The Structure of an Investment Portfolio in Two-step Problem of Optimal Investment with One Risky Asset Via the Probability Criterion

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Abstract. At paper we investigate problem of the investment portfolio selection from one risky asset and one risk-free asset. We use the probability criterion for the investment portfolio selection. The possibility of rebalancing of the investment portfolio is used for diversification of the portfolio. We find an approximate analytical solution of the problem using the law of total probability. The investment portfolio is selected for various distributions of returns. We give an example.

Keywords: the two-step problem, the probability criterion, optimal control, the structure of the investment portfolio, diversification.

1 Introduction

The problem of optimal investment is the problem of assets selection for investment, so that the investor's capital will be maximal at the some instant of time in future. Therefore the cost function is investor's capital. Due to the fact that at the some instant of time prices of some assets are unknown, returns from these assets are random variables. Consequently, investor's capital will become a random variable with the purchase of these assets. In such a way optimization is possible after applying some criterions to the cost function: for an example, probabilistic [1], VaR [2], the Markowitz problem [3]. At paper we will use the probability criterion because the value of the probability criterion will give the probability of exceeding of threshold amount desired by the investor.

The two-step problem is considered for allowing rebalancing of the investment portfolio at the some instant of time. As shown in [4], optimal control at the first step is similar to logarithmic strategy. Note that logarithmic strategy provide maximal average growth of the investor's capital. At the same time there are not so many articles about two-step problem with the probability criterion or VaR criterion [4]-[6] because of complexity of such problems. More frequently one-step problems with various criterions and bounds [7]-[8] and multi-step problems with expectation or variance as criterion [9]-[10] are investigated. Another way to form the investment portfolio is usage of machine learning techniques [11]-[12].

Searching for optimal control, as done in [4]-[5] by means of dynamic programming method, is hard enough. That's why we need to find more simple approach for obtaining of some approximate solution, which will be close enough to optimal solution. One of such approaches is using of piecewise constant control at the second step rather than positional control. Such approach we will consider.

We will choose one risk-free asset and some risky asset (having non-zero variance) for the investment portfolio selection. Obviously, the structure of the investment portfolio depends on distributions parameters and distribution law. We will analyze the structure of the investment portfolio for various distributions of risky asset return.

2 Statement of the Problem

Assume that the investor have two assets for investment at the each trading period: risk-free asset with constant return b_0 and risky asset (stock, bond) with return $-1+\tilde{X}_1$ at the first step and risky asset with return $-1+\tilde{X}_2$ at the second step. Variables $\tilde{X}_1$ and $\tilde{X}_2$ are independent identically distributed random variables with existing first and second moments and density function $f(x)$. Variables $\tilde{X}_1$, $\tilde{X}_2$ characterize the ratio of sale price of risky asset to the purchase price. State some realizations of sample mean $\overline{x}_n$ and sample variance $\overline{s}_n^2$ of $-1+\tilde{X}_1$. Assume $u_{0,i}$ is the fraction of the investor's capital invested in risk-free asset at the trading period i and $u_{1,i}$ is the fraction of the investor's capital invested in risky asset at the trading period i, C_1 is initial investor's capital and φ is amount desired by the investor. Assume «short-sales» are banned and investor's capital is invested fully at the each trading period. The dynamics of investor's capital is then described by

$$C_{j+1} = C_j \left(1 + u_{0,j} b_0 + u_{1,j}(-1+\tilde{X}_j)\right), j = 1, 2,$$

and control $u_j \stackrel{\text{def}}{=} \text{col}(u_{0,j}, u_{1,j})$ is selected at the each step from set

$$U \stackrel{\text{def}}{=} \{y_0, y_1 : y_0 + y_1 = 1, y_0 \geq 0, y_1 \geq 0\}.$$

We consider next sequence of segments which are all set of possible values of investor's capital C_2

$$s_0 \stackrel{\text{def}}{=} (-\infty, C^1(N)), s_1 \stackrel{\text{def}}{=} [C^1(N), C^2(N)), s_2 \stackrel{\text{def}}{=} [C^2(N), C^3(N)), \ldots,$$

$$s_N \stackrel{\text{def}}{=} [C^N(N), C^{N+1}(N)), s_{N+1} \stackrel{\text{def}}{=} [C^{N+1}(N), +\infty),$$

where

$$C^i(N) = 2C_1(i-1)\frac{m}{N}, i = \overline{1, N+1},$$

and

$$m = \int\limits_{-\infty}^{+\infty} x f(x) dx,$$

where N is a priori specified natural number, which characterizes the number of segments of the decomposition, m is expectation of random variable $\tilde{X}_1$. Such selection of values $C^i(N)$ is connected with the fact that fineness of s_i tends to zero subject to $N \to \infty$. Segments s_0 and s_{N+1} is saved constant, as segment s_0 characterizes infeasible on practice values of investor's capital, as without debts we cannot obtain capital is less than zero. Segment s_{N+1} characterizes values with small probability because obtaining return higher than 100 per cents is almost impossible in practice.

