COMBINATORIAL INTERPRETATION OF BI^{s}NOMIAL COEFFICIENTS AND GENERALIZED CATALAN NUMBERS

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ABSTRACT

We provide a combinatorial interpretation of bi^{s}nomial coefficients, by using paths that lie on hypergrids. We also give a generalization of Catalan numbers, called as s-Catalan, through using s-Pascal triangle. Two identities of s-Catalan numbers are derived.

Keywords Bi^{s}nomial coefficients · s-Pascal triangle · Generalized Pascal Formula · Hypergrids · s-Catalan numbers.

1 Introduction

Bi^{s}nomial coefficients were introduced for the first time in 1730, by Abraham de Moivre [7], in his study to answer to the following question: "Considering L dices with $(s+1)$ numbered faces. If they are thrown randomly, what would be the chance of the sum of exhibited numbers to be equal to k ?", see also Hall and Knight [16]. Some years later, Euler [8, 9], studied these coefficients and derived a number of properties, as formulae (4), (6) below. In 1876, André [1] used combinations on words to establish several other properties.

Recently, the authors [3], published a paper that focused on a historical introduction of bi^{s}nomial coefficient, as well as a presentation of some new arithmetical properties of these numbers. First, we need to introduce some definitions and concepts concerning bi^{s}nomial coefficients, s-Pascal triangle and Catalan numbers.

1.1 Bi^{s}nomial Coefficients

Definition 1.1 *Let $s \geq 1, n \geq 0$ be integers and let $k \in \{0, 1, \ldots, sn\}$. The bi^{s}nomial coefficient denoted by $\binom{n}{k}_s$, is the coefficient of x^k in the following development*

$$(1 + x + x^2 + \cdots + x^s)^n = \sum_{k \geq 0} \binom{n}{k}_s x^k. \tag{1}$$

For $k < 0$ or $k > sn$, we have, $\binom{n}{k}_s = 0$. For $s = 1$, we get the classical binomial coefficient $\binom{n}{k}_1 = \binom{n}{k}$. In the literature of bi^{s}nomial coefficients, we often meet the following well known properties

- Expression of bi^{s}nomial coefficients in terms of binomial coefficients,

$$\binom{n}{k}_s = \sum_{j_1 + j_2 + \cdots + j_s = k} \binom{n}{j_1}\binom{j_1}{j_2} \cdots \binom{j_{s-1}}{j_s}. \tag{2}$$

- de Moivre alternate summation,

$$\binom{n}{k}_s = \sum_{j=0}^{\lfloor k/(s+1) \rfloor} (-1)^j \binom{n}{j}\binom{k - j(s+1) + n - 1}{n - 1}. \tag{3}$$

- Symmetry relation,

$$\binom{n}{k}_s = \binom{n}{sn-k}_s. \tag{4}$$

- Generalized Pascal Formula,

$$\binom{n}{k}_s = \sum_{m=0}^{s} \binom{n-1}{k-m}_s. \tag{5}$$

- Diagonal recurrence relation,

$$\binom{n}{k}_s = \sum_{m=0}^{n} \binom{n}{m} \binom{m}{k-m}_{s-1}. \tag{6}$$

By definition, Pascal triangle is the triangular array of binomial coefficients, where each of their elements is calculated by using Pascal Formula, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. We consider a generalization of Pascal triangle denoted by s-Pascal triangle, as the array of bi^{s}nomial coefficients that are generated by using Relation (5). For example, Table 1 gives the 3-Pascal triangle in the left justified form. We find the first values of bi^{s}nomial coefficients in SLOANE [22], through using the codes A027907, A008287 and A053343, for, $s = 2$, $s = 3$ and $s = 4$, respectively.

