A Unified Optimization Algorithm For Solving "Regret-Minimizing Representative" Problems

Suraj Shetiya[‡], Abolfazl Asudeh[†], Sadia Ahmed[‡], Gautam Das[‡]
[‡]University of Texas at Arlington; [†]University of Illinois at Chicago
[‡]{suraj.shetiya@mavs, sadia.ahmed78@mavs, gdas@cse}.uta.edu, [†]asudeh@uic.edu

ABSTRACT

Given a database with numeric attributes, it is often of interest to rank the tuples according to linear scoring functions. For a scoring function and a subset of tuples, the *regret* of the subset is defined as the (relative) difference in scores between the top-1 tuple of the subset and the top-1 tuple of the entire database. Finding the *regret-ratio minimizing set* (RRMS), i.e., the subset of a required size k that minimizes the maximum regret-ratio across all possible ranking functions, has been a well-studied problem in recent years. This problem is known to be NP-complete and there are several approximation algorithms for it. Other NP-complete variants have also been investigated, e.g., finding the set of size k that minimizes the *average regret ratio* over all linear functions. Prior work have designed customized algorithms for different variants of the problem, and are unlikely to easily generalize to other variants.

In this paper we take a different path towards tackling these problems. In contrast to the prior, we propose a unified algorithm for solving different problem variants. Unification is done by localizing the customization to the design of variant-specific subroutines or "oracles" that are called by our algorithm. Our unified algorithm takes inspiration from the seemingly unrelated problem of *clustering* from data mining, and the corresponding K-MEDOID algorithm. We make several innovative contributions in designing our algorithm, including various techniques such as linear programming, edge sampling in graphs, volume estimation of multi-dimensional convex polytopes, and several others. We provide rigorous theoretical analysis, as well as substantial experimental evaluations over real and synthetic data sets to demonstrate the practical feasibility of our approach.

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1. INTRODUCTION

Data-driven decision making is challenging when there are multiple criteria to be considered. Consider a database of n tuples with d numeric attributes. In certain cases, "experts" can come up with a (usually linear) function to combine the criteria into a "goodness score" that reflects their preference for the tuples. This function can then be used for ranking and evaluating the tuples [1–4]. However, devising such a function is challenging [5, 6], hence not always a reasonable option, especially for ordinary non-expert users [7]. For instance consider a user who wants to book a hotel in Miami, FL. She wants to find a hotel that is affordable, is close to a beach, and has a good rating. It is not reasonable to expect her to come up with a ranking function, even though she may roughly know what she is looking for. Therefore, she will probably start exploring different options and may end up spending several confusing and frustrating hours before she can finalize her decision. Alternatively, one could remove the set of "dominated" tuples [8], returning a Pareto-optimal [9] (a.k.a. *skyline* [7, 8]) set, which is the smallest set guaranteed to contain the "best" choice of the user, assuming that her preference is monotonic [8]. In the case where user preferences are further restricted to linear ranking functions, only the *convex hull* of the dataset needs to be returned.

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The problem with the skyline or convex hull is that they can be very large themselves, sometimes being a significant portion of the data [10, 11], hence they lose their appeal as a small representative set for facilitating decision making. Consequently, as outlined in § 7, there has been extensive effort to reduce the size of the set. Nanongkai et al. [10] came up with the elegant idea of finding a small set that may not contain the absolute "best" for any possible user (ranking function), but guarantees to contain a *satisfactory* choice for each possible function. To do so, they defined the notion of "*regret-ratio*" of a representative subset of the dataset for any given ranking function as follows: it is the relative score difference between the best tuple in the database and the best tuple in the representative set. Given $k < n$, the task is to find the *regret-ratio minimizing set* (RRMS), i.e., a subset of size k that minimizes the maximum regret-ratio across all possible ranking functions. This problem is shown to be NP-complete, even for a constant (larger than two) number of criteria (attributes) [12]. Other researchers have also considered different versions of the problem formulation. For instance Chester et. al. [13] generalize the notion of regret from the comparison of the the actual top-1 of database to the top-k. More recently, in [14, 15] the goal was to compute the representative set that minimizes the *average regret-ratio* across all possible functions, instead of minimizing the max regret-ratio. All these variants have been shown to be NP-complete.

Given their intractable nature, there has been significant effort in designing efficient heuristics and approximation algorithms for these problems. The RRMS problem has been investigated in several papers [11, 12, 16], and several approximation algorithms have been designed; the algorithms in [11, 12] run in polynomial time and can approximate the max-regret ratio within any user-specified accuracy threshold. The average regret-ratio problem has been investigated in [14], and a different greedy approach has been proposed, which achieves a constant (not user-specified though) approximation factor with high probability.

1.1 Technical Highlights

In this paper, we make the following observations about the previous works: (a) the proposed algorithms are dependent on the specific problem formulation, and do not seem to generalize to different variants, and (b) the focus has been on designing approximation algorithms, and not on optimal algorithms. We take a different route towards solving these problems, and our work makes two important contributions:

Firstly, we develop a *unified algorithm* called *URM* that works across different formulations of the problem, including max [10] and average [14] regret minimizing sets. The unified algorithm makes calls to a subroutine (we refer to it as an "oracle"), and it is this subroutine/oracle that needs to be customized for each problem variant. Thus the customization is localized to the design of the oracle.

Secondly, we make a connection between the various regret minimizing problems and the seemingly unrelated classical data mining problem of *clustering* and the well-known K-MEDOID [17] algorithm. Most variants of clustering are NP-hard, yet the K-MEDOID algorithm is extremely popular in practice and is based on a hill-climbing approach to find a local optima. One of the main technical highlights of our contributions is to take inspiration from, and design our unified algorithm based on the K-MEDOID algorithm, even though the regret minimization problems seemingly appear quite different from clustering problems. One of the consequences of our approach is that the well-known advantages of the K-MEDOID algorithm transfer over to our unified algorithm. For example, the K-MEDOID algorithm has the *any-time* property; given more time it can be repeatedly restarted from different random starting configurations, which gives it the ability to improve the local optima that it has discovered thus far. Our unified algorithm also has this property, which can be useful in time-sensitive applications, including a query answering system.

To achieve our two contributions, several novel and challenging technical problems had to be solved. The K-MEDOID algorithm provided inspiration, but was not easily adaptable for our case. Instead, we had to carefully model our problem as that of optimally partitioning the space ranking functions into k convex geometric regions, such that for each region exactly one tuple from the database is the "representative", i.e., it has the highest (max, or average, depending on the problem variant being solved) score for any function in that region. At a high level, our algorithm first chooses k tuples randomly as our initial representative set. Then it only examines these k tuples, and partitions the function space into k convex regions such that the tuple associated with each region outscores the remaining $k-1$ tuples for any ranking function within its region. Then, the database is examined to update the best representative for each region, and the process iterates until a local optima is reached.

The process of examining the database for updating the best representative for a region required us to develop variant-specific oracles. For the case of the max regret-ratio, we propose different innovative strategies for designing an efficient oracle. For large regions, we design a threshold-based algorithm based on function-space discretization. For narrow regions, we model the problem as an instance of edge sampling from a weighted graph where edge weights are determined by solving (constant-sized) linear programs.

For the average regret-ratio case, designing an oracle was challenging. As another of our innovative technical highlights, we show that it is reducible to the classical problem of computing the volume of polytopes defined by the intersection of $O(n)$ halfspaces. Unfortunately, this approach has a time complexity of $\Omega(n^d)$ [18]. Even if we assume that the dimension of the database is fixed, this high complexity makes this approach only of theoretical interest. Consequently, we propose an alternative approach based on Monte-Carlo sampling of the function space.

We provide proof of convergence of our unified algorithm, as well as provide detailed theoretical analyses of our oracles. We also conduct extensive empirical experiments on both real and synthesis datasets that show the efficiency and effectiveness of our proposal.

2. PRELIMINARIES

Data Model: We consider a database $\mathbb{D}$ in the form of n tuples t_1 to t_n, defined over d numeric attributes A_1 to A_d. We use the notation $t_i[j]$ to show the value of t_i on attribute A_j. Without loss of generality, we assume that attribute values are normalized and standardized as the non-negative real numbers, $\mathbb{R}^+$. The numeric attributes are used for scoring and ranking the tuples. Additionally, the dataset may also include non-ordinal attributes that may be used in filtering, but not in ranking. Finally, for each attribute A_i, we assume that the larger attribute values are preferred. The values x of an attribute A_i with smaller-preferred nature require a straight forward transformation such as $(\max(A_i) - x)/(\max(A_i) - \min(A_i))$.

EXAMPLE 1. *Consider a dataset of* 10 *tuples, defined over the attributes A_1 to A_3, as shown below. The values of the attributes are normalized in range [0,100] and for all attributes the higher values are preferred. In this example $t_4[3]$ refers to the value of tuple t_4 on attribute d_3 which is equal to* 75.

tuple	A_1	A_2	A_3	*tuple*	A_1	A_2	A_3
t_1	*60*	*80*	*77*	t_6	*56*	*65*	*91*
t_2	*55*	*75*	*63*	t_7	*61*	*78*	*80*
t_3	*75*	*60*	*59*	t_8	*90*	*60*	*58*
t_4	*68*	*70*	*75*	t_9	*86*	*68*	*74*
t_5	*80*	*75*	*73*	t_{10}	*77*	*67*	*82*

Ranking Model: The database tuples are ranked using the scores assigned by a *ranking function*. For every tuple $t \in \mathbb{D}$, a ranking function $f : \mathbb{R}^d \to \mathbb{R}^+$ assigns a non-negative score to t. A tuple t_i outranks t_j based on f if its score is larger than the one of t_j. Following the literature in regret-minimizing context [10–13], we consider the class of linear ranking functions in this paper[1]. The score of a tuple t based on a linear function f with a weight vector $\vec{w} = \{w_1, w_2, \cdots, w_d\}$ is computed as:

$$f_{\vec{w}}(t) = \sum_{i=1}^{d} w_i \cdot t[i] \tag{1}$$

In the rest of paper, we simplify $f_{\vec{w}}(t)$ to $f(t)$ when $\vec{w}$ is clear in the context. As an example, let us consider Example 1, while choosing $\vec{w} = \langle .25, .5, .25\rangle$ for the ranking function. Using this function, $f(t_1) = .25 \times 60 + .5 \times 80 + .25 \times 77 = 74.25$. Computing the score of other tuples, the ordering of the tuples in $\mathbb{D}$ based this function is $\{t_5, t_1, t_7, t_9, t_{10}, t_4, t_6, t_2, t_8, t_3\}$.

