given by

$$\bar{\mu}_A(x) = \begin{cases} [0.3, 0.8] & \text{if } x \in \{0, b\}, \\ [0.1, 0.5] & \text{otherwise.} \end{cases}$$

It is easy to check that A is an i-v fuzzy subalgebra of $\mathcal{K}$.

Lemma 3.5. *If A is an i-v fuzzy subalgebra of $\mathcal{K}$, then $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$ for all $x \in \mathcal{K}$.*

Proof. For every $x \in \mathcal{K}$, we have

$$\begin{aligned} \bar{\mu}_A(0) &= \bar{\mu}_A(x * x) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(x)\} \\ &= \text{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(x), \mu_A^U(x)]\} \\ &= [\mu_A^L(x), \mu_A^U(x)] = \bar{\mu}_A(x), \end{aligned}$$

completing the proof. □

Theorem 3.6. *Let A be an i-v fuzzy subalgebra of $\mathcal{K}$. If there is a sequence $\{x_n\}$ in $\mathcal{K}$ such that*

$$\lim_{n \to \infty} \bar{\mu}_A(x_n) = [1, 1],$$

then $\bar{\mu}_A(0) = [1, 1]$.

Proof. Since $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$ for all $x \in \mathcal{K}$, we have $\bar{\mu}_A(0) \geq \bar{\mu}_A(x_n)$ for every positive integer n. Note that

$$[1, 1] \geq \bar{\mu}_A(0) \geq \lim_{n \to \infty} \bar{\mu}_A(x_n) = [1, 1].$$

Hence $\bar{\mu}_A(0) = [1, 1]$. □

Theorem 3.7. *An i-v fuzzy set $A = [\mu_A^L, \mu_A^U]$ in $\mathcal{K}$ is an i-v fuzzy subalgerba of $\mathcal{K}$ if and only if μ_A^L and μ_A^U are fuzzy subalgebras of $\mathcal{K}$.*

Proof. Suppose that μ_A^L and μ_A^U are fuzzy subalgebras of $\mathcal{K}$. Let $x, y \in \mathcal{K}$. Then

$$\begin{aligned} \bar{\mu}_A(x * y) &= [\mu_A^L(x * y), \mu_A^U(x * y)] \\ &\geq [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}] \\ &= \text{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \end{aligned}$$

Hence A is an i-v fuzzy subalgebra of $\mathcal{K}$.

Conversely, assume that A is an i-v fuzzy subalgerba of $\mathcal{K}$. For any $x, y \in \mathcal{K}$, we have

$$\begin{aligned}[\mu_A^L(x*y), \mu_A^U(x*y)] &= \bar{\mu}_A(x*y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} \\ &= \text{rmin}\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= [\min\{\mu_A^L(x), \mu_A^L(y)\}, \min\{\mu_A^U(x), \mu_A^U(y)\}].\end{aligned}$$

It follows that $\mu_A^L(x*y) \geq \min\{\mu_A^L(x), \mu_A^L(y)\}$ and $\mu_A^U(x*y) \geq \min\{\mu_A^U(x), \mu_A^U(y)\}$. Hence μ_A^L and μ_A^U are fuzzy subalgebras of $\mathcal{K}$. □

Theorem 3.8. *Let A be an i-v fuzzy set in $\mathcal{K}$. Then A is an i-v fuzzy subalgebra of $\mathcal{K}$ if and only if the non-empty set*

$$\bar{U}(A; [\delta_1, \delta_2]) := \{x \in \mathcal{K} \mid \bar{\mu}_A(x) \geq [\delta_1, \delta_2]\}$$

is a subalgebra of $\mathcal{K}$ for every $[\delta_1, \delta_2] \in D[0,1]$.

We then call $\bar{U}(A; [\delta_1, \delta_2])$ the *i-v level subalgebra* of A.