Table 1: Triangle of bi^{3}nomial coefficients $\binom{n}{k}_3$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1												
1	1	1	1	1									
2	1	2	3	4	3	2	1	0					
3	1	3	6	10	12	12	10	6	3	1			
4	1	4	10	20	31	40	44	40	31	20	10	4	1

1.2 Catalan Numbers

For a well introduction to Catalan numbers, their properties and combinatorial interpretations, the reader can refer to Stanley [23], Kochy [19]. Catalan numbers, named after the Belgian mathematician Eugène Charles Catalan (1814-1894), are defined as follows,

$$C_n = \frac{1}{n+1}\binom{2n}{n},\ n \in \mathbb{Z}^+. \tag{7}$$

The generating function of these numbers is,

$$C(x) = \sum_n C_n x^n = \frac{1-\sqrt{1-4x}}{2x}. \tag{8}$$

Catalan numbers are given in Sloane [22], by using the code A000108, the first elements are,

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, \ldots.$$

These numbers could be generated by subtracting the mentioned columns of Pascal triangle, as given in Table 2. This permit us to get the three Formulae, (9), (10), (11).

$$C_n = \binom{2n}{n} - \binom{2n}{n+1},\ n \geq 0. \tag{9}$$

$$C_n = \binom{2n-1}{n} - \binom{2n-1}{n+1},\ n \geq 1. \tag{10}$$

$$C_{n+1} = \binom{2n}{n} - \binom{2n}{n+2},\ n \geq 0. \tag{11}$$

In the following section, we give combinatorial interpretations of both bi^{s}nomial coefficients and generalized Pascal Formula, through using oriented paths that moving on Hypergrids.

Table 2: Right part of Pascal triangle.

1								
	1							
2		1						
	3		1					
6		4		1				
	10		5		1			
20		15		6		1		
	35		21		7		1	
70		56		28		8		1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$				
$\binom{2n}{n}$	$\binom{2n-1}{n}$	$\binom{2n}{n+1}$	$\binom{2n-1}{n+1}$	$\binom{2n}{n+2}$				

2 Combinatorial interpretation of bi^{s}nomial coefficients

Freund [10], gave a combinatorial interpretation of bi^{s}nomial coefficients $\binom{n}{k}_s$, as the number of different ways of distributing k objects among n cells, where each cell contains at most s objects, see also, Bondarenko [4]. Recently, A. Bazeniar et $al.$, [2], provided an interpretation of these numbers, as the number of lattice paths that connect the two points of a grid, $(0, 0)$ and $(k, n-1)$, for $0 \leq k \leq sn$, by taking at most s vertices in the eastern direction. We begin by giving some definitions and terminologies that we need in the rest of this paper.

2.1 Definitions and Notations

We denote by $H_{n,s}$, an hypergrid of dimension n, (we consider n ordered directions), such that each axis contains s vertices without counting the vertex of origin O. As particular cases, for $s = 1$ and $n \geq 4$, hypergrids are called $hypercubes$, whereas, for $n = 2$ and $s \geq 2$, we talk about $grids$.

Definition 2.1 *Let $n, s, p \in \mathbb{Z}^+$, $i \in \{1, 2, \ldots, n\}$. An up-oriented path lying on the hypergrid $H_{n,s}$, is a path of a finite length, such that*

1. *it starts from the vertex O,*
2. *when the path reaches the vertex U by taking the i^{th} direction, it should reach a vertex V by taking the $(i+p)^{th}$ direction.*

We denote by $p_{i_1,i_2,\ldots,i_n}$, an up-oriented path lying on the hypergrid $H_{n,s}$, that reached,

- i_1 vertices by taking the 1^{st} direction,
- i_2 vertices by taking the 2^{nd} direction,
- $\vdots$
- i_n vertices by taking the n^{th} direction,

with $0 \leq i_m \leq s$, for $m \in \{1, 2, \ldots, n\}$.

We represent the up-oriented path $p_{i_1,i_2,\ldots,i_n}$ by the linear form,

$$\underbrace{11\cdots1}_{i_1 \text{ times}}\underbrace{22\cdots2}_{i_2 \text{ times}}\cdots\underbrace{nn\cdots n}_{i_n \text{ times}},$$

or by the power form, $1^{i_1}2^{i_2}\cdots n^{i_n}$. We denote by the number k, the length of $p_{i_1,i_2,\ldots,i_n}$, such that $k = i_1 + i_2 + \cdots + i_n$, as well as $P_{n,k,s}$ the set of all $p_{i_1,i_2,\ldots,i_n}$ of length k that lie on the hypergrid $H_{n,s}$.