In a d-dimensional space every tuple is presented as a point, and every linear function can be modeled as a origin-starting ray that passes through the point specified by its weight vector. For example, the function f with the weight vector $\vec{w} = \langle .25, .5, .25\rangle$ is modeled as the ray that starts from the origin and passes through the point $\langle .25, .5, .25\rangle$. The ordering of tuples based on f is specified by the ordering their projections on the ray of f (please refer to [5] for further details). As a result, the universe of origin-starting rays in the first quadrant of the d dimensional space forms the universe of linear functions. We call this the function space.

[1]Note that a large class of non-linear functions can fit this model after a straight-forward linearization [19].

A tuple t_i of a database is said to dominate tuple t_j if each of the attribute values for t_i is not smaller than that of t_j's while there exists an attribute A_k where $t_i[k] > t_j[k]$. For instance, in Example 1, $t_1 : \langle 60, 80, 77\rangle$ dominates $t_2 : \langle 55, 75, 63\rangle$. The set of tuples from the database which are not dominated tuples in the database is known as the *skyline* (or Pareto-optimal) [7–9].

A maxima representative, or simply a *representative*, is a subset $S \subseteq \mathbb{D}$ that is used for finding the maximum of $\mathbb{D}$ for any arbitrary ranking function f. Skyline is the minimal representative of $\mathbb{D}$ that guarantees the containment of the maximum of any function in the class of monotonic ranking functions. That is the reason it is popular for multi-criteria decision making (in the absence of a ranking function). Similarly, *convex-hull* is the minimal representative that contains the maximum for the class of linear ranking functions.

However, the skyline or convex-hull may contain a large portion of the database, which diminishes their applicability as a small set for decision making [10,11]. For instance, the skyline of Example 1 is $\{t_1, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}\}$, which includes 80% of the tuples.

2.1 Regret optimization measures

Regret ratio is a measure of dissatisfaction of a user when she sees results returned from the subset instead of the entire database. Let f be a ranking function, then regret ratio is defined as the ratio of the difference in ranking function scores between the top database tuple and the top tuple from the representative set to the score of the top database tuple:

DEFINITION 1 (REGRET RATIO). *Given a function f, let tuple t be $argmax_{\forall t \in \mathbb{D}} f(t)$. The regret ratio of set $S \subseteq \mathbb{D}$ for a ranking function f can be computed as*

$$rr(f, S) = \min_{t' \in S} \frac{f(t) - f(t')}{f(t)} \tag{2}$$

For instance, in Example 1, consider the set $S = \{t_2, t_3, t_4\}$ and the function f with the weight vector $\vec{w} = \langle .25, .5, .25\rangle$. t_5 is the max for this function ($f(t_5) = 75.75$) while t_4 is the tuple with the max score ($f(t_4) = 70.75$) in S. Therefore, $rr(f, S)$ in this example is $(75.75 - 70.75)/75.75 \simeq .066$.

For a universe of ranking functions, an aggregate over the regret ratio of each ranking function is considered as the regret ratio of the representative set for the universe.

In this paper, we provide *a unified* model that can handle a wide variety of aggregates (any ℓ^p norm):

DEFINITION 2. *ℓ^p norm regret measure : Given a database $\mathbb{D}$, a representative $S \subseteq \mathbb{D}$, a real number $p \geq 1$, and a set of ranking functions $\mathcal{F}$, the regret-ratio for the ℓ^p norm is defined as:*

$$RR_{\mathcal{F}}(S) = Agg^p_{\forall f \in \mathcal{F}} rr(f, S)$$

where Agg^p is the ℓ^p norm aggregate measure.

While different aggregates can be used here, so far the literature has considered *maximum* and *average* regret ratio minimizing sets. Without limiting our proposal to a specific value of norm, we especially show the adaptation of our framework for the existing aggregates, i.e., (i) ℓ_∞: maximum regret ratio and (ii) ℓ_1: average regret ratio. We will lay out the formal definitions of these measures in the remainder of this section.

DEFINITION 3. *Maximum (ℓ^∞ norm) Regret Ratio: Given a database $\mathbb{D}$ and a set of functions $\mathcal{F}$, the maximum regret ratio [10] of a set $S \subseteq \mathbb{D}$ is the maximum value of regret ratio for the set S over the set of all possible ranking functions $\mathcal{F}$. That is,*

$$RR_{\mathcal{F}}(S) = \sup_{f \in \mathcal{F}} rr(f, S) \tag{3}$$

While the maximum regret ratio looks at the worst case regret ratio for a set of functions, a different measure for user dissatisfaction is the average regret ratio.

DEFINITION 4. *Average (ℓ^1 norm) Regret Ratio: Given a database $\mathbb{D}$, a set of functions $\mathcal{F}$, and a probability distribution $\eta(.)$ where $\eta(f)$ is the probability of each function $f \in \mathcal{F}$, the average regret ratio [14] of a set $S \subseteq \mathbb{D}$ is defined as*

$$ARR_{\mathcal{F}}(S) = \int_{f \in \mathcal{F}} \eta(f)\, rr(f, S)\, df \tag{4}$$

Even though the regret ratio notions are defined for general classes of functions, the majority of the existing work consider $\mathcal{F}$ as the class of linear ranking functions (Eq. 1) [10–14, 20]. Also, for average regret ratio, the uniform distribution is considered as the probability distribution of ranking functions $\eta(.)$ [14]. We follow the literature on these. In the rest of the paper, we simplify the notations $RR_{\mathcal{F}}(S)$ and $ARR_{\mathcal{F}}(S)$ to $RR(S)$ and $ARR(S)$ for the class of linear ranking functions ($\mathcal{L}$).

2.2 Problem Definition

In this paper, we consider the problem of finding a compact representative of size k from a database such that the regret optimization measure is minimized. Formally:

ℓ^p **REGRET RATIO REPRESENTATIVE PROBLEM:**
Given a dataset $\mathbb{D}$, a set of ranking functions $\mathcal{F}$, and a value k, find the representative S of $\mathbb{D}$ for $\mathcal{F}$ such that $|S| = k$ and ℓ^p norm regret measure (Definition 2) of S is minimized.

In particular, the problem for the max. (resp. avg.) regret ratio is to find a set $S \subseteq \mathbb{D}$ of size k that minimizes the max. (resp. avg.) regret ratio. For any constant number of dimensions larger than 2, the maximum regret ratio problem is NP-complete [12]. Similarly, the problem for average regret ratio is proven to be NP-complete [14].

The two problems we consider in this paper are the max and average regret ratio optimizing set problems. In section 3, we describe the Regret Minimizing Framework algorithm, a general framework to solve the regret ratio class of problems, followed by sections 4 and 5 which discuss the details of the oracles for the Maximum and Average regret ratio problems respectively. We show empirical results for both these problems in section 6.

3. UNIFIED REGRET MINIMIZATION

3.1 Overview

In this section, we propose our *Unified Regret Minimizer (URM)* algorithm which is inspired by the K-MEDOID algorithm. Before providing the details of the algorithm, it is necessary to explain two central ideas in this paper: (i) representative and (ii) region of a tuple t, defined in definition 5 and definition 6 respectively.

DEFINITION 5. *Given a function f and a set of tuples $S \subseteq \mathbb{D}$, a tuple $t \in S$ is a representative for f if it has the maximum score, based on f among elements of S. Formally:*

$$\rho(f, S) = argmax_{t \in S} f(t)$$

DEFINITION 6 (REGION OF THE TUPLE t). *Given a set S and a tuple $t \in S$, the region of t is the set of functions for which t is the representative. That is,*

$$R_t(S) = \{f \in \mathcal{L} \mid \rho(f, S) = t\}$$

For example, Fig. 1 shows the regions of the tuples $\{t_1, t_5, t_7, t_9\}$ in Example 1.

Having the necessary definitions in place, we now provide a brief overview of the K-MEDOID and then show the transformation of our problem to it. Recall that given a database $\mathbb{D}$, the output size k and a regret measure, the *URM* algorithm finds a set of k representative tuples from $\mathbb{D}$, such that the regret measure is minimized.

K-MEDOID *algorithm:* An iterative algorithm that partitions a set of n objects into k clusters, such that the distance between objects belonging to a cluster and their cluster center are minimized. While many clustering algorithms (such as *kNN* [21]) have cluster centers that may not belong to the set of n objects, the K-MEDOID algorithm sets itself apart by choosing the cluster centers from the set of n objects. More details about the algorithm is provided in [17].

Although the clustering problem and the K-MEDOID algorithm seem different from our regret minimization problems, the notion of *region of a tuple*, provided in Definition 6 is the key in the problem transformation. As we shall show in the following, we see the problem as a partitioning of the function space into k convex regions while the tuples are the centroids.

Before providing further technical details in § 3.2, we would like to highlight a key difference between the nature of the problems: While the K-MEDOID algorithm deals with a countable set of objects n, the *URM* algorithm deals with an infinite set of objects (functions). The K-MEDOID algorithm clusters n objects (n being a discrete and finite number) into k clusters, each of which contains finite number of objects. In contrast, as we shall elaborate next, the *URM* algorithm clusters the continuous space of ranking functions, which contains an infinite number of functions into k convex regions of ranking functions.

