Reconstruction of images smeared uniformly and non-uniformly

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Abstract. In the work, the following two variants of the direct and inverse problems about image smearing along a rectilinear trajectory are compared: 1) Uniform smearing, the same at all points of the image (smear Δ = const). This variant is described by a set of one-dimensional Fredholm integral equations (IEs) of the first kind of convolution type with direction of x axis along the smear trajectory and y is perpendicular to a smear, as well as by one two-dimensional IE of convolution type, moreover the axis x is directed horizontally and y vertically down. IEs are solved by Tikhonov regularization (TR) method (since the problem for solving them is ill-posed) and Fourier transform (FT). 2) Non-uniform image smearing of several moving objects (smear $\Delta = \Delta\ (x)$). This variant is described by IE of general type and solved by TR method and quadrature method (in case of set of one-dimensional IEs) or cubature method (in case of one two-dimensional IE). It is shown that in case of non-uniform smear, use of a set of one-dimensional IEs is preferable to a two-dimensional IE. The results of numerical experiments are obtained.

Keywords: Smeared image, Rectilinear smear, Uniform and non-uniform smeares, Integral equations, Tikhonov regularization method, MatLab.

1 Introduction

Consider one of the actual problems of distorted image processing – the elimination of image smearing via mathematical processing ([1–6], et al.). Smearing may be due to a shift of the image recording device – IRD (digital photo camera, videocamera, tracking device) or the motion of the object itself (one person or several people, cars, aircraftes) during the exposure. The problem of mathematical elimination of smear consists of two problems: a direct problem (smear modeling) and an inverse problem (smear elimination).

To date, in a number of publications, the variant of rectilinear uniform image smearing is considered in detail [1, 4–8], but the rectilinear non-uniform smearing is not considered in detail [1] and the arbitrary (non-uniform non-rectilinear) smearing is considered altogether briefly (the “blind” deconvolution method [6, p. 192]).

The purpose of this work is a comparative consideration of two variants for straight-line image smearing – uniform and non-uniform one. Example: a smeared image of runners on a track, running at the same, as well as at different speeds obtained by a fixed IRD. Note that in [1], the case was considered when an IRD during the exposure moved rectilinear with some (known) speed $v(t)$. In this paper, we consider the case when the smear $\Delta(x)$ of the objects themselves is known.

First, we recall the well-known case of uniform rectilinear smear [4, 5, 7–9].

2 The mathematical description of uniform rectilinear image smearing

Consider the direct and inverse problems.

2.1 The direct problem

The *direct problem* of uniform and rectilinear smear is described by an integral [4, 9]:

$$g_y(x) = \frac{1}{\Delta} \int_x^{x+\Delta} w_y(\xi)\, d\xi, \tag{1}$$

where $\Delta = \text{const}$ is smear value; the x and ξ axes are directed along a smear, and the y axis is perpendicular to a smear (y plays the role of a parameter); w_y is the given non-smeared image, and g_y is the calculated smeared image in each y-line. To calculate g according to (1), we developed m-function smearing.m [9], as well as smear.m, a simplified version of smearing.m when the smear angle $\theta = 0$, while in MatLab there are m-functions fspecial.m and imfilter. m [6] for modeling g.

2.2 The inverse problem

The *inverse* (more important and complex) *problem* can be solved in *two approaches.*

In the *first approach*, a set of one-dimensional Fredholm integral equations (IEs) of the first kind of convolution type (for each value of *y*) is solved to eliminate the smearing [4, 7–9]:

$$\int_{-\infty}^{\infty} h(x-\xi)\, w_y(\xi)\, d\xi = g_y(x), \quad -\infty < x < \infty, \tag{2}$$

where

$$h(x) = \begin{cases} 1/\Delta, & -\Delta \le x \le 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

IE (2) is obtained from relatio (1). Here, *h* is mathematically the kernel of IE, and physically and technically it is the *point spread function* (PSF) [2, 5, 9, 10]. The PSF is what each point of the object turns into on the image when smearing (in a stroke). In the smearing problem, the function *h* is usually differential or spatially invariant, which means that the smear is uniform and the smear value Δ is the same at all points of the image.

The problem of solving IE (2) is ill-posed [11, 12]. Therefore, we use the stable Tikhonov regularization method (TRM) with Fourier transform (FT) [5, 9, 11]:

$$w_{\alpha y}(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_{\alpha y}(\omega)\, e^{-i\omega\xi}\, d\omega , \tag{4}$$

where

$$W_{\alpha y}(\omega) = \frac{H(-\omega)\, G_y(\omega)}{|H(\omega)|^2 + \alpha\, \omega^{2p}} \tag{5}$$

is the regularized spectrum, or FT of the solution; $H(\omega) = F(h(x))$ and $G_y(\omega) = F(g_y(x))$ are the Fourier spectra of functions $h(x)$ and $g_y(x)$, where F is a FT symbol; $\alpha > 0$ is the regularization parameter; $p \ge 0$ is the regularization order (usually $p = 1$ or 2). To select the parameter α, a number of methods have been developed: the discrepancy principle, the method of teaching example-images, etc. [9, p. 236], [11]. The calculation of a restored image by the formulas (4)–(5) is carried out according to the developed m-function desmearingf.m [9, p. 137, 330].