Example 2.1 *In Figure 1, we differentiate an up-oriented path from ordinary paths that lie on the grid $H_{2,3}$, as follows,*

- *The first path on the left is an up-oriented path because the directions are taken in an increasing order, then, we have, $p_{3,1} = 1112 = 1^3 2^1$.*

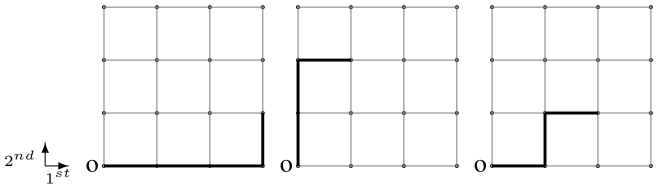

Figure 1: An up-oriented path and ordinary paths on the grid $H_{2,3}$.

- *The second and the third paths to the right, are not up-oriented paths due to a disorder on directions of the two paths.*

The following theorem gives a combinatorial interpretation of bi^{s}nomial coefficients by counting the cardinality of the set $P_{n,k,s}$.

Theorem 2.1 *For $n, k, s \in \mathbb{Z}_+$, with $0 \leq k \leq sn$, we have, $\#P_{n,k,s} = \binom{n}{k}_s$.*

Proof 2.1 *For $n = 0, 1, 2$, it is easy to verify the statement. We Suppose it true for n, let us prove it for the dimension $(n+1)$. By using Relation (5), we get,*

$$\begin{aligned}
&\binom{n+1}{k}_s = \sum_{m=0}^{s} \binom{n}{k-m}_s \\
&= \binom{n}{k}_s + \binom{n}{k-1}_s + \binom{n}{k-2}_s + \cdots + \binom{n}{k-s}_s \\
&= \sum_{i_{n+1}=0}^{s} \#\left\{1^{i_1}2^{i_2}\cdots n^{i_n} \,|\, \sum_{m=1}^{n} i_m = k - i_{n+1}; i_1, i_2, \ldots, i_n \leq s\right\} \\
&= \sum_{i_{n+1}=0}^{s} \#\left\{1^{i_1}2^{i_2}\cdots n^{i_n}(n+1)^{i_{n+1}} \,|\, \sum_{m=1}^{n} i_m = k - i_{n+1}; i_1, i_2, \ldots, i_n \leq s\right\} \\
&= \#\left\{1^{i_1}2^{i_2}\cdots n^{i_n}(n+1)^{i_{n+1}} \,|\, \sum_{m=1}^{n+1} i_m = k; i_1, i_2, \ldots, i_n, i_{n+1} \leq s\right\}. \\
&= \#P_{n+1,k,s}.
\end{aligned}$$

Example 2.2 *In Figure 2, we count four possible up-oriented paths of length 3 in the hypercube $H_{4,1}$. In Table 3, we distinguish these paths accordingly to their linear and power forms.*

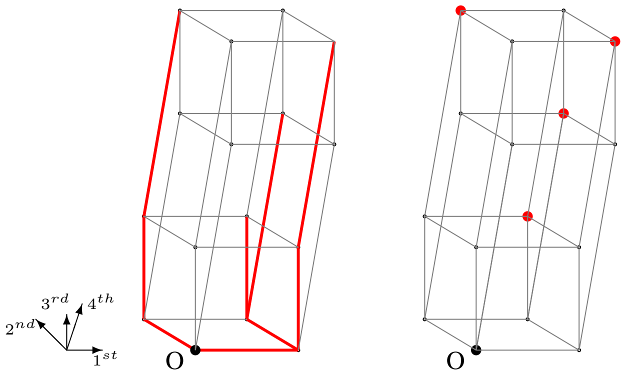

Figure 2: The up-oriented paths of length 3 in the hypergrid $H_{4,1}$ and their final vertices.