The *URM* algorithm forms the basis for a unified framework to find compact representatives for a variety of regret measures. It does this by abstracting the various notions of regret measures into an *oracle*. At a high level, *URM* operates as follows. It is initialized with a set of k tuples, which form the compact representatives for the first iteration of the algorithm. These k tuples are used to partition the entire function space into k convex regions, such that for each region the corresponding tuple is the representative. In each iteration, we replace the k tuples with a potentially new set of k tuples that improves the regret measure. This is done by choosing, for each convex region of functions, the tuple from the database $\mathbb{D}$ that is the best representative for that region (this is accomplished by making calls to the oracle). The function space is repartitioned into k new convex regions, and the iterations continue until we converge to a local optima (i.e., the k tuples do not change).

3.2 Algorithm development

The key in the design of *URM* is that a set of k representative tuples *partition* the function space into k *convex* regions, such that each of the tuples t_i is the "representative" for all functions of its region R_i. This enables adopting the K-MEDOID technique for clustering the function space.

THEOREM 1. *Consider the set $S : \{t_1, \cdots, t_k\}$ as the compact representative of database $\mathbb{D}$ and the region of functions that are represented by t_i be R_i. Then the followings hold:*

1. *The regions $R_1, \cdots, R_k$ partition the function space.*
2. *For each tuple t_i, R_i is convex.*
3. *Each region R_i is the intersection of $(k-1)$ half-spaces.*

PROOF. Consider an arbitrary ranking function f. From Eq. 2, the regret ratio $r = rr(f, S)$ is the regret ratio of the set S for the

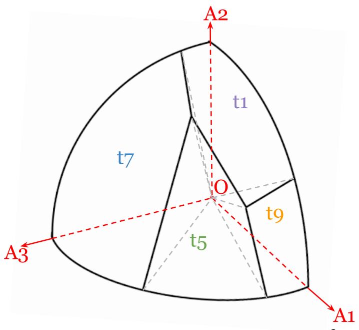


Figure 1: In Example 1, Tuples in a representative $\{t_1, t_5, t_7, t_9\}$ partition the function space into convex regions. For a representative of size k, each region is formed as the intersection of $(k-1)$ half spaces. Each tuple can be thought of as being in charge of its own region.

function f. The tuple from the set S which achieves this regret ratio is $t = \text{argmin}\ \forall_{i \le k}(f(t) - f(t_i))/f(t)$.
This tuple $t \in S$ is the representative for the function f. The region R_i consists of all the functions that have t_i as the representative. As the above property is applicable to the set of all ranking functions, regions $R_1, \cdots, R_k$ partition the function space into k regions. This proves property 1.

The region of functions for which the tuple t_i is ranked higher than tuple t_1 is a half-space denoted by H^+_{i1}, which is defined by the inequality,

$$\sum_{i=1}^{k} w_i \cdot (t_i[i] - t_1[i]) \ge 0 \tag{5}$$

The hyper-plane represented by the left-hand side of Eq. 5 is denoted as ordering exchange hyper-plane as it divides the space into two regions, one in which tuple t_i is better than t_1 and vice versa in the other. As Eq. 5 represents a half-space and half-spaces are convex, H^+_{i1} is a convex region. Similarly, let H^+_{ij} be the region of functions where t_i is better than t_j. As R_i is the region of functions where t_i is the representative, we need to consider the function space where t_i is better than all the other tuples, which is given by,

$$R_1 = \cap_{j=1}^{k} H^+_{ij} \tag{6}$$

R_i is the intersection of $k-1$ half-spaces which proves property 3.

We know that, the intersection of convex regions is convex. As half-spaces are convex, region R_i, which is an intersection of $(k-1)$ half-spaces is convex. This proves property 3. □

An interesting observation from Theorem 1 is that each region is described as a list of $(k-1)$ half-spaces (equations), which is *independent of the number of dimensions*.

As an example let us consider Fig. 1, which shows the regions for the tuples $\{t_1, t_5, t_7, t_9\}$ in Example 1. Table 1 shows the three half-spaces that define each of the regions. For instance, the region of t_1 is described by the half-spaces between t_1 and t_5, t_7, and t_9. Each of these regions are drawn in Fig. 1. One can verify that these regions are convex and partition the function space.

Theorem 1 shows that the regions of a set of tuples $S : \{t_1, .., t_k\}$ *partition* the function space into k non-overlapping convex clusters. Now we show how this enables adopting the K-MEDOID algorithm. Consider a database $\mathbb{D}$ and an initial set $S : \{t_1, .., t_k\}$ of k representative tuples in the first iteration of the algorithm. Let the region of functions R_i for tuple t_i be represented as the intersection of $(k-1)$ half-spaces. Based on Definition 6, t_i is preferred over all tuples $t_j \in S \backslash \{t_i\}$ for any function $f \in R_i$. *But it does not necessarily mean that t_i is preferred over all tuples $t_j \in \mathbb{D} \backslash \{t_i\}$.* So, for each i, there may be another tuple in the dataset that can introduce a smaller regret compared to t_i for R_i. This is a key observation in designing the subsequent iterations of the algorithm. To do so, we rely on the existence of the "regret optimization oracle", that finds

Table 1: Table containing the inequalities that define each region of the representative $\{t_1, t_5, t_7, t_9\}$.

Region	Half-space 1	Half-space 2	Half-space 3	Color
Region for t_1	$t_1 - t_5$: $-20A_1 + 5A_2 + 4A_3 \geq 0$	$t_1 - t_7$: $-A_1 + 2A_2 - 3A_3 \geq 0$	$t_1 - t_9$: $-26A_1 + 12A_2 + 3A_3 \geq 0$	Purple
Region for t_5	$t_5 - t_1$: $20A_1 - 5A_2 - 4A_3 \geq 0$	$t_5 - t_7$:$19A_1 - 3A_2 - 7A_3 \geq 0$	$t_5 - t_9$: $-6A_1 + 7A_2 - A_3 \geq 0$	Green
Region for t_7	$t_7 - t_1$: $A_1 - 2A_2 + 3A_3 \geq 0$	$t_7 - t_5$: $-19A_1 + 3A_2 + 7A_3 \geq 0$	$t_7 - t_9$: $-25A_1 + 10A_2 + 6A_3 \geq 0$	Blue
Region for t_9	$t_9 - t_1$ $26A_1 - 12A_2 - 3A_3 \geq 0$	$t_9 - t_5$: $6A_1 - 7A_2 + A_3 \geq 0$	$t_9 - t_7$: $25A_1 - 10A_2 - 6A_3 \geq 0$	Orange

Algorithm 1 *URM* Algorithm

```
    Input: $\mathbb{D}$, Regret Oracle $Orc$, Initial set of $k$ tuples $initial$
    Output: Representative $S$
 1: $S \leftarrow initial$
 2: repeat
 3:     $S' \leftarrow$ new List
 4:     for tuple $t$ in $S$ do
 5:         $R_t \leftarrow$ RegionOf($t$, $S \setminus \{t\}$)[2]
 6:         $t' \leftarrow Orc(\mathbb{D}, R_t)$ // the best tuple for $R_t$
 7:         Add $t'$ to $S'$
 8:     end for
 9:     $S \leftarrow S'$
10: until Convergence
11: return $S$
```

the best tuple in the database for R_i. This oracle is dependent on the variant of the regret measure we are seeking to optimize.

Regret optimization oracle: The regret optimization oracle is the variant-specific part of our overall approach. It is used to find a representative tuple from the database which best optimizes a specific regret measure for a convex region of ranking functions R_i (e.g., max regret, average regret, more general ℓ_p-norm regret, etc). Formally, given a database $\mathbb{D}$, a particular variant of regret measure of interest, and a convex region of functions R_i defined by the intersection of half-spaces, the regret optimization oracle finds the tuple (and the corresponding regret measure) from the database which minimizes the regret measure over all functions in R_i.

For now, we assume that such a variant-specific oracle exists, and we proceed with describing how our unified algorithm *URM* can leverage such an oracle (details about the design of the oracle for different problem variants are deferred to § 4 and § 5). At every iteration of the algorithm, for every region of functions R_i, the regret optimization oracle is used to find the new representative for the region R_i. The set of k tuples obtained from k calls to the regret optimization oracle form the new set of representatives for the next iteration of the algorithm. The iterations continue until the representative set of tuples ceases to change.

The pseudo code of the *URM* algorithm is given in Algorithm 1. Also, Algorithm 2 shows the pseudo code for finding the region R_i for a tuple t_i, in the form of the intersection of $(k-1)$ half-spaces.

Algorithm 2 RegionOf

```
    Input: A set of $k$ tuples $S$ and a tuple $t \in S$
    Output: $R_t$ in the form of intersection of $(k-1)$ half spaces
 1: $R_t \leftarrow$ new set
 2: for tuple $t_j$ in $S \setminus \{t\}$ do
        // $H_j$ is defined as as $\sum_{i=1}^{d} H_j[i] \times w_i \geq 0$
 3:     $H_j \leftarrow$ new list of size $d$
 4:     for $i \leftarrow 1$ to $d$ do $H_j[i] = t[i] - t_j[i]$
 5:     Add $H_j$ to $R_t$
 6: end for
 7: return $R_t$
```

[2]Note that the region of a tuple can be empty unless it belongs to the convex hull. In case of a region of a tuple being empty we add a random tuple in its place.

Compared to the existing literature for regret ratio minimizing problem, the *URM* algorithm has some unique and important features. *URM* provides a unified framework for the ℓ^p and ℓ^∞ classes of regret ratio measures. Our iterative algorithm has the any-time property where the user may stop at any time/iteration and still find a set of compact representatives. The any-time property can be very useful in case of real time query answering systems and other time-sensitive applications.

3.3 Proof of convergence for ℓ^p and ℓ^∞ norms

A critical requirement for the iterative algorithm is the convergence guarantee. Here, we prove that our algorithm converges to a local optima, for the ℓ^p and ℓ^∞ norm class of regret ratio measures.