We will find control at the second step u_2 as piecewise-constant strategy depending on segment s_i. Thus the problem of searching for optimal control is described by

$$\tilde{P}_\varphi(u_1(C_1), u_2(s_0, s_1, \ldots, s_N, s_{N+1})) \stackrel{\text{def}}{=}$$
$$\stackrel{\text{def}}{=} \mathcal{P}\{C_3(u_1(C_1), u_2(s_0, s_1, \ldots, s_N, s_{N+1})) \geq \varphi\}. \quad (1)$$

Formulate the problem

$$(\tilde{u}_1^\varphi(\cdot), \tilde{u}_2^\varphi(\cdot)) =$$
$$= \arg \max_{u_1(C_1) \in U, u_2(s_0, s_1, \ldots, s_N, s_{N+1}) \in U} \tilde{P}_\varphi(u_1(C_1), u_2(s_0, s_1, \ldots, s_N, s_{N+1})). \quad (2)$$

3 The Search of Approximate Value of the Probability Functional

According to the law of total probability [13] we get

$$\mathcal{P}\{C_3 \geq \varphi\} = \sum_{i=0}^{N+1} \mathcal{P}\{C_2 \in s_i\} \mathcal{P}\{C_3 \geq \varphi | C_2 \in s_i\}. \quad (3)$$

Using the definition of conditional probability we have

$$\mathcal{P}\{C_2 \in s_i\} \mathcal{P}\{C_3 \geq \varphi | C_2 \in s_i\} = \mathcal{P}\{\{C_2 \in s_i\} \cdot \{C_3 \geq \varphi\}\} =$$
$$= \mathcal{P}\{\{C_2 \in s_i\} \cdot \{C_2(1 + u_{0,2}(s_i) b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2))\} \geq \varphi\}.$$

There are following nestings for $i = \overline{1, N}$

$$\{C_2 \in s_i\} \cdot \{C^i(N)(1 + u_{0,2}(s_i) b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\} \subset$$
$$\subset \{C_2 \in s_i\} \cdot \{C_2(1 + u_{0,2}(s_i) b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\} \subset$$
$$\subset \{C_2 \in s_i\} \cdot \{C^{i+1}(N)(1 + u_{0,2}(s_i) b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\}.$$

Consequently,

$$\mathcal{P}\{C_2 \in s_i\} \mathcal{P}\{C^i(N)(1 + u_{0,2}(s_i) b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\} \leq$$

$$\leq \mathcal{P}\{\{C_2 \in s_i\} \cdot \{C_2(1 + u_{0,2}(s_i)b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\} \leq$$
$$\leq \mathcal{P}\{C_2 \in s_i\}\mathcal{P}\{C^{i+1}(N)(1 + u_{0,2}(s_i)b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\}.$$

Note $C^{i+1}(N) - C^i(N) \to 0$ subject to $N \to \infty$, therefore we use the new functional for obtaining approximate value of functional (1), written also as (3)

$$\hat{P}_\varphi(u_1(C_1), u_2(s_0, s_1, \ldots, s_N, s_{N+1})) \stackrel{\text{def}}{=}$$
$$\stackrel{\text{def}}{=} \sum_{i=1}^{N+1} \mathcal{P}\{C_2 \in s_i\}\mathcal{P}\{C^i(N)(1 + u_{0,2}(s_i)b_0 + u_{1,2}(s_i)(-1 + \tilde{X}_2)) \geq \varphi\}. \quad (4)$$

And formulate the new problem

$$(\hat{u}_1^\varphi(\cdot), \hat{u}_2^\varphi(\cdot)) =$$
$$= \arg \max_{u_1(C_1)\in U, u_2(s_0,s_1,s_2,\ldots,s_N,s_{N+1})\in U} \hat{P}_\varphi(u_1(C_1), u_2(s_0, s_1, \ldots, s_N, s_{N+1})). \quad (5)$$

The solution of problem (5) is approximate solution of problem (2).