Proof. Assume that A is an i-v fuzzy subalgebra of $\mathcal{K}$ and let $[\delta_1, \delta_2] \in D[0,1]$ be such that $x, y \in \bar{U}(A; [\delta_1, \delta_2])$. Then

$$\bar{\mu}_A(x*y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} \geq \text{rmin}\{[\delta_1, \delta_2], [\delta_1, \delta_2]\} = [\delta_1, \delta_2],$$

and so $x*y \in \bar{U}(A; [\delta_1, \delta_2])$. Thus $\bar{U}(A; [\delta_1, \delta_2])$ is a subalgebra of $\mathcal{K}$.

Conversely, assume that $\bar{U}(A; [\delta_1, \delta_2])$ $(\neq \emptyset)$ is a subalgebra of $\mathcal{K}$ for every $[\delta_1, \delta_2] \in D[0,1]$. Suppose there exist $x_0, y_0 \in \mathcal{K}$ such that

$$\bar{\mu}_A(x_0 * y_0) < \text{rmin}\{\bar{\mu}_A(x_0), \bar{\mu}_A(y_0)\}.$$

Let $\bar{\mu}_A(x_0) = [\gamma_1, \gamma_2]$, $\bar{\mu}_A(y_0) = [\gamma_3, \gamma_4]$, and $\bar{\mu}_A(x_0 * y_0) = [\delta_1, \delta_2]$. Then

$$[\delta_1, \delta_2] < \text{rmin}\{[\gamma_1, \gamma_2], [\gamma_3, \gamma_4]\} = [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}].$$

Hence $\delta_1 < \min\{\gamma_1, \gamma_3\}$ and $\delta_2 < \min\{\gamma_2, \gamma_4\}$. If we take

$$[\lambda_1, \lambda_2] = \frac{1}{2}(\bar{\mu}_A(x_0 * y_0) + \text{rmin}\{\bar{\mu}_A(x_0), \bar{\mu}_A(y_0)\}),$$

then

$$\begin{aligned}[\lambda_1, \lambda_2] &= \frac{1}{2}([\delta_1, \delta_2] + [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}]) \\ &= [\frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}), \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\})].\end{aligned}$$

It follows that

$$\min\{\gamma_1, \gamma_3\} > \lambda_1 = \frac{1}{2}(\delta_1 + \min\{\gamma_1, \gamma_3\}) > \delta_1$$

and

$$\min\{\gamma_2, \gamma_4\} > \lambda_2 = \frac{1}{2}(\delta_2 + \min\{\gamma_2, \gamma_4\}) > \delta_2$$

so that $[\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \bar{\mu}_A(x_0 * y_0)$. Therefore $x_0 * y_0 \notin \bar{U}(A; [\lambda_1, \lambda_2])$. On the other hand,

$$\bar{\mu}_A(x_0) = [\gamma_1, \gamma_2] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2]$$

and

$$\bar{\mu}_A(y_0) = [\gamma_3, \gamma_4] \geq [\min\{\gamma_1, \gamma_3\}, \min\{\gamma_2, \gamma_4\}] > [\lambda_1, \lambda_2],$$

and so $x_0, y_0 \in \bar{U}(A; [\lambda_1, \lambda_2])$. It contradicts that $\bar{U}(A; [\lambda_1, \lambda_2])$ is a subalgebra of $\mathcal{K}$. Hence $\bar{\mu}_A(x*y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$ for all $x, y \in \mathcal{K}$. This completes the proof. □

Theorem 3.9. *Every subalgebra of $\mathcal{K}$ can be realized as an i-v level subalgerba of an i-v fuzzy subalgebra of $\mathcal{K}$.*

Proof. Let $\mathcal{M}$ be a subalgebra of $\mathcal{K}$ and let A be an i-v fuzzy set on $\mathcal{K}$ defined by

$$\bar{\mu}_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in \mathcal{M}, \\ [0,0] & \text{otherwise,} \end{cases}$$

where $\alpha_1, \alpha_2 \in (0,1]$ with $\alpha_1 < \alpha_2$. It is clear that $\bar{U}(A; [\alpha_1, \alpha_2]) = \mathcal{M}$. We will show that A is an i-v fuzzy subalgebra of $\mathcal{K}$. Let $x, y \in \mathcal{K}$. If $x, y \in \mathcal{M}$, then $x * y \in \mathcal{M}$ and so

$$\bar{\mu}_A(x * y) = [\alpha_1, \alpha_2] = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