THEOREM 2. *In each successive iteration of the URM algorithm, the regret measure for the set S : $\{t_1, \cdots, t_k\}$ improves for the ℓ^p norm regret measure.*

PROOF. Let us consider a set S : $\{t_1, \cdots, t_k\}$ as the compact representatives before the iteration and let set S' : $\{t'_1, \cdots, t'_k\}$ be the compact representatives after the iteration. The ℓ^p norm for the set S can be described by

$$RR = \sqrt[p]{\int_{f\in F} (rr(f,S))^p} = \sqrt[p]{\sum_{i=1}^{k} \int_{f\in R_i} (rr(f,t_i))^p}$$

During the iteration, for every region R_i the regret minimization oracle finds a tuple from the database $\mathbb{D}$ that minimizes the ℓ^p norm regret measure. Hence, we know that,

$$\overset{k}{\underset{i=1}{\forall}} \int_{f\in R_i} (rr(f,S))^p \geq \int_{f\in R'_i} (rr(f,t'_i))^p \tag{7}$$

As the equation for RR' can be written as,

$$RR' = \sqrt[p]{\sum_{i=1}^{k} \int_{f\in R_i} (rr(f,t'_i))^p} \tag{8}$$

Using Eq. 7 in Eq. 8, we get

$$\sqrt[p]{\sum_{i=1}^{k} \int_{f\in R_i} (rr(f,t_i))^p} \geq \sqrt[p]{\sum_{i=1}^{k} \int_{f\in R_i} (rr(f,t'_i))^p}$$

As the region of t_i differs from that of t'_i, we can conclude

$$rr(f,t'_i)) \geq (rr(f,S') \Rightarrow RR \geq RR' \tag{9}$$

□

A similar proof for ℓ^∞ norm can be proved.

THEOREM 3. *In each successive iteration of the URM algorithm, the regret measure for the set S : $\{t_1, \cdots, t_k\}$ improves for the ℓ^∞ norm regret measure(maximum regret ratio).*

PROOF. Let us consider a set S : $\{t_1, \cdots, t_k\}$ as the compact representatives before the iteration and let set S' : $\{t'_1, .., t'_k\}$ be the compact representatives after the iteration. The ℓ^∞ norm regret measure for the set S can be described by

$$RR = \max_{i=i}^{k} \sup_{f \in R_i} rr(f, t_i)$$

During the iteration, for every region R_i the regret minimization oracle finds a tuple from the database $\mathbb{D}$ that minimizes the ℓ^∞ norm regret measure. Hence, we know that

$$\overset{k}{\underset{i=1}{\forall}} \sup_{f \in R_i} rr(f, t_i) \geq \sup_{f \in R_i} rr(f, t'_i) \tag{10}$$

Using equations 9 and 10 we get,

$$\max_{i=i}^{k} \sup_{f \in R_i} rr(f, t_i) \geq \max_{i=i}^{k} \sup_{f \in R_i} rr(f, t'_i) \geq \sup_{f \in F} rr(f, S')$$
$$RR \geq RR'$$

□

3.4 Running Example

To provide a better understanding of the algorithm, in this section we provide a run of the algorithm over Example 1, while considering max as the target regret ratio measure. Let the set $S = \{t_1, t_5, t_7, t_9\}$ be the initial representative for the algorithm. The region of each tuple is highlighted in Table 1. Based on Definition 6, each tuple t_i is in charge of its own region R_i, highlighted in Fig. 1, i.e., t_i is the best tuple in S for all the ranking functions lying inside R_i. In the next iteration the *URM* algorithm goes through each of these regions to find better representatives. To do this, it calls the oracle *MaxO*. We shall provide the details of this oracle in § 4. For each region R_i, the oracle finds the best tuple in the entire dataset that has the minimum value of max regret ratio the functions in R_i. After the first iteration, the representative changes to $\{t_1, t_5, t_6, t_9\}$. This new representative creates a different partitioning of the function space, shown in Fig. 2. We report the max regret ratio scores of this iteration in Table 2. As we can see, the new representative has a lower score of max regret ratio, which is due to the convergence property of the algorithm, discussed in § 3.3. While in this example, only one of the tuples changed during the iteration, in practice more than one tuple may change during an iteration. The algorithm then calls the oracle *MaxO* to find the best tuples for the new regions and continues until convergence.

4. MAX REGRET RATIO ORACLE

So far in this paper, we discussed the regret-minimizing problem in general, assuming the existence of an oracle for computing the regret. In this section we focus on the original (and dominant) measure of regret-ratio: max regret-ratio [10]. Given a database $\mathbb{D}$ and a region of ranking functions R, the oracle *MaxRROrc* finds the tuple that has the least maximum regret ratio score for all functions in R. The region of functions R is formulated as the intersection of half spaces, $R = \{H_1, .., H_{k-1}\}$. The objective is to design an efficient oracle for this case.

We first model the problem into a weighted directed complete graph, and use it for calculating the max regret ratio. Then, we propose three optimization techniques to make the oracle efficient and scalable.

4.1 Graph Transformation

In order to find the tuple with the least max regret ratio in a region R, we transform the problem into a graph exploration instance. Consider a weighted directed complete graph G, as illustrated in Fig. 3, with n nodes and $n(n-1)$ edges such that:

- Every tuple $t \in \mathbb{D}$ translates to the node t in G.
- The weight $w_{t \to t'}$ of an edge $t \to t'$ (from node t to node t') is equal to the max regret ratio of replacing t' with t in the region R.

In order to compute the weight of an edge $t \to t'$ we use the (fixed size – independent of n) linear programming (LP) shown in Eq. 11.

Having the graph G constructed, the max regret ratio of assigning a tuple t as the representative (removing all other tuples $t' \in \mathbb{D} \backslash \{t\}$) of the region R, is the maximum weight of its outgoing edges. This is proved in Theorem 4.

THEOREM 4. *Given the graph G of a database $\mathbb{D}$, a space of ranking functions R the max regret ratio of a tuple $t \in \mathbb{D}$ in R is:*

$$RR(t, \mathbb{D}, R) = \max_{\forall t' \in \mathbb{D} \backslash \{t\}} w_{t \to t'}$$

PROOF. Let $f' \in R$ be the ranking function for which the tuple t has the maximum regret ratio score. Let $r'_{f'}$ be the maximum regret ratio score and tuple $t_{f'}$ be the tuple which has the maximum score from the database $\mathbb{D}$ for the function f'. By definition, the equation for maximum regret ratio $r'_{f'}$ is

$$r_{f'} = \frac{f'(t_{f'}) - f'(t)}{f'(t_{f'})}$$

An important observation that we will use in this proof is that the equation for $r_{f'}$ is dependent only on the tuple $t_{f'}$ from the database. The computation of edge weight $w_{t \to t'}$ can be formulated as,

$$w_{t \to t'} = \sup_{f \in R} \frac{f(t') - f(t)}{f(t')}$$

As we compute the edge weights between tuple t and all the other tuples from the database D, the maximum value is computed when the comparison between tuples t and $t_{f'}$. Hence, $RR(t, \mathbb{D}, R)$ is equal to $r_{f'}$. □

The representative tuple for a region of functions R is the node that has the minimum max-weight over its outgoing edges. Therefore, after constructing the graph G, the oracle can make a pass over the graph and find the representative node to assign to the region.

Given that the LPs have a constant size, the construction of the graph G and finding the representative tuple for a region R based on it has the time complexity of $\mathcal{O}(n^2)$.

Even though the construction of graph G enables a polynomial algorithm for the max regret ratio oracle, it is still a quadratic algorithm which is not efficient and scalable in practice. Therefore, in the rest of this section we propose different approaches for making the oracle more efficient.

The idea is to find the representative tuple of the region without the complete construction of G. For instance, following the convergence property of the *URM* algorithm, we know that, at every iteration, the max regret ratio of the representative of a region R is not more than the one for the representative from the previous iteration. We can exploit this property by using the regret ratio value of the representative for the region R_i as a threshold during the regret ratio value computation for the region R_i. That is, for any node in graph G, we ignore computing the weights of its outgoing edges, as soon as we find a edge that is not smaller than the threshold for this region. Following the idea of not computing the weights of all edges in G, next, we propose a threshold-based algorithm for the max regret ratio oracle.

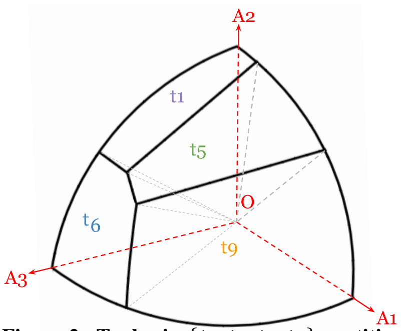


Figure 2: Tuples in $\{t_1, t_5, t_6, t_9\}$ partition the function space into convex regions. This is the updated representative after one iteration of *URM*, with $\{t_1, t_5, t_7, t_9\}$ as the initial set. The new representative has the max regret ratio 0.0444.

Table 2: An iteration of the *URM* algorithm corresponding to the max regret ratio problem for example 1. Regret ratio for each of the tuples in representative is shown. The last row gives the max regret ratio score for the representatives.

Initial Rep.		Rep. after one Iter.	
Tuple	**RR**	**Tuple**	**RR**
t_1	0.0	t_1	0.0103
t_5	0.0223	t_5	0.0177
t_7	0.1208	t_6	0.0230
t_9	0.0938	t_9	0.0444
	0.1208		**0.0444**

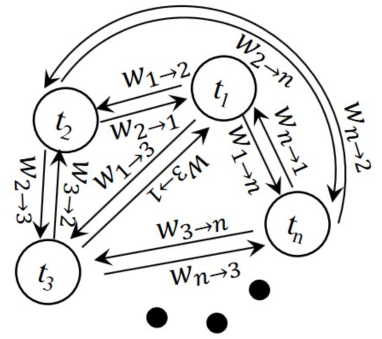


Figure 3: Graph transformation for max regret ratio oracle

$$\begin{aligned} &\max \quad r && (11)\\ &s.t. \quad && r \geq 0\\ & && \Sigma_{j=1}^{d} w_j \times t'[j] = 1\\ & && \Sigma_{j=1}^{d} w_j \times (t'[j] - t[j]) \geq r\\ &\forall 0 < \ell < k: && \Sigma_{j=1}^{d} H_\ell[j] \times w_j \geq 0 \end{aligned}$$

4.2 Threshold-based Algorithm

Threshold-based algorithms are proven to be effective in practice and are the de-facto solution for many important problems such as top-k query processing [2, 22]. When the objective value is minimization, the idea is to sort the tuples based on a lower bound on their objective values, ascending. Then starting from the top of the list, while maintaining a threshold, we continue processing the tuples in the sorted order until the lower bound value of the remaining tuples in the list is larger than the current threshold. The algorithm can stop then, as the objective value of the remaining tuples cannot be less that the best known threshold.