In the *second approach*, a two-dimensional Fredholm IE of the first kind of convolution type (cf. (2)) is used to eliminate the smearing (and the defocusing) [3, 4–9]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\xi, y-\eta)\, w(\xi,\eta)\, d\xi\, d\eta = g(x,y), \quad -\infty < x, y < \infty, \tag{6}$$

where x and ξ axes are horizontal, and y and η are vertically down. In this case, the PSF h will be displayed on the plane (x,y) as a narrow strip (Fig. 1) [9, p. 112]:

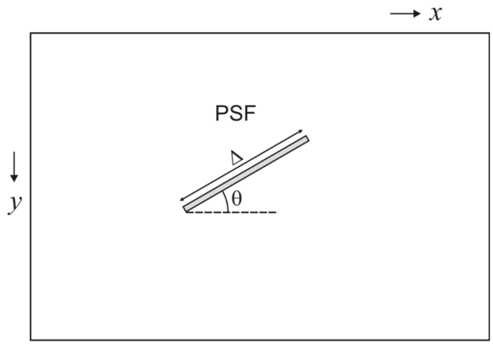

Fig. 1. The point spread function (PSF) $h(x,y)$ in the form of a narrow strip of length Δ at angle θ.

In this approach, the calculation of the direct problem is based on the m-functions fspecial.m and imfilter.m [6]. And the solution of two-dimensional IE (6) (inverse problem) by the TR method and two-dimensional FT is equal to $w_\alpha(x,y) = F^{-1}\big(W_\alpha(\omega_1,\omega_2)\big)$, where F^{-1} is the inverse Fourier transform (IFT) or

$$w_\alpha(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_\alpha(\omega_1,\omega_2)\, e^{-i(\omega_1 x+\omega_2 y)}\, d\omega_1\, d\omega_2\,. \tag{7}$$

In (7), $W_\alpha(\omega_1,\omega_2)$ is regularized spectrum (two-dimensional FT) of the solution, equal to (cf. (5))

$$W_\alpha(\omega_1,\omega_2) = \frac{H^*(\omega_1,\omega_2)\, G(\omega_1,\omega_2)}{|H(\omega_1,\omega_2)|^2 + \alpha\,(\omega_1^2+\omega_2^2)^p}\,, \tag{8}$$

where $H(\omega_1,\omega_2) = F\big(h(x,y)\big)$, $G(\omega_1,\omega_2) = F\big(g(x,y)\big)$. MatLab has the m-functi-on deconvreg.m [6] for solving the IE (6) by the TR and FT methods according to (7)–(8). We give the well-known formulas (1)–(8) in order to compare different approaches.

3 The mathematical description of non-uniform rectilinear image smearing

Based on relations (1)–(8), we consider the non-uniform rectilinear image smearing along the smear trajectory. Suppose that from a smeared picture, we determined in some way the dependence $\Delta = \Delta(x)$ of the smear Δ on coordinate x, directed along the smear.

3.1 The direct problem

In this case, the PSF h will not be difference, or spatially invariant and the *direct problem* will be written as (cf. (1)):

$$g_y(x) = \frac{1}{\Delta(x)} \int_x^{x+\Delta(x)} w_y(\xi)\, d\xi . \tag{9}$$

To calculate g according to (9), we developed the m-function smear_n.m.

3.2 The inverse problem

The *inverse problem* in case of the *first approach* is written as a set of one-dimensional Fredholm integral equations of the first kind of general type (for each value of y) [9, p. 125]:

$$Aw_y \equiv \int_a^b h(x,\xi)\, w_y(\xi)\, d\xi = g_y(x), \quad c < x < d , \tag{10}$$

where A is an integral operator; $[a,b]$ and $[c,d]$ are limits for ξ and x. PSF h will be written as:

$$h(x,\xi) = \begin{cases} 1/\Delta(x), & x \le \xi \le x + \Delta(x), \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

To solve IE (10), the FT cannot be applied, but the *quadrature method* is well suited and leads IE (10) to a system of linear algebraic equations (SLAE) [9, p. 126]:

$$Aw_y = g_y , \tag{12}$$

where A is the matrix associated with h (the same to all y-rows), w_y is the desired vector, g_y is the right-hand side of the SLAE. A stable solution of the SLAE (12) is given by the Tikhonov regularization method [9, p. 126]

$$(\alpha I + A^T A)\, w_{y\alpha} = A^T g_y \,, \tag{13}$$

where $\alpha > 0$ is the regularization parameter, I is the unit matrix, A^T is the transposed matrix, and $w_{y\alpha}$ is the regularized solution in y-row equal to

$$w_{y\alpha} = (\alpha I + A^T A)^{-1} A^T g_y \,. \tag{14}$$

For computer implementation of formulas (10)–(14), the m-function desmearq_n.m was developed.