Table 3: Linear and power forms of the up-oriented paths of length 3 in the hypergrid $H_{4,1}$.

i_1	i_2	i_3	i_4	Linear forms	Power forms
1	1	1	0	123	$1^1 2^1 3^1 4^0$
1	1	0	1	124	$1^1 2^1 3^0 4^1$
1	0	1	1	134	$1^1 2^0 3^1 4^1$
0	1	1	1	234	$1^0 2^1 3^1 4^1$

In fact, $\#\left\{1^{i_1}2^{i_2}3^{i_3}4^{i_4};\ i_1 + i_2 + i_3 + i_4 = 3;\ i_1, i_2, i_3, i_4 \leq 1\right\} = \binom{4}{3}_1 = \binom{4}{3} = 4.$

In the following subsection, by using Theorem 2.1, we derive a combinatorial interpretation of generalized Pascal Formula over hypergrids.

2.2 Combinatorial interpretation of generalized Pascal Formula

Definition 2.2 *We denote by* J_{n-1}*, the projection map on the hypergrid* $H_{n-1,s}$*, defined as,*

$$J_{n-1}: \begin{array}{ccc} P_{n,k,s} & \to & \bigcup_{m=0}^{s} P_{n-1,k-m,s} \\ p_{i_1,i_2,\ldots,i_n} & \mapsto & p_{i_1,i_2,\ldots,i_{n-1}} \end{array}$$

Theorem 2.2 *The generalized Pascal Formula,* $\binom{n}{k}_s = \sum_{m=0}^{s} \binom{n-1}{k-m}_s$*, can be interpreted over hypergrids by the following bijection,* $P_{n,k,s} \overset{J_{n-1}}{\approx} \bigcup_{m=0}^{s} P_{n-1,k-m,s}$.

Proof 2.2 *Obviously, the map* J_{n-1} *is surjective by definition, so,* $J_{n-1}(P_{n,k,s}) = \bigcup_{m=0}^{s} P_{n-1,k-m,s}$*. On one hand, by Theorem 2.1, we have,* $\#P_{n,k,s} = \binom{n}{k}_s$*. On the other hand, for all* $m_1, m_2 \in \{0, 1, \ldots, s\}$*, such that* $m_1 \neq m_2$*, it is clear that,* $P_{n-1,k-m_1,s} \bigcap P_{n-1,k-m_2,s} = \emptyset$*, so,* $\#J_{n-1}(P_{n,k,s}) = \#\bigcup_{m=0}^{s} P_{n-1,k-m,s} = \sum_{m=0}^{s} \#P_{n-1,k-m,s} = \sum_{m=0}^{s} \binom{n-1}{k-m}_s = \binom{n}{k}_s$*. Consequently, we have proved that the two sets* $P_{n,k,s}$ *and* $\bigcup_{m=0}^{s} P_{n-1,k-m,s}$*, have the same cardinality, then, they are in bijection.*

Example 2.3 *For* $n = 4, k = 6, s = 3$*, the generalized Pascal Formula,* $\binom{4}{6}_3 = \sum_{m=0}^{3} \binom{3}{6-m}_3 = 10 + 12 + 12 + 10$*, is interpreted over hypergrids by the following bijection,* $P_{4,6,3} \overset{J_3}{\approx} P_{3,6,3} \cup P_{3,5,3} \cup P_{3,4,3} \cup P_{3,3,3}$*, see Table 4.*

Table 4: The bijection $P_{4,6,3} \overset{J_3}{\approx} \bigcup_{m=0}^{3} P_{3,6-m,3}$.

$P_{4,6,3}$	$P_{3,6,3}$	$P_{4,6,3}$	$P_{3,5,3}$	$P_{4,6,3}$	$P_{3,4,3}$	$P_{4,6,3}$	$P_{3,3,3}$
111222	111222	111234	11123	111244	1112	111444	111
111223	111223	222334	22233	123344	1233	222444	222
112223	112223	111334	11133	112344	1123	333444	333
111233	111233	112234	11223	223344	2233	233444	233
111333	111333	112334	11233	122344	1223	133444	133
222333	222333	122234	12223	112244	1122	123444	123
122233	122233	113334	11333	113344	1133	223444	223
122333	122333	112224	11222	233344	2333	122444	122
112333	112333	123334	12333	122244	1222	112444	112
112233	112233	223334	22333	133344	1333	113444	113
		122334	12233	222344	2223		
		111224	11122	111344	1113		

3 Generalized Catalan numbers

In this section, our aim is to generalize Catalan numbers by using s-Pascal triangle, as well as to extend their identities corresponding to this generalization. First, we recall some generalizations of Catalan numbers.