If $x, y \notin \mathcal{M}$, then $\bar{\mu}_A(x) = [0,0] = \bar{\mu}_A(y)$ and thus

$$\bar{\mu}_A(x * y) \geq [0,0] = \text{rmin}\{[0,0], [0,0]\} = \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

If $x \in \mathcal{M}$ and $y \notin \mathcal{M}$, then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2]$ and $\bar{\mu}_A(y) = [0,0]$. It follows that

$$\bar{\mu}_A(x * y) \geq [0,0] = \text{rmin}\{[\alpha_1, \alpha_2], [0,0]\} = \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

Similarly for the case $x \notin \mathcal{M}$ and $y \in \mathcal{M}$, we get $\bar{\mu}_A(x*y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$. Therefore A is an i-v fuzzy subalgebra of $\mathcal{K}$, and the proof is complete. □

Theorem 3.10. *Let $\mathcal{M}$ be a subset of $\mathcal{K}$ and let A be an i-v fuzzy set on $\mathcal{K}$ which is given in the proof of Theorem 3.9. If A is an i-v fuzzy subalgebra of $\mathcal{K}$, then $\mathcal{M}$ is a subalgebra of $\mathcal{K}$.*

Proof. Assume that A is an i-v fuzzy subalgebra of $\mathcal{K}$. Let $x, y \in \mathcal{M}$. Then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2] = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2].$$

This implies that $x * y \in \mathcal{M}$. Hence $\mathcal{M}$ is a subalgebra of $\mathcal{K}$. □

Theorem 3.11. *If A is an i-v fuzzy subalgebra of $\mathcal{K}$, then the set*

$$\mathcal{K}_{\bar{\mu}_A} := \{x \in \mathcal{K} \mid \bar{\mu}_A(x) = \bar{\mu}_A(0)\}$$

is a subalgebra of $\mathcal{K}$.

Proof. Let $x, y \in \mathcal{K}_{\bar{\mu}_A}$. Then $\bar{\mu}_A(x) = \bar{\mu}_A(0) = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = \text{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(0)\} = \bar{\mu}_A(0).$$

Combining this and Lemma 3.5, we get $\bar{\mu}_A(x * y) = \bar{\mu}_A(0)$, i.e., $x * y \in \mathcal{K}_{\bar{\mu}_A}$. Hence $\mathcal{K}_{\bar{\mu}_A}$ is a subalgebra of $\mathcal{K}$. □

Definition 3.12 (Biswas [1])**.** Let f be a mapping from a set $\mathcal{K}$ into a set $\mathcal{K}'$. Let B be an i-v fuzzy set in $\mathcal{K}'$. Then the *inverse image* of B, denoted by $f^{-1}[B]$, is the i-v fuzzy set in $\mathcal{K}$ with the membership function given by $\bar{\mu}_{f^{-1}[B]}(x) = \bar{\mu}_B(f(x))$ for all $x \in \mathcal{K}$.

Lemma 3.13 (Biswas [1]). *Let f be a mapping from a set $\mathcal{K}$ into a set $\mathcal{K}'$. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be i-v fuzzy sets in $\mathcal{K}$ and $\mathcal{K}'$, respectively. Then*
(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$,
(ii) $f(m) = [f(m^L), f(m^U)]$.

Theorem 3.14. *Let f be a homomorphism from a BCK-algebra $\mathcal{K}$ into a BCK-algebra $\mathcal{K}'$. If B is an i-v fuzzy subalgebra of $\mathcal{K}'$, then the inverse image $f^{-1}[B]$ of B is an i-v fuzzy subalgebra of $\mathcal{K}$.*

Proof. Since $B = [\mu_B^L, \mu_B^U]$ is an i-v fuzzy subalgebra of $\mathcal{K}'$, it follows from Theorem 3.7 that μ_B^L and μ_B^U are fuzzy subalgebras of $\mathcal{K}'$. Using Proposition 3.1, we know that $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy subalgebras of $\mathcal{K}$, Hence, by Lemma 3.13 and Theorem 3.7, we conclude that $f^{-1}[B] = [f^{-1}[\mu_B^L], f^{-1}[\mu_B^U]]$ is an i-v fuzzy subalgebra of $\mathcal{K}$. □