4 The Solution of the Problem at the First Step and at the Second Step

For solving (5) we solve

$$P_\varphi^i(u_{0,2}, u_{1,2}) \stackrel{\text{def}}{=}$$
$$\stackrel{\text{def}}{=} \mathcal{P}\{C^i(N)(1 + u_{0,2}b_0 + u_{1,2}(-1 + \tilde{X}_2)) \geq \varphi\} \to \max_{u_{0,2}+u_{1,2}=1, u_{0,2}\geq 0, u_{1,2}\geq 0}.$$

because controls at the first step and at the second step are independent and probabilities $\mathcal{P}\{C_2 \in s_i\}$ are non-negative. Solution of last problem we can find in [1]:

$$P_i = \max_{u_{0,2}+u_{1,2}=1, u_{0,2}\geq 0, u_{1,2}\geq 0} P_\varphi^i(u_{0,2}, u_{1,2}) =$$
$$= \begin{cases} 1, & \varphi \leq C^i(N)(1 + b_0), \\ 1 - F(\varphi/C^i(N)), & \varphi > C^i(N)(1 + b_0), \end{cases}$$

where

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

To find strategies at the first step we solve problem

$$P_\varphi(u_{0,1}, u_{1,1}) \stackrel{\text{def}}{=} \sum_{i=1}^{N+1} \mathcal{P}\{C_2 \in s_i\}P_i \to \max_{u_{0,1}+u_{1,1}=1, u_{0,1}\geq 0, u_{1,1}\geq 0}. \quad (6)$$

Reduce the dimension of the problem by substitution $u_{0,1} = 1 - u_{1,1}$ and simplify problem (6)

$$\sum_{i=1}^{N+1} \mathcal{P}\{C_1(1 + b_0 - u_{1,1}(1 + b_0) + u_{1,1}\tilde{X}_1) \in s_i\}P_i \to \max_{0 \le u_{1,1} \le 1}.$$

Note at the point $u_{11} = 0$ value of function $P_\varphi(u_{0,1}, u_{1,1})$ is equal to

$$P_\varphi(1, 0) = \sum_{i=1}^{N+1} \mathcal{P}\{C_1(1 + b_0) \in s_i\}P_i.$$

Find value of $P_\varphi(u_{0,1}, u_{1,1})$ subject to $u_{1,1} > 0$. Note

$$\mathcal{P}\{C_1(1 + b_0 - u_{1,1}(1 + b_0) + u_{1,1}\tilde{X}_1) \le a\} =$$

$$= F\left(\frac{a - C_1(1 + b_0 - u_{1,1}(1 + b_0))}{C_1 u_{1,1}}\right),$$

If $i = \overline{1, N}$ we have

$$\mathcal{P}\{C_1(1 + b_0 - u_{1,1}(1 + b_0) + u_{1,1}\tilde{X}_1) \in s_i\} =$$

$$= \mathcal{P}\{C^i(N) \le C_1(1 + b_0 - u_{1,1}(1 + b_0) + u_{1,1}\tilde{X}_1) < C^{i+1}(N)\} =$$

$$= F\left(\frac{C^{i+1}(N) - C_1(1 + b_0 - u_{1,1}(1 + b_0))}{C_1 u_{1,1}}\right) -$$

$$-F\left(\frac{C^{i}(N) - C_1(1 + b_0 - u_{1,1}(1 + b_0))}{C_1 u_{1,1}}\right).$$

If $i = N + 1$ we have

$$\mathcal{P}\{C_1(1 + b_0 - u_{1,1}(1 + b_0) + u_{1,1}\hat{X}_1) \in s_{N+1}\} =$$

$$= \mathcal{P}\{C^{N+1}(N) \le C_1(1 + b_0 - u_{1,1}(1 + b_0) + u_{1,1}\tilde{X}_1) < +\infty\} =$$

$$= 1 - \left(\frac{C^{N+1}(N) - C_1(1 + b_0 - u_{1,1}(1 + b_0))}{C_1 u_{1,1}}\right).$$

Based on the above, we get

$$P_\varphi(u_{0,1}, u_{1,1}) =$$

$$= \sum_{i=1}^{N} F\left(\frac{C^{i+1}(N) - C_1(1 + b_0 - u_{1,1}(1 + b_0))}{C_1 u_{1,1}}\right)(-P_{i+1} + P_i) + P_{N+1} -$$

$$-F\left(\frac{C^{1}(N) - C_1(1 + b_0 - u_{1,1}(1 + b_0))}{C_1 u_{1,1}}\right)P_1.$$

To find the solution of problem (6) we do close-meshed computational grid from interval $0 \le u_{1,1} \le 1$ and find the maximal value at grid nodes: node which has the maximal value is approximate solution of the first step [14].