Note that the quadrature method with Tikhonov's regularization (14) can also be used to solve a IE of convolution type (2) with PSF (3). The m-function desmearingq.m has been developed for this.

The *inverse problem* in case of the *second approach* can be written in the form of a two-dimensional Fredholm integral equation of the first kind of general type (cf. (6)):

$$Aw \equiv \int_a^b \int_c^d h(x,\xi,y,\eta)\, w(\xi,\eta)\, d\xi\, d\eta = g(x,y), \quad a \le x \le b, \quad c \le y \le d \,. \tag{15}$$

Equation (15) can be solved by a *quadrature method* (more precisely, cubature) (cf. [13, p. 167]). According to this method, each of the integrals in (15) is replaced by a finite sum on discrete node grids with respect to x, ξ, y, η and we obtain a SLAE with a four-dimensional matrix A and a two-dimensional right-hand side g. To solve such a SLAE, one needs to transform the four-dimensional matrix A into a two-dimensional one, two-dimensional right-hand side g to transform into a one-dimensional one, and the resulting one-dimensional solution w to transform into a two-dimensional one. Although the (successful) attempt to solve a two-dimensional IE by the cubature method took place [13, p. 167–169], nevertheless, this is a cumbersome method and its use for restoration of non-uniform smeared image is problematic.

It is also possible to apply for solving IE (15) an *iteration method*, for example, the Friedman iterative regularization method [13, p. 272], [14], which is simpler than the cubature method, but it requires a good choice of the initial approximation, knowledge of the parameter of the method ν, the number of iterations, etc.

As a result, it should be recognized that the most effective method is (10)–(14), based on line-by-line image processing, according to which one-dimensional IE (10) and SLAE (12) need to be solved with a two-dimensional matrix.

4 Illustrative example

The following *numerical example* was solved. The original image of seven runners on the track is shown in Fig. 2.

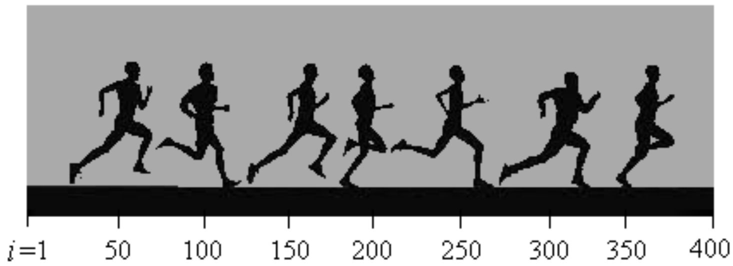

Fig. 2. Runners, file runners.jpg 123×400 px (i is counting number).

4.1 The direct problem

Fig. 3a shows the result of the direct problem – uniform horizontal smearing according to (1) via m-function smear.m. The magnitude of smear $\Delta = 20$ px.

We can see that image smearing is significant. The smearing is performed with diffusing the image edges ('diffusion' option) to reduce the Gibbs effect – the effect of false waves [9].

4.2 The inverse problem of uniform image smearing

Fig. 3b shows the result of solving the inverse problem – the line-by-line restoration of the image by the quadrature method with Tikhonov's regularization according to (14) using the developed m-function desmearq.m. Regularization parameter $\alpha = 10^{-6}$ (chosen by selection). We see that despite the considerable smearing (Fig. 3a), the image is well restored, and without the Gibbs effect due to diffusing the image edges (in Fig. 3a).

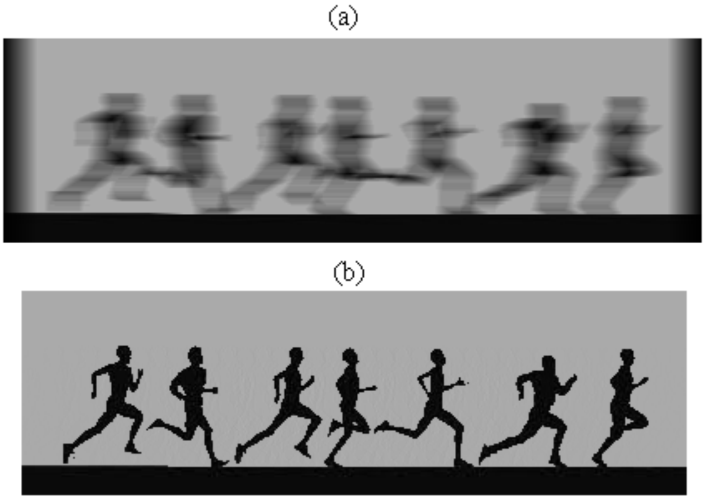

Fig. 3. The direct and inverse problems of uniform image smearing. a – smeared image of runners 123×420; b – restored image 123×400.