We apply a similar strategy here. But, first, we need to find the lower bounds on the max regret ratio of the tuples in region R. We apply two strategies for finding the lower bound: (i) function sampling and (ii) edge sampling, explained in § 4.2.1 and § 4.2.2, respectively. The objective is to construct a sorted vector $\mathcal{V}$ based on the lower bound values on the max regret ratio of the tuples.

4.2.1 Lower bound based on function sampling

The first strategy is to use function sampling for finding the lower bounds. We use the existing work [5] for sampling unbiased functions from the region R. Using a set of N IID function samples drawn from R, we construct a n by N table $\mathcal{T}$ that every row in it is a tuple and every column is one of the sampled functions. Every cell $\mathcal{T}[i, j]$ is the regret ratio of the tuple t_i on the sampled function f_j (Fig. 4). In order to identify the cell values in $\mathcal{T}$, we first make a pass over the matrix and fill every cell $\mathcal{T}[i, j]$ with the value of $f_j(t_i)$. Also, for every column j, we keep track of its maximum value max_j. After finishing the first pass over the matrix, we do a second pass replacing each cell value $\mathcal{T}[i, j]$ by $(max_j - \mathcal{T}[i, j])/max_j$.

After the table $\mathcal{T}$ is constructed, the lower bound on the max regret ratio of each tuple t_i is the max value on row $\mathcal{T}[i]$. That is,

$$lower_{RR}(t_i, R) = \max_{j=1}^{N} \mathcal{T}[i, j] \qquad (12)$$

Algorithm 3 uses the above idea and returns a sorted vector of tuples based on the lower bound estimation of their max regret ratio.

tid	f_1	$\cdots$	f_j	$\cdots$	f_N
t_1					
$\vdots$					
t_i			$rr_{f_j}(t_i)$		
$\vdots$					
t_n					

Figure 4: Illustration of table $\mathcal{T}$

Algorithm 3 SortedLB$_{FS}$

```
    Input: Database $\mathbb{D}$, Set of Function Samples $F$
    Output: Sorted vector of tuples based on the lower bound of their max regret ratio
 1: for $j \leftarrow 1$ to $|F|$ do
 2:     $max_j \leftarrow 0$
 3:     for $i \leftarrow 1$ to $n$ do
 4:         $\mathcal{T}[i, j] \leftarrow F[j](t_i)$
 5:         if $\mathcal{T}[i, j] > max_j$ then $max_j \leftarrow \mathcal{T}[i, j]$
 6:     end for
 7:     for $i \leftarrow 1$ to $n$ do $\mathcal{T}[i, j] \leftarrow \frac{max_j - \mathcal{T}[i,j]}{max_j}$
 8: end for
 9: for $i \leftarrow 1$ to $n$ do $\mathcal{V}[i] \leftarrow (i, \max_{j=1}^{N} \mathcal{T}[i, j])$
10: return Sorted($\mathcal{V}$) on second column
```

Making two passes over $\mathcal{T}$ and then sorting the vector $\mathcal{V}$, Algorithm 3 is in $\mathcal{O}(n(N + \log n))$. Note that, considering a fixed sampling budget, the algorithm is linearithmic.

4.2.2 Lower bound based on edge sampling

Function sampling works well for fat regions i.e. regions which have a large volume where sampling from these regions is easy. But if the space of function space is a thin region then function sampling from it would end up being costly. We propose weighted sampling of edges of G for these cases.

Recall that the max regret ratio of a tuple t_i in a region R is the max of the weights of its outgoing edges in G (c.f. Theorem 4). A loose lower bound on the max regret ratio value can be obtained by uniformly sampling a few edges from the graph and computing their weights. To make it more effective, we use a weighted sampling of the edges. In order to obtain the weights for the sampling process, we start with a set of ranking functions, chosen within the region R. To obtain the functions, a linear program is used to find the Chebyshev center [23], which is the center of largest inscribed hyper sphere inside of R. Using a normal distribution, we sample a few functions and transform these functions to lie on the surface of the hyper sphere described by the Chebyshev center. We use the sum of the scores of the tuples as a guidance for the weighted

Algorithm 4 SortedLB$_{ES}$

```
    Input: Database $\mathbb{D}$, Number of samples $N$, Number of edge samples $N_{edge}$, Convex region of functions $R$
    Output: Sorted vector of tuples based on the lower bound of their max regret ratio
 1: $center, radius \leftarrow$ Compute Chebyshev Center for region $R$
 2: $F \leftarrow Sample(N)$ // draw $N$ samples
 3: for $j \leftarrow 1$ to $|F|$ do
 4:     for $i \leftarrow 1$ to $n$ do $\mathcal{P}[i] \leftarrow \mathcal{P}[i] + F[j](t_i)$
 5:     $total \leftarrow total + \mathcal{P}[i]$
 6: end for
 7: for $i \leftarrow 1$ to $n$ do $\mathcal{P}[i] \leftarrow \frac{\mathcal{P}[i]}{total}$ // Normalize $\mathcal{P}$
 8: for $i \leftarrow 1$ to $n$ do
        // draw $N_{edge}$ samples from distribution $\mathcal{P}$
 9:     $E \leftarrow Sample(\mathcal{P}, N_{edge})$
10:     for $j \leftarrow 1$ to $N_{edge}$ do
11:         $w \leftarrow max(w,$ compute $w_{i \rightarrow E[j]}$ based on Eq. 11$)$
12:     end for
13:     $\mathcal{V}[i] \leftarrow (i, w)$
14: end for
15: return Sorted($\mathcal{V}$) on second column
```

Algorithm 5 MaxO

```
    Input: Database $\mathbb{D}$, Convex region of functions $R$, Sampling budget $N$, representative $t$, threshold $\tau$
    Output: The representative tuple for $R$
 1: $F \leftarrow$ Sample($R$,$N$) // draw $N$ samples from $R$
 2: if $F$ is not empty then $\mathcal{V} \leftarrow$ SortedLB$_{FS}$($\mathbb{D}$, $F$)
 3: else $\mathcal{V} \leftarrow$ SortedLB$_{ES}$($\mathbb{D}$, $R$, $N$)
 4: for $i \leftarrow 1$ to $n$ do
 5:     if $\mathcal{V}[i, 2] \geq \tau$ then break
 6:     $rr \leftarrow 0$
 7:     for $j \leftarrow 1$ to $n$ where $j \neq \mathcal{V}[i, 1]$ do
 8:         $w \leftarrow$ compute $w_{\mathcal{V}[i,1] \rightarrow j}$ based on Eq. 11
 9:         if $w > rr$ then $rr \leftarrow w$
10:         if $rr \geq \tau$ then continue
11:     end for
12:     if $rr < \tau$ then $\tau \leftarrow rr$; $t \leftarrow \mathcal{V}[i, 1]$
13: end for
14: return $t$
```

sampling. The idea is that the tuples with the higher scores are more likely of being representative of the region. Hence, we use the normalized vector of the score aggregates for the tuples, as the probability distribution for edge sampling.

Weighted sampling is performed using these weights to obtain a few edges of the graph. Computing the weights of these edges gives us a tighter lower bound value for max regret ratio. These lower bound values are then used in a similar manner to the threshold based algorithm § 4.2.1. Algorithm 4 shows the pseudo code for the weighted edge sampling algorithm.

Having the sorted list of tuples $\mathcal{V}$, we can now design our threshold based algorithm (Algorithm 5). Starting from the first tuple in the list, the algorithm computes the weights for the outgoing edges the current tuple in graph G (Eq. 11). The max regret ratio of the current tuple is the max of its outgoing edges. While making a pass over $\mathcal{V}$ and computing the max regret ratio of the tuples, the algorithm keeps track of the best known solution (the least value of max regret ratio) as the threshold, and stops as soon as the lower bound values of the remaining tuples are higher than the threshold.

5. AVERAGE REGRET RATIO ORACLE

In this section, we shift our focus to a different measure of regret ratio, namely, average regret-ratio (ARR). An exact solution for the ARR computation oracle requires to partition the region R into the "maxima sub-regions" such that a specific tuple t_i is the maxima for each and every of the functions in each sub-region. Let $T_M \subseteq \mathbb{D}$ be the set of tuples that have the maximum score for at least one function in R. Also let $n' \leq n$ be the size of T_M. For each tuple $t \in T_M$, the set of functions for which t is the maxima is a convex region, defined by the intersection of $n' - 1 = O(n)$ half-spaces. For every function f in $R_{t'}$ (the maxima sub-region of a tuple $t' \in T_M$), the regret ratio of a tuple $t \in \mathbb{D}$ is $(f(t') - f(t))/f(t')$. This, in the end, provides an exact solution for computing the ARR of a tuple in a region R. However, it involves a volume computation under the regret ratio curves across the sub-regions. Unfortunately, even though the total number of sub-regions is in $O(n)$ and each sub-region is defined as the intersection of a linear number of half spaces, the computation of ARR within each of the sub-regions is not computationally feasible in higher dimensions. That is because, as proven by Dyer et al. [18], the exact computation of the volume of a convex shape described as the intersection of a linear number of half-spaces is #P-hard.