Definition 3.15 (Biswas [1]). Let f be a mapping from a set $\mathcal{K}$ into a set $\mathcal{K}'$. Let A be an i-v fuzzy set in $\mathcal{K}$. Then the *image* of A, denoted by $f[A]$, is the i-v fuzzy set in $\mathcal{K}'$ with the membership function defined by

$$\bar{\mu}_{f[A]}(y) = \begin{cases} \text{rsup}_{z \in f^{-1}(y)} \bar{\mu}_A(z), & \text{if } f^{-1}(y) \neq \emptyset, \forall\, y \in \mathcal{K}', \\ [0,0], & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Theorem 3.16. *Let f be a homomorphism from a BCK-algebra $\mathcal{K}$ into a BCK-algebra $\mathcal{K}'$. If A is an i-v fuzzy subalgebra of $\mathcal{K}$, then the image $f[A]$ of A is an i-v fuzzy subalgebra of $\mathcal{K}'$.*

Proof. Assume that A is an i-v fuzzy subalgebra of $\mathcal{K}$. Note that $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy subalgebra of $\mathcal{K}$ if and only if μ_A^L and μ_A^U are fuzzy subalgebras of $\mathcal{K}$. It follows from Proposition 3.2 that the images $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy subalgebras of $\mathcal{K}'$. Combining Theorem 3.7 and Lemma 3.13, we conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy subalgebra of $\mathcal{K}'$. □

4. Interval-valued fuzzy ideals

Definition 4.1. An i-v fuzzy set A in $\mathcal{K}$ is called an *interval-valued fuzzy ideal* (briefly, *i-v fuzzy ideal*) of $\mathcal{K}$ if
(IV1) $\bar{\mu}_A(0) \geq \bar{\mu}_A(x)$,
(IV2) $\bar{\mu}_A(x) \geq \text{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\}$
for all $x, y \in \mathcal{K}$.

Example 4.2. Let $\mathcal{K} = \{0, a, b, c\}$ be a BCK-algebra in Example 3.4. Let A be an i-v fuzzy set in $\mathcal{K}$ defined by $\bar{\mu}_A(0) = [0.5, 0.8]$, $\bar{\mu}_A(a) = \bar{\mu}_A(b) = [0.3, 0.6]$ and $\bar{\mu}_A(c) = [0.2, 0.3]$. Routine calculations give that A is an i-v fuzzy ideal of $\mathcal{K}$.

Proposition 4.3. *Let A be an i-v fuzzy ideal of $\mathcal{K}$. If the inequality $x * y \leq z$ holds in $\mathcal{K}$, then $\bar{\mu}_A(x) \geq \text{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}$.*

Proof. Assume that the inequality $x * y \leq z$ holds in $\mathcal{K}$. Then

$$\bar{\mu}_A(x * y) \geq \text{rmin}\{\bar{\mu}_A((x * y) * z), \bar{\mu}_A(z)\} = \text{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(z)\} = \bar{\mu}_A(z),$$

which implies from (IV2) that $\bar{\mu}_A(x) \geq \text{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\} \geq \text{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}$. The proof is complete. □

Lemma 4.4. *For any i-v fuzzy ideal A of $\mathcal{K}$, if $x \le y$ in $\mathcal{K}$ then $\bar{\mu}_A(x) \ge \bar{\mu}_A(y)$.*

Proof. Assume that $x \le y$ in $\mathcal{K}$. Then

$$\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\} = \mathrm{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(y)\} = \bar{\mu}_A(y).$$

This completes the proof. □

Theorem 4.5. *Any i-v fuzzy ideal of $\mathcal{K}$ is an i-v fuzzy subalgebra of $\mathcal{K}$.*

Proof. Let A be an i-v fuzzy ideal of $\mathcal{K}$. Since $x * y \le x$ for all $x, y \in \mathcal{K}$, it follows from Lemma 4.4 that $\bar{\mu}_A(x) \le \bar{\mu}_A(x * y)$. Hence by (IV2), we have

$$\bar{\mu}_A(x * y) \ge \bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\} \ge \mathrm{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}.$$

Therefore A is an i-v fuzzy subalgebra of $\mathcal{K}$. □

The converse of Theorem 4.5 need not be true in general.