Find distribution parameters of random variables $\tilde{X}_1$ and $\tilde{X}_2$ using relations of the method of moments [13] for often used in financial mathematics normal, log-normal, uniform distributions.

If $\tilde{X}_1$ is normal, then density function $f(x)$ is equal to

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Consequently, $\mathbf{M}[\tilde{X}_1] = \mu$, $\mathbf{D}[\tilde{X}_1] = \sigma^2$, where $\mathbf{M}[\tilde{X}_1]$ is mean, and $\mathbf{D}[\tilde{X}_1]$ is variance. Thus using realizations of return we obtain next relations for μ and σ^2

$$\begin{cases} \mathbf{M}[-1+\tilde{X}_1] = -1+\mu = \overline{x}_n, \\ \mathbf{D}[-1+\tilde{X}_1] = \sigma^2 = \overline{s}_n^2. \end{cases}$$

If $\tilde{X}_1$ is log-normal, then density function $f(x)$ is equal to

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right\}$$

subject to $x > 0$ and 0, otherwise. Thus $\mathbf{M}[\tilde{X}_1] = \exp\{\mu + \sigma^2/2\}$, $\mathbf{D}[\tilde{X}_1] = (\exp\{\sigma^2\} - 1)\exp\{2\mu + \sigma^2\}$

$$\begin{cases} \mathbf{M}[-1+\tilde{X}_1] = -1+\exp\{\mu+\sigma^2/2\} = \overline{x}_n, \\ \mathbf{D}[-1+\tilde{X}_1] = (\exp\{\sigma^2\}-1)\exp\{2\mu+\sigma^2\} = \overline{s}_n^2. \end{cases}$$

solving this system we get

$$\sigma^2 = \ln\left\{\frac{\overline{s}_n^2}{(\overline{x}_n+1)^2} + 1\right\},$$

$$\mu = \ln(\overline{x}_n + 1) - \frac{\sigma^2}{2}.$$

If $\tilde{X}_1$ is distributed by uniform law, then density function $f(x)$ is equal to

$$f(x) = \frac{1}{B-A}$$

subject to $x \in [A, B]$ and 0, otherwise. Therefore $\mathbf{M}[\tilde{X}_1] = (A + B/2)$, $\mathbf{D}[\tilde{X}_1] = (B-A)^2/12$

$$\begin{cases} \mathbf{M}[-1+\tilde{X}_1] = -1+\frac{A+B}{2} = \overline{x}_n, \\ \mathbf{D}[-1+\tilde{X}_1] = \frac{(B-A)^2}{12} = \overline{s}_n^2. \end{cases}$$

Solving last system we get

$$A = 1 + \overline{x}_n - \sqrt{3}\overline{s}_n,$$

$$B = 1 + \overline{x}_n + \sqrt{3}\overline{s}_n.$$

5 Example

Compare for accuracy obtained relations with exact solution provided in [5]. Assume initial investor's capital $C_1 = 1$, desired capital level $\varphi = 1,08$, return from risk-free asset $b_0 = 0,03$, X_1 is uniformly distributed random variable with parameters $A = 0$, $B = 2,2$ at the example No 1 and $A = 0$, $B = 2,3$ at the example No 2. We find approximate strategies of the first step and the value of probability functional (6) at this strategy for various value N. Mesh width is $0,01$. We will mark exact solution by bold font.

Table 1. Comparison approximate strategy with exact solution

No. example	N	$u_{0,1}$	$u_{1,1}$	$\mathcal{P}(C_3 \geq \varphi)$	Time of computations, sec.
1		**0.8326**	**0.1674**	**0.732**	
	100	0.81	0.19	0.7199	1,14
	500	0.82	0.18	0.7276	4,19
	1000	0.83	0.17	0.7306	7,51
	1500	0.83	0.17	0.7316	11,85
	3000	0.83	0.17	0.7318	25,21
2		**0.8326**	**0.1674**	**0.7548**	
	100	0.8	0.2	0.7416	1,87
	500	0.83	0.17	0.7539	5,03
	1000	0.83	0.17	0.7542	8,11
	1500	0.83	0.17	0.7543	11,51
	3000	0.83	0.17	0.7544	26,9

Note that exact solution was found in positional strategy class. But nonetheless as follows from table No 1 approximate strategy and approximate value of the probability functional is almost identical with exact solution when $N \geq 1000$.