Fortunately, although exact volume computation is usually costly, Monte-carlo methods [24] combined with tail inequalities [25] provide strong estimation methods for the problem. Following this, Zeighami et. al [15] have shown that the ARR value can be approximated using N samples of ranking functions with an error bound of ϵ and a confidence of $1 - \sigma$, where the relation between N, ϵ, and σ is shown in the following equation.

$$\epsilon = \sqrt{\frac{3}{N} \ln \frac{1}{\sigma}} \tag{13}$$

Essentially, [15] shows that *discretizing* the continuous function space to a set of N uniform samples, and using the samples for finding the maxima representatives guarantee an error ϵ, with the confidence interval of $1 - \sigma$, as specified in Eq. 13. We follow this in the design of the ARR oracle. That is, to use the discrete set of N uniform function samples for finding the representative tuples.

Consider a database $\mathbb{D}$ and a set of representative tuples S and their regions. For each of the regions, we want to find the tuple $t \in \mathbb{D}$ for which the ARR score is the lowest. For any region R_i, we select the functions which lie inside R_i. We use this set of ranking functions to estimate the ARR score of each of the tuples in the database. Next, for each region, the tuple with the lowest ARR score replaces the previous representative of that region. The algorithm is given in 6.

6. EXPERIMENTS

6.1 Experimental setup

Datasets: For evaluating our algorithms, we have used the following datasets. We generated two synthetic datasets - *Surface* and *Scaled* along the lines of *Sphere* and *SkyPoints* in [12].

- (Real dataset) *Colors [26]:* This is one of the commonly used datasets for the evaluation of regret ratio and skyline problems [10, 12, 27]. In our experiments, we have used the *Color Histogram* dataset. It contains 68,040 tuples, each being a color image. For every image, it contains 32 attributes, where each attribute is the density of a color in the entire image.
- (Synthetic) *Surface:* We generated the Surface dataset by uniformly sampling points on the surface of a unit hypersphere. Therefore, by construction, all the points in the Surface dataset belong to the skyline. The dataset contain 20,000 tuples, over 12 attribute, in range $[0, 1]$.
- (Synthetic) *Scaled:* For the Scaled dataset, we uniformly generated points inside a unit hypersphere. Since, in a high dimensional space, a large portion of the total volume of a hypersphere

Algorithm 6 AvgO

```
    Input: $\mathbb{D}$, Convex region $R$, Function samples $F$ in $R$
    Output: The representative tuple for $R$
 1: for $j \leftarrow 1$ to $|F|$ do $max[j] \leftarrow 0$
 2: for $i \leftarrow 1$ to $n$ do
 3:     for $j \leftarrow 1$ to $|F|$ do
 4:         $scores[i,j] \leftarrow F[j](t_i)$
 5:         if $max[j] < scores[i,j]$ then $max[j] \leftarrow scores[i,j]$
 6:     end for
 7: end for
 8: $min \leftarrow \infty$
 9: for $i \leftarrow 1$ to n do
10:     $arr[i] \leftarrow \sum_{j \leftarrow 1}^{|S|} \frac{max[j]-scores[i,j]}{max[j]}$
11:     if $min > arr[i]$ then $min \leftarrow arr[i]; t \leftarrow i$
12: end for
13: return t
```

lies near the surface area, most of the points in the *Scaled* dataset is also present in the skyline. The scaled dataset contains a set of 20,000 tuples, each defined over 12 attributes in range $[0, 1]$.

- (Real dataset) *DOT [28]:* The flight on-time dataset is published by the US Department of Transportation(DOT). It records, for all flights conducted by the 14 US carriers in January 2015, attributes such as scheduled and actual departure time, taxiing time and other detailed delay metrics. The dataset consists of 457,013 tuples and 7 ordinal attributes.
- (Real dataset) *NBA [29]:* NBA dataset contains the points for the combination of player,team,season up to 2009. It contains 21,961 tuples and 17 ordinal attributes: *gp, minutes, pts, oreb, dreb, reb, asts, stl, blk, turnover, pf, fga, fgm, fta, ftm, tpa, tpm.*

Hardware and Platform: All our experiments were performed on a Core-i7 machine running Ubuntu 16.04 with 64 GB of RAM. The algorithms were implemented in Python.

Evaluations: In order to asses the performance of our algorithm, we focus on two main criteria namely, efficiency and efficacy. Concretely, we evaluate based on the following metrics - (i) the quality of the results produced, i.e. the regret ratio measure of the result (ii) the amount of time taken by the algorithm.

Algorithms Evaluated: For the max regret ratio representative problem, we have used HD-RRMS as the baseline [11], one of the recent and advanced algorithms for finding the max regret ratio representative that guarantees a tunable additive approximation. To do so, it discretizes the ranking function space and models the problem as a discrete matrix min-max problem. A combination of binary search technique and transforming the problem into fixed-size set covers are applied for solving the problem. To make the problem practical, the greedy approximation algorithm is used for the set cover instances. As we shall later show in this section, the extra approximation induced by the greedy set cover in the HD-RRMS algorithm shows up in our results and can be seen in the HD-RRMS curve when the regret ratio does not reduce with time in some cases. For the average regret ratio case, we have implemented the GREEDY-SHRINK algorithm [15]. This algorithm starts with the entire database as its representative and iteratively removes the tuple for which the increase in the average regret ratio is the least. This process is continued until k tuples are left. In addition to GREEDY-SHRINK, we also compute the global minima, using N samples from the function space, drawn based on Eq. 13 with $\epsilon = 0.01$ and a confidence as 0.999. We filtered the dataset to only the skyline points. Using the N samples, a table containing the regret-ratio values for the filtered tuples is generated, similar to table 4. For every combination of k tuples as a representative, we compute the average regret-ratio value. It is important to note here that existing algorithms are all approximation algorithms and do not exhibit the *anytime* property. As a result, *URM* is not directly comparable to the existing work. In our experiments we focus on specific properties of our algorithm.

We also compare our algorithm with several skyline reducing algorithms (which are discussed in more detail in § 7). We have implemented the KRSPGREEDY algorithm from [30], EIQUE algorithm from [31], NAIVEGREEDY algorithm from [32], *ϵ-ADR* greedy algorithm from [33] and SKYCOVER algorithm from [34].

6.2 Summary of experimental results

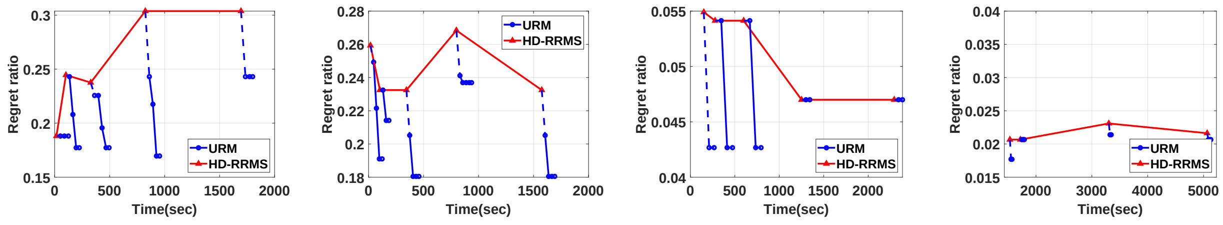
At a high level, the experiments verified the quality and efficiency of our proposal. We consider regret ratio score as the measure of quality. For the max regret ratio representative problem, *URM* improves the quality of the representatives provided by HD-RRMS when used as a starting point. When provided with a time budget *URM* qualitatively outperforms HD-RRMS algorithm. In case of the average regret ratio representative problem, *URM* qualitatively outperforms GREEDY-SHRINK when provided with the same time budget. Our experiments show that *URM* converges to the local minima very fast which increases the chance of discovering the global optima. In addition, the experiments demonstrate some of the useful properties of our approach, namely, (a) *anytime* property - even if the execution of the algorithm is terminated at any point of time, the algorithm will still have a representative (b) provision for getting better results - by restarting with a different set of initial points, *URM* can achieve better representatives.

6.3 Results for Max Regret-Ratio

Results when initial representative is given: Similar to other clustering-inspired algorithms, the quality of the results produced by the *URM* algorithm depends on the initial set of tuples. Often feeding the output of an existing approximation algorithm as the input to these iterative algorithms yield good results. In this set of experiments, we have used the compact representative returned by HD-RRMS as the initial set of tuples to the *URM* algorithm. To be fair in the assessment, while creating the plots for *URM*, we have taken the time to generate the initial points into consideration. Concretely, we have added the running time of HD-RRMS to the running time of *URM* when creating the plots for the *URM* algorithm. We run the experiments on the *Surface*, *Colors*, *NBA*, *DOT*,and the *Scaled* datasets. For *Surface*, *Colors*, *DOT* and *Scaled* datasets we run *URM* for 4 and 5 dimensions with 20k points each. We use the same configuration used in [11] for *NBA* and *DOT* datasets. While we use 20k tuples with 4, 5 and 6 dimensions for *NBA* dataset, we use 400k tuples with 4 and 5 dimensions for *DOT* dataset. Fig. 5 and 6 plot the obtained max regret ratio to the time taken by each algorithm for *Surface*, *Colors*, *Scaled* and the *DOT* datasets for dimensions 4 and 5, respectively. Fig. 7 compares the *URM* and HD-RRMS algorithms for the *NBA* datasets.