Example 4.6. The i-v fuzzy subalgebra A in Example 3.4 is not an i-v fuzzy ideal, since $\bar{\mu}_A(a) < \mathrm{rmin}\{\bar{\mu}_A(a * b), \bar{\mu}_A(b)\}$.

We give a condition for an i-v fuzzy subalgebra to be an i-v fuzzy ideal.

Theorem 4.7. *An i-v fuzzy subalgebra A of $\mathcal{K}$ is an i-v fuzzy ideal of $\mathcal{K}$ if and only if for all $x, y \in \mathcal{K}$, the ineguality $x * y \le z$ implies $\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}$.*

Proof. ($\Rightarrow$) It follows from Proposition 4.3.

($\Leftarrow$) Suppose that A is an i-v fuzzy subalgebra of $\mathcal{K}$ satisfying that if $x * y \le z$ in $\mathcal{K}$ then $\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}$. Since $x * (x * y) \le y$, it follows from the hypothesis that

$$\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\}.$$

Hence A is an i-v fuzzy ideal of $\mathcal{K}$.

Definition 4.8. Let $\mathcal{K}$ be a BCK-algebra with condition (S). An i-v fuzzy set A in $\mathcal{K}$ is called an *interval-valued fuzzy $\circ$-subalgebra* (briefly, *i-v fuzzy $\circ$-subalgebra*) of $\mathcal{K}$ if $\bar{\mu}_A(x \circ y) \ge \mathrm{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$ for all $x, y \in \mathcal{K}$.

Theorem 4.9. *Let $\mathcal{K}$ be a BCK-algebra with condition (S). Then every i-v fuzzy ideal of $\mathcal{K}$ is an i-v fuzzy $\circ$-subalgebra of $\mathcal{K}$.*

Proof. Let A be an i-v fuzzy ideal of $\mathcal{K}$. Since $(x \circ y) * x \le y$ for all $x, y \in \mathcal{K}$, it follows from Lemma 4.4 and (IV2) that

$$\bar{\mu}_A(x \circ y) \ge \mathrm{rmin}\{\bar{\mu}_A((x \circ y) * x), \bar{\mu}_A(x)\} \ge \mathrm{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(x)\}.$$

Hence A is an i-v fuzzy $\circ$-subalgebra of $\mathcal{K}$. □

Proposition 4.10. *Let $\mathcal{K}$ be a BCK-algebra with condition (S) and let A be an i-v fuzzy ideal of $\mathcal{K}$. If the inequality $x \le y \circ z$ holds in $\mathcal{K}$, then $\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}$.*

Proof. Assume that the inequality $x \le y \circ z$ holds in $\mathcal{K}$. Using (IV1), (IV2) and Theorem 4.9, we have

$$\begin{aligned} \bar{\mu}_A(x) &\ge \mathrm{rmin}\{\bar{\mu}_A(x * (y \circ z)), \bar{\mu}_A(y \circ z)\} \\ &\ge \mathrm{rmin}\{\bar{\mu}_A(0), \bar{\mu}_A(y \circ z)\} \\ &= \bar{\mu}_A(y \circ z) \ge \mathrm{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}. \end{aligned}$$

The proof is complete. □

We give a condition for an i-v fuzzy set in a BCK-algebra with condition (S) to be an i-v fuzzy ideal.

Theorem 4.11. *Let A be an i-v fuzzy set in a BCK-algebra $\mathcal{K}$ with condition (S). Then A is an i-v fuzzy ideal of $\mathcal{K}$ if and only if it satisfies:*

(IV3) $x \le y \circ z$ *in* $\mathcal{K}$ *implies* $\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(y), \bar{\mu}_A(z)\}$.