Now analyze the structure of the optimal investment portfolio obtained with using derived above relations. Assume $\overline{x}_n = 0,1$, and $\overline{s}_n = 0,15$ and $\varphi = 1,08$, $b_0 = 0,03$, $C_1 = 1$ again. We find the investment portfolio for normal distribution at the example No 3, for log-normal distribution at the example No 4, for uniform distribution at the example No 5. We find approximate strategies of the first step and the value of probability functional (6) at this strategy for various value N again. Mesh width is 0,01.

As follows from table No 2 growth of value N does not allow to significantly increase the value of criterion $\mathcal{P}(C_3 \geq \varphi)$ after $N \geq 1000$. At the same time the value of $\mathcal{P}(C_3 \geq \varphi)$ criterion and the structure of the investment portfolio is almost identical although in each example various distributions were used.

Table 2. The structure of the optimal investment portfolio

No. example	N	$u_{0,1}$	$u_{1,1}$	$\mathcal{P}(C_3 \geq \varphi)$	Time of computations, sec.
3	100	0.47	0.53	0.7804	11,54
	500	0.54	0.46	0.7993	48,77
	1000	0.56	0.44	0.8048	92,08
	1500	0.57	0.43	0.8067	142,54
	3000	0.57	0.43	0.8071	257,17
4	100	0.42	0.58	0.7265	2,54
	500	0.49	0.51	0.782	7,44
	1000	0.52	0.48	0.7875	15,67
	1500	0.53	0.47	0.7895	28,46
	3000	0.53	0.47	0.7899	93,67
5	100	0.45	0.55	0.7684	2,9
	500	0.54	0.46	0.7839	10,52
	1000	0.57	0.43	0.7885	14,74
	1500	0.57	0.43	0.79	17,56
	3000	0.57	0.43	0.7904	35,04

6 Conclusion

In this work we have studied the two-step problem of optimal investment with one risky asset using the probability as optimality criterion. We have found function which approximates criterial function and have proposed the algorithm to optimize this function. Various cases of distribution of returns were investigated and we have found the structure of the optimal investment portfolio is almost identical despite of one or another distribution.

References

1. Kan Yu.S., Kibzun A.I. *Zadachi stokhasticheskogo programmirovaniya s veroyatnostnymi kriteriyami* (Problems of Stochastic Programming with Probabilistic Criteria). Moscow: Fizmatlit, 2009.
2. Jorion P. Value at Risk: The New Benchmark for Managing Financial Risk. Irwin Professional Publishing, 1997.
3. Markowitz H.M. Portfolio selection.// The Journal of Finance, 1952. V. 7, No. 1, P. 77-91.
4. Kibzun A.I., Ignatov A.N. The two-step problem of investment portfolio selection from two risk assets via the probability criterion// Autom. Remote Control. 2015. V. 76. No. 7. P. 1201–1220.
5. Grigor'ev P.V., Kan Yu.S. Optimal Control of the Investment Portfolio with Respect to the Quantile Criterion // Autom. Remote Control. 2004. V. 65. No. 2. P. 319–336.
6. Kibzun A.I., Kuznetsov E.A. Optimal Control of the Portfolio // Autom. Remote Control. 2001. V. 62. No. 9. P. 1489–1501.

7. Benati S. The optimal portfolio problem with coherent risk measure constraints // European Journal of Operational Research, 2003. V. 150. P. 572–584.
8. Benati S., Rizzi R. A mixed integer linear programming formulation of the optimal of the optimal mean/Value-at-Risk portfolio problem // European Journal of Operational Research, 2007. V. 176. P. 423–434.
9. Calafiore G. Multi-period portfolio optimization with linear control policies. // Automatica. 2008. V.44. I. 10. P. 2463–2473.
10. Skaf J., Boyd S. Multi-period portfolio optimization with constraints and transactions costs. https://stanford.edu/~boyd/papers/dyn_port_opt.pdf. 2009.
11. Li B., Zhao P., Hoi S.C.H., et al. PAMR: Passive Aggressive Mean Reversion Strategy for Portfolio Selection // Machine Learning. 2012. V. 87. No. 2. P. 221–258.
12. Li B., Hoi S.C.H. On-Line Portfolio with Moving Average Reversion // Int. Conf. on Machine Learning (ICML). Edinburgh, Scotland, 2012.
13. Kibzun A.I., Goryainova E.R., Naumov A.V. *Teoriya veroyatnostei i matematicheskaya statistika. Bazovyi kurs s primerami i zadachami* (Probability Theory and Mathematical Statistics. Basic Course with Examples and Problems). Moscow: Fizmatlit. 2007.
14. Polyak B.T. *Vvedenie v optimizatsiyu* (Introduction to Optimization). Moscow: Nauka, 1983.