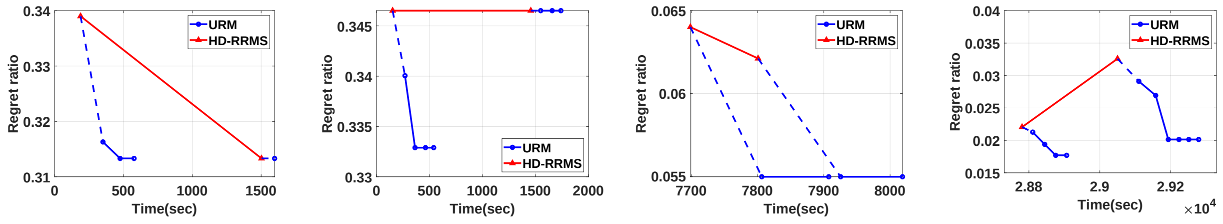
To get different results from HD-RRMS, we used different values of γ for discretization. Every red point shows the experimental result of one individual run of the HD-RRMS algorithm. While HD-RRMS is expected to provide better results as γ increased, in several settings, one can see an increase in the max regret ratio of the generated output. The reason is that HD-RRMS uses the greedy approach for solving the (theoretically fixed-size) set cover instances. This adds one more level of approximation to the algorithm which, in the end, results in the non-decreasing behavior of it in some cases.

In the figures, each red point is connected to a blue point by a dotted line, which represents feeding the HD-RRMS algorithm's output to the *URM* algorithm. The string of blue points connected by the solid blue line show the performance of the *URM* algorithm



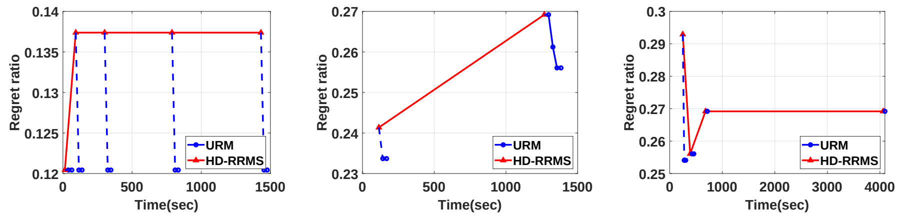
(a) Surface [4D] (b) Scaled [4D] (c) Colors [4D] (d) DOT [4D]

Figure 5: Maximum regret ratio when *URM* uses the compact representative produced by HD-RRMS as the initial set [4D, K=5].



(a) Surface [5D] (b) Scaled [5D] (c) Colors [5D] (d) DOT [5D]

Figure 6: Maximum regret ratio when *URM* uses the compact representative produced by HD-RRMS as the initial set [5D, K=5].



(a) NBA [4D] (b) NBA [5D] (c) NBA [6D]

Figure 7: Maximum regret ratio when *URM* uses the compact representative produced by HD-RRMS as the initial set for NBA dataset.

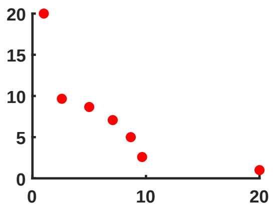
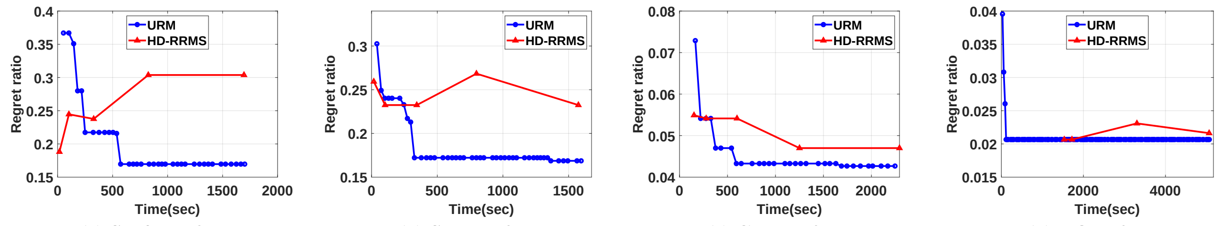
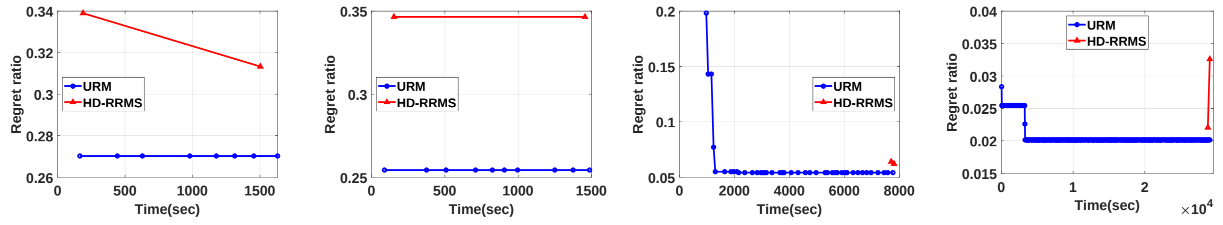


Figure 8: 2D Dataset Example



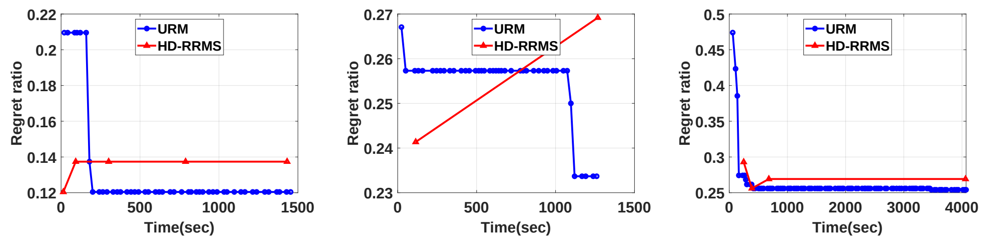
(a) Surface [4D] (b) Scaled [4D] (c) Colors [4D] (d) DOT [4D]

Figure 9: Maximum regret ratio when both *URM* and HD-RRMS uses a fixed time budget [4D, K=5].



(a) Surface [5D] (b) Scaled [5D] (c) Colors [5D] (d) DOT [5D]

Figure 10: Maximum regret ratio when both *URM* and HD-RRMS uses a fixed time budget [5D, K=5].



(a) NBA [4D] (b) NBA [5D] (c) NBA [6D]

Figure 11: Maximum regret ratio when both *URM* and HD-RRMS uses a fixed time budget for NBA dataset.

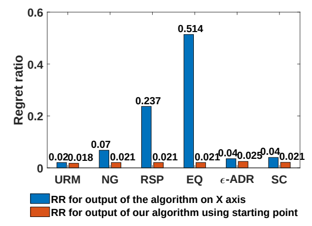
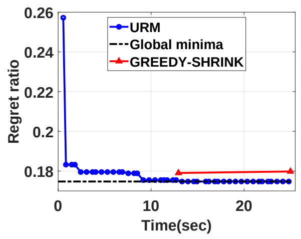
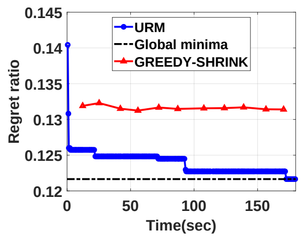


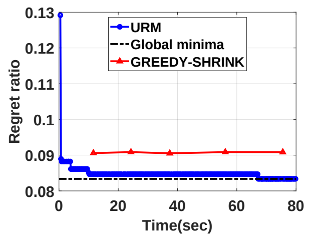
Figure 12: Comparing skyline reducing alg. with *URM* for DOT dataset with 4 dimensions



(a) Colors, Set size = 4 [9D]



(b) Colors, Set size = 5 [9D]



(c) Colors, Set size = 6 [9D]

Figure 13: Average regret ratio, we are comparing our results against the global best representatives [9D].

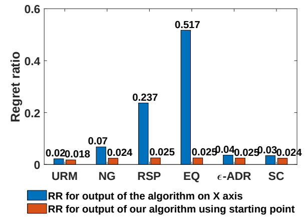


Figure 14: Comparing skyline reducing alg. with *URM* for DOT dataset with 5 dimensions

over the subsequent iterations. First, our experiments show that *URM* quickly reaches the local minima of regret ratio in this setting. Also, in most of the experiments, *URM* improves the results of the HD-RRMS algorithm. The ones where *URM* could not improve the result of HD-RRMS were the ones that HD-RRMS had, by chance, discovered a local optima. Apart from the performance of the algorithm, it is interesting to see the existence of multiple local optima and the impact of the starting point on the optima discovered. For instance in Fig. 5b, each setting discovered a different local optima. The fast convergence of *URM* enables multiple runs of the algorithm with multiple starting points, which increases the chance of discovering the global optima.

Results on a fixed time budget: While *URM* is an *anytime* algorithm, HD-RRMS is not. That is, HD-RRMS returns a representative only after it is finished. Therefore, in order to make a fair assessment, we run the HD-RRMS algorithm with different values of γ and allocate the exact same amount of time taken by the HD-RRMS algorithm as the time budget for our *URM* algorithm. We run our experiments with *Surface*, *Colors*, *NBA*, *DOT*,and the *Scaled* datasets. This set of experiments demonstrate two important properties of the *URM* algorithm. First, as we can see from Fig. 9, 10 and 11 the regret ratio scores reduce monotonically. This is due to the convergence property of the algorithm, proved in § 3.3. In addition, the *URM* algorithm produces reasonably good regret ratio scores even when the allocated time budget is small. This means that a user can still get a reasonably good set of representatives even if she terminates the algorithm before it finishes. Another important result is that *URM* actually allows the user to find a better representative. With a fixed time budget, we restart the algorithm with a random set of starting points and repeat this process until our time budget expires. Essentially, with subsequent repetition of the *URM* algorithm, we find a different set of compact representatives with a better score of max regret-ratio. Another important property that our algorithm exhibits is the any time property. As *URM* is an iterative hill climbing algorithm, the best representative among the ones it has already visited can be consumed by the user even before the end of the time budget. This can be seen in Fig. 9 and Fig. 10 where if we were to stop the algorithm at any point of time we would get a representative marked by the blue circles. First, looking at the figures, one can see the monotonically decreasing behaviour of *URM*, compared to HD-RRMS. Also, HD-RRMS needs to finish at least once or will not provide any output. For instance, in Fig. 10(c), HD-RRMS did not provide any result for a time budget less than 8000 seconds. In contrast, having the anytime property, *URM* has an output to offer at any point of time.