Proof. $(\Rightarrow)$ It follows from Proposition 4.10.

$(\Leftarrow)$ Let A be an i-v fuzzy set in $\mathcal{K}$ satisfying (IV3). Since $0 \le x \circ x$ for all $x \in \mathcal{K}$, it follows from (IV3) that $\bar{\mu}_A(0) \ge \mathrm{rmin}\{\bar{\mu}_A(x), \bar{\mu}_A(x)\} = \bar{\mu}_A(x)$ for all $x \in \mathcal{K}$. Note that $x \le (x * y) \circ y$ for all $x, y \in \mathcal{K}$. Hence by (IV3), we obtain $\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\}$, which proves (IV2). Consequently, A is an i-v fuzzy ideal of $\mathcal{K}$. □

Theorem 4.12. *Let A be an i-v fuzzy set in $\mathcal{K}$. Then A is an i-v fuzzy ideal of $\mathcal{K}$ if and only if the non-empty set*

$$\bar{U}(A; [\gamma_1, \gamma_2]) := \{x \in \mathcal{K} \mid \bar{\mu}_A(x) \ge [\gamma_1, \gamma_2]\}$$

is an ideal of $\mathcal{K}$ for all $[\gamma_1, \gamma_2] \in D[0, 1]$.

Proof. Assume that A is an i-v fuzzy ideal of $\mathcal{K}$ and let $[\gamma_1, \gamma_2] \in D[0, 1]$ be such that $x \in \bar{U}(A; [\gamma_1, \gamma_2])$. Then $\bar{\mu}_A(0) \ge \bar{\mu}_A(x) \ge [\gamma_1, \gamma_2]$, i.e., $0 \in \bar{U}(A; [\gamma_1, \gamma_2])$. Let $x, y \in \mathcal{K}$ be such that $x * y \in \bar{U}(A; [\gamma_1, \gamma_2])$ and $y \in \bar{U}(A; [\gamma_1, \gamma_2])$. Then $\bar{\mu}_A(x * y) \ge [\gamma_1, \gamma_2]$ and $\bar{\mu}_A(y) \ge [\gamma_1, \gamma_2]$. It follows from (IV2) that

$$\bar{\mu}_A(x) \ge \mathrm{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\} \ge \mathrm{rmin}\{[\gamma_1, \gamma_2], [\gamma_1, \gamma_2]\} = [\gamma_1, \gamma_2]$$

so that $x \in \bar{U}(A; [\gamma_1, \gamma_2])$. Hence $\bar{U}(A; [\gamma_1, \gamma_2])$ is an ideal of $\mathcal{K}$. Conversely, suppose that $\bar{U}(A; [\gamma_1, \gamma_2]) (\ne \emptyset)$ is an ideal of $\mathcal{K}$ for all $[\gamma_1, \gamma_2] \in D[0, 1]$. Assume that there exists $a \in \mathcal{K}$ such that $\bar{\mu}_A(0) < \bar{\mu}_A(a)$. Let $\bar{\mu}_A(0) = [0^L, 0^U]$ and $\bar{\mu}_A(a) = [a^L, a^U]$. Then $0^L < a^L$ and $0^U < a^U$. If we take $[\delta_1, \delta_2] := \frac{1}{2}(\bar{\mu}_A(0) + \bar{\mu}_A(a))$, then $[\delta_1, \delta_2] = [\frac{1}{2}(0^L + a^L), \frac{1}{2}(0^U + a^U)]$. Hence $0^L < \delta_1 < a^L$ and $0^U < \delta_2 < a^U$, which imply that