4.3 The inverse problem of non-uniform image smearing

The next step is *non-uniform image smearing*. We suppose that the runners run at different speeds v, which means that they have different smears $\Delta = v \cdot T$ during the exposure time T. On the basis of Figure 2, we determine the boundaries between the runners and the values of the smears Δ runners (see Table). As a result, a smear $\Delta(x)$ or $\Delta(i)$ is represented as a piecewise constant function. Each runner has its own smear value within his range.

Table

Number of runner	1	2	3	4	5	6	7
Range of values i	$1 \le i \le 75$	$75 < i \le 127$	$127 < i \le 182$	$182 < i \le 213$	$213 < i \le 274$	$274 < i \le 339$	$339 < i \le 400$
Value of Δ, px	5	8	11	14	17	20	23

Fig. 4a shows an image smeared piecewise non-uniformly, namely, the smearing increases from the left runner, for which smear is 5 px, to the right runner, for which smear is 23 px, i.e. smearing is substantially non-uniform. This smearing is performed according to (9) using the m-function smear_n.m (with diffusing the edges).

Fig. 4b shows the result of image restoration (the inverse problem) by the quadrature method with Tikhonov's regularization according to (10)–(14) using the m-function desmearq_n.m. Regularization parameter $\alpha = 5 \cdot 10^{-3}$ (chosen by selection). Fig. 4b shows that the images of the runners is restored, but with an uneven background.

4.4 The alignment of image background

The *background alignment* in Fig. 4b has been done: the intensity values in Fig. 4b more than 100 are replaced by 170 (this is the background value). Fig. 4c shows the final result of image restoration after background alignment. We see that the images of the runners are restored quite satisfactorily (cf. Fig. 3b).

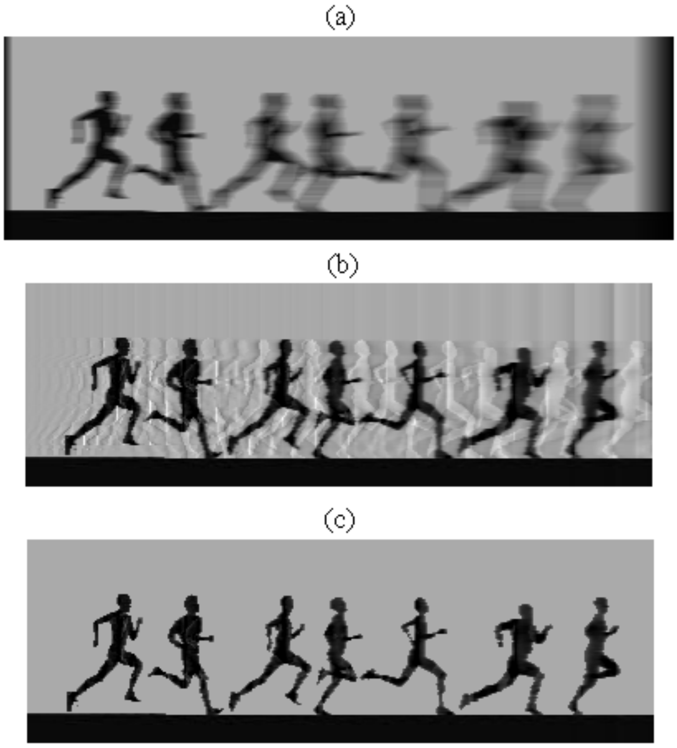

Fig. 4. The direct and inverse problems of non-uniformly image smearing. a – smeared image of runners 123×405; b – restored image 123×379; c – restored image with aligned background 123×379.

5 Conclusion and future plans

The described technique can be used in practice for restoring group images of several objects (people, airplanes, cars) moving at different speeds and therefore received different smears Δ on the image during the exposure by a fixed IRD. In subsequent publications, the question about a method for determining non-uniform smear $\Delta\ (x)$, as well as general case of smear $\Delta = \Delta\ (x,\ y)$ will be considered. Example: a smeared image of a stream of cars on a wide highway moving at different speeds. A comparison will also be made with the technique described in [1], which considers the non-uniform shift of a IRD. Finally, a variant of the regularization method with the variable regularization parameter $\alpha = \alpha(x)$ will be considered. This variant should take into account the different degrees of image distortion in Fig. 4a.

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