Proof of concept by an example: Using the NBA dataset, we highlight a concrete example. The dataset contains performance records of different Basketball players across different seasons. Consider the case where the decision criteria are Points, Rebounds, Assists and Steals. More than 60 tuples belonged to the skyline. Offering such a large set to the user is overwhelming. Instead, using the *URM* algorithm while setting the output size to 6 we find the set $\{$ Wilt Chamberlain - 1967, Don Buse -1975, Nate Archibald - 1972, Michael Jordan - 1987, Wilt Chamberlain - 1961, John Stockton - 1988$\}$ with maximum regret-ratio of 0.058. That is, the user can make selection between these 6 tuples, yet be sure that the score (quality) of its selection is not more than 5.8% percent worse than the optimal choice. For example, for the ranking function Assists+Steals, the optimal tuple is *John Stockton 1990* with the score of 1398. The tuple *John Stockton - 1988* in the representative set has the score of 1381 which shows how close to optimal it is.

6.4 Results for Avg Regret-Ratio

Comparison against the global minima: *URM* is an iterative algorithm where the regret ratio score monotonically decreases over time. That is, it produces better results as time passes. In contrast, the GREEDY-SHRINK algorithm generates results only at program termination. For a fair comparison between the two algorithms, we run GREEDY-SHRINK with different parameter values of ϵ and σ. We also demonstrate the ability of our algorithm to find the global minima given sufficient time. Therefore, in addition to the GREEDY-SHRINK algorithm we compare against the global minima, the computation of which is described in § 6.1. Concretely, we let the *URM* algorithm finish multiple iterations and recorded the regret ratio at the end of each iteration. Finally, we have compared the progression of these scores against the global minima along with the results from GREEDY-SHRINK. For this set of experiments, we have used 20,000 points from the *Colors* dataset, in a 9 dimensional space. As discussed in § 5, instead of computing the exact volumes, we approximate the average regret ratio score using N samples with an ϵ of 0.01 and a confidence of 0.999 as described in 13. We present the results of our experiments in Fig. 13. The black dotted line indicates the global minima while the red triangles show the output from the GREEDY-SHRINK algorithm. The *URM* algorithm outperforms the GREEDY-SHRINK algorithm for different output sizes of k. In all cases *URM* finds a representative with small value of average regret ratio in a small amount of time. In fact, our experiments show that it is possible to reach the global minima of average regret ratio in a short time.

6.5 Results for skyline reducing algorithms

To compare our algorithm with skyline reducing algorithms, we have implemented the algorithms KRSPGREEDY [30], ϵ-*ADR* [33], EIQUE [31], NAIVEGREEDY [32], and SKYCOVER [34]. NAIVEGREEDY takes the skyline and a starting tuple as input to output a representative. To be fair in comparison, we passed every point from the skyline as a starting point and chose the set with least regret ratio. ϵ-*ADR* and SKYCOVER take an error parameter ϵ as the input and output the smallest set size satisfying ϵ-*ADR* optimization criteria. To find a set of size k or smaller, we find a large ϵ which satisfies the set size criteria and apply a binary search to get the smallest ϵ satisfying the size requirement. The results for the DOT dataset with dimensions 4 and 5 are presented in Fig. 12 and Fig. 14. We denote KRSPGREEDY as *RSP*, NAIVEGREEDY as *NG*, EIQUE as *EQ*, ϵ-*ADR* greedy as ϵ-*ADR* and SKYCOVER as *SC* in the plots. The blue bar shows the regret ratio of the output of the skyline reducing algorithms, with the name of the algorithm on the *X* axis. The orange bars show the regret ratio of the output of our (*URM*) algorithm when initialized with the corresponding algorithm's output. As expected *URM* outperforms other algorithms.

7. RELATED WORK

Over the past few decades, a major amount of research has focused on the generation of a representative of the dataset to assist users with multi-criteria decision making. Most of these published works can be grouped into skyline discovery, skyline reduction and *regret* based compact representative computations. In this section, in addition to discussing the related works in these three categories we also provide a brief summary of the clustering techniques that have inspired the *URM* algorithm.

Regret minimizing representatives: As an effective solution to the problem of finding compact representatives, Nanongkai et al. [10] introduced regret-ratio minimizing representative. The notion of regret introduced in the paper deals with the amount of dissatisfaction the user would express when she is shown the top item from the representative set instead of the top item from the dataset when provided with the user preference. Many different variations of the original regret-minimizing representative problem have been studied since this paper [16, 35]. One specific variant, finding a representative set that minimizes the maximum regret ratio, has been extensively studied by several researchers. Agarwal et al. [12] proved that this problem is NP-complete for dimensions larger than 2. Asudeh et al. [11] introduce function space discretization and the transformation of the problem to set-cover instances. [11, 12] propose approximation algorithms with similar tunable approximation guarantees for the problem. Another interesting variant of this problem is finding the average regret ratio problem. It was introduced by Zeigami et al. [14, 15]. However, it is important to note that while others have studied and proposed solutions for single variants of the regret measures, we have proposed an unifying algorithm that can work with a class of regret measures.

Skyline and convex hull: In the pursuit of efficiency, finding a small set of representatives of the entire dataset has been of key interest in recent years. In the absence of user preferences, skyline [8, 36, 37] and convex hull based algorithms obtain a part of the dataset which behave like representatives. While convex hull is more compact than the skyline, the computation of a convex hull is significantly more costly in large dimensions. Bentley et al have studied approximation algorithms with tight approximation ratios for convex hulls [38]. However, as we move to higher dimensions, most points in the dataset appear on the skyline or convex hull. As a result, even though these skyline/convex hull based algorithms are effective in lower dimensions, the large size of skyline and convex hull render them ineffective as representatives.

Skyline reducing algorithms: Existing works have studied the problem of reducing the size of the skyline in various scenarios [30–34, 39–41]. To the best of our knowledge, none, except the regret-minimizing literature, incorporate user customizable functions for skyline reduction nor can be directly translated to the regret ratio problem. [30] and [39] reduce the skyline size by choosing a subset of size *k* of it which maximizes the number of dominated tuples by this subset. [32] proposes distance-based skyline, a subset of size k that minimizes the sum of the distances between the points and their closest representative. [41] and [31] find a subset of size *k* with maximum diversity. [42] propose the concept of skyline ordering, which is skyline-based partitioning of a given data set such that an order exists among the partitions. This ordering is then exploited to reduce the set size to *k*. We note that all of the aforementioned have a different objective function than regret-ratio and cannot provide any guarantee on how good it is for an arbitrary function.

Similarly, [33] and [34] propose a new measure, ϵ-*ADR* query, which chooses a subset of the skyline such that the tuples of the skyline when scaled by (1+ ϵ), dominate the rest of the skyline tuples. We illustrate the difference between the optimization measure of ϵ-*ADR* query and regret ratio with an example. Consider a dataset with *n* points (in 2D), out of which (n-2) are equi-angularly placed on the surface of a circle with radius *10* and the two other points are $\langle 20, 1\rangle$ and $\langle 1, 20\rangle$. Note that all the points belong to the skyline. Figure 8 shows this example for $n = 7$. The optimal set for size 2 is (the convex hull) $\{\langle 20, 1\rangle, \langle 1, 20\rangle\}$ with max regret ratio 0. On the other hand, ϵ-*ADR* query chooses a subset of the skyline such that the tuples of the skyline when scaled by (1+ ϵ), dominate the rest of the skyline tuples. As both $\langle 20, 1\rangle$ and $\langle 1, 20\rangle$ have one of their attributes set to *1*, the scaling required for these tuples to dominate the other tuples is large making them less attractive to the ϵ-*ADR* query. For the example of Figure 8, ϵ-*ADR* [33] returns $\{\langle 8.66, 5.0\rangle, \langle 5.0, 8.66\rangle\}$. Note that, instead of $\langle 20, 1\rangle$ and $\langle 1, 20\rangle$, we could add points $\langle X, 1\rangle$ and $\langle 1, Y\rangle$ with X and Y such that there exists a tuple $\langle a, b\rangle$ from the database which has the property $\frac{X}{a} < b$ and $\frac{Y}{b} < a$. For example, we can increase X and Y to *49* as this dataset contains the point $\langle 7.07, 7.07\rangle$ for which $\frac{49}{7.07} = 6.93 < 7.07$.

Clustering: Clustering have long been studied to partition data into similar groups. One of the major variants in this field is the K-MEDOID algorithm [17]. Like many other clustering algorithms, K-MEDOID is also an iterative algorithm that partitions the points into clusters by minimizing the distance between the points from the cluster center. However, the difference between the K-MEDOID algorithm and other clustering techniques is that K-MEDOID always chooses the cluster centroids from the existing data points. In the literature, there have been several variations of K-MEDOID that have explored several avenues of choosing the best centroid. Few of the most prominent are PAM, CLARA [43] and CLARANS [44].

8. FINAL REMARKS

In this paper, we proposed a unified algorithm for solving a variety of "regret-minimizing representative" problems for different aggregate norms. Unlike the existing work that design approximation algorithms for a specific variant of problem, we design an iterative optimization algorithm to search for optima for the general problem. To do so, we make connection with the problem of *clustering* from data mining, and the corresponding K-MEDOID algorithm. We propose several innovative technical contributions in designing our algorithm and provide theoretical analysis as well as empirical experiments that demonstrate the effectiveness of our proposal. The experiments also highlight useful properties of our approach that makes it a suitable practical algorithm.

Our work gives rise to several interesting optimization opportunities that deserve further study. For example, currently in *URM*, we update all regions concurrently. An interesting heuristic to update the regions one after the other with a specific ordering which might improve our algorithm. In particular, at every iteration, we could find the representative tuple with the smallest region by volume as a good candidate for being replaced. This, however, is challenging to develop and implement, as each update requires k expensive volume computations. We consider this, along with other possible optimizations and open problems for future work.

9. ACKNOWLEDGEMENTS

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