$$\bar{\mu}_A(0) = [0^L, 0^U] < [\delta_1, \delta_2] < [a^L, a^U].$$

This shows that $0 \notin \bar{U}(A; [\delta_1, \delta_2])$, which leads to a contradiction. Therefore $\bar{\mu}_A(0) \ge \bar{\mu}_A(x)$ for all $x \in \mathcal{K}$. Now suppose that there are $a, b \in \mathcal{K}$ such that $\bar{\mu}_A(a) < \mathrm{rmin}\{\bar{\mu}_A(a*b), \bar{\mu}_A(b)\}$. Let $\bar{\mu}_A(a) = [a^L, a^U]$, $\bar{\mu}_A(a * b) = [(a * b)^L, (a * b)^U]$ and $\bar{\mu}_A(b) = [b^L, b^U]$. Put $[\beta_1, \beta_2] := \frac{1}{2}(\bar{\mu}_A(a) + \mathrm{rmin}\{\bar{\mu}_A(a * b), \bar{\mu}_A(b)\})$. Then

$$[\beta_1, \beta_2] = [\frac{1}{2}(a^L + \min\{(a * b)^L, b^L\}), \frac{1}{2}(a^U + \min\{(a * b)^U, b^U\})],$$

and so $a^L < \beta_1 < \min\{(a * b)^L, b^L\}$ and $a^U < \beta_2 < \min\{(a * b)^U, b^U\}$. It follows that

$$\bar{\mu}_A(a) = [a^L, a^U] < [\beta_1, \beta_2] < [\min\{(a * b)^L, b^L\}, \min\{(a * b)^U, b^U\}]$$

so that $a \notin \bar{U}(A; [\beta_1, \beta_2])$. But $\bar{\mu}_A(a * b) = [(a * b)^L, (a * b)^U] > [\beta_1, \beta_2]$ and $\bar{\mu}_A(b) = [b^L, b^U] > [\beta_1, \beta_2]$, i.e., $a * b \in \bar{U}(A; [\beta_1, \beta_2])$ and $b \in \bar{U}(A; [\beta_1, \beta_2])$. This leads to a contradiction. Consequently, $\bar{\mu}_A(a) \ge \mathrm{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\}$ for all $x, y \in \mathcal{K}$. This completes the proof. □

Theorem 4.13. *An i-v fuzzy set* $A = [\mu_A^L, \mu_A^U]$ *in* $\mathcal{K}$ *is an i-v fuzzy ideal of* $\mathcal{K}$ *if and only if* μ_A^L *and* μ_A^U *are fuzzy ideals of* $\mathcal{K}$.

Proof. Assume that A is an i-v fuzzy ideal of $\mathcal{K}$. For any $x \in \mathcal{K}$, we have

$$[\mu_A^L(0), \mu_A^U(0)] = \bar{\mu}_A(0) \geq \bar{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)].$$

It follows that $\mu_A^L(0) \geq \mu_A^L(x)$ and $\mu_A^U(0) \geq \mu_A^U(x)$. Let $x, y \in \mathcal{K}$. Then

$$\begin{aligned} [\mu_A^L(x), \mu_A^U(x)] &= \bar{\mu}_A(x) \geq \operatorname{rmin}\{\bar{\mu}_A(x * y), \bar{\mu}_A(y)\} \\ &= \operatorname{rmin}\{[\mu_A^L(x * y), \mu_A^U(x * y)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= [\min\{\mu_A^L(x * y), \mu_A^L(y)\}, \min\{\mu_A^U(x * y), \mu_A^U(y)\}], \end{aligned}$$

and so $\mu_A^L(x) \geq \min\{\mu_A^L(x * y), \mu_A^L(y)\}$ and $\mu_A^U(x) \geq \min\{\mu_A^U(x * y), \mu_A^U(y)\}$. Hence μ_A^L and μ_A^U are fuzzy ideals of $\mathcal{K}$.

Conversely, suppose that μ_A^L and μ_A^U are fuzzy ideals of $\mathcal{K}$. Then the non-empty level sets $U(\mu_A^L; \alpha_1)$ and $U(\mu_A^U; \alpha_2)$ are ideals of $\mathcal{K}$ where $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 \leq \alpha_2$. Noticing that $\bar{U}(A; [\alpha_1, \alpha_2]) = U(\mu_A^L; \alpha_1) \cap U(\mu_A^U; \alpha_2)$ which is an ideal of $\mathcal{K}$, and applying Theorem 4.12, we know that A is an i-v fuzzy ideal of $\mathcal{K}$. □

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