Stanley, [23], Koshy, [19] and Grimaldi, [15], collect many combinatorial interpretations of Catalan numbers through using: paths, parenthesis, words or binary numbers, binary trees, ... In 1791, before Eugène Charles Catalan studied these numbers, Fuss, [11], introduced Fuss-numbers, given under many expressions, as, $F(k,n) = \frac{1}{(k-1)n+1}\binom{kn}{n}$, see [14], or, $F(k,n) = \frac{1}{kn+1}\binom{kn+1}{n}$, see [19], also as follows, $F(k,n) = \frac{1}{n}\binom{kn}{n-1}$, see, [15, 17]. We mention that, for $k = 2$, $F(2,n)$ gives the Catalan numbers. A combinatorial interpretation of these numbers is given as the number of paths from $(0,0)$ to $(n,(k-1)n)$, which take steps of the set $\{(0,1),(1,0)\}$, that lie below the line $y = (k-1)x$, see [20].

Raney numbers, [21], are defined as $R(k,r,n) = \frac{r}{kn+r}\binom{kn+r}{n}$, this is a generalization of Fuss-numbers, as we have, $R(k,1,n) = F(k,n)$. $R(k,r,n)$ counts the forests composed by r ordered rooted trees, with k components and n vertices, see [23].

Hilton and Pedersen, [17], presented a solution to the well known ballot problem, as well as they gave a generalization of Catalan numbers. They showed that the number of paths lie completely below the line $y = x$, which connect the two points $(1, 0)$ and (a, b), for $a > b$ two integers, is equal to the number $\frac{a-b}{a+b}\binom{a+b}{a}$. As a particular case, for $a = n + 1$ and $b = n$, we get the Catalan numbers.

Gessel,[12], called $S(m, n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$ as Super-Catalan numbers. This is a generalization of Catalan numbers, as we have, $S(1, n)/2 = C_n$. Gessel and Xin, [13], presented a combinatorial interpretation of these numbers for $m = 2, 3$, by using the famous Dyck paths.

Koç et $al.$, [18], gave the following generalization, $C(n, m) = \frac{n-m+1}{n+1}\binom{n+m}{n}$, with $m \leq n$. As a particular case, $C(n, n)$ gives Catalan numbers. They showed that $C(n, m)$ is the number of paths from $(0, 0)$ to (n, m) through using right step and up-step without moving upper the line $x = y$.

3.1 s-Catalan numbers

In the rest of this paper we consider an odd integer s. First, we define central bi^{s}nomial coefficients as a generalization of central binomial coefficients, as follows

Definition 3.1 *For $n \in \mathbb{Z}^+$, central bi^{s}nomial coefficients are given by the following form, $\binom{2n}{sn}_s$.*

Remark 3.1 *Central bi^{s}nomial coefficients divide s-Pascal triangle into two symmetric parts, as in the classical case, for $s = 1$.*

Definition 3.2 *For $n \geq 0$, we define s-Catalan numbers as*

$$C_{n,s} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s. \tag{12}$$

The values which correspond to the s-Catalan numbers appeared in physics of particles theory (under another appellation), especially, in the issues related to $spin$ multiplicities, see the two recent papers of, E. Cohen et $al.$, [5] and T. Curtright et $al.$, [6].

We get the s-Catalan numbers by subtracting from the middle column of the s-Pascal triangle, $\binom{2n}{sn}_s$, its next column to the right of the same level, $\binom{2n}{sn+1}_s$. For $s = 3$, Table 5 and Table 6, give the first numbers of 3-Catalan numbers as follows,

$1, 1, 4, 34, 364, 4269, 52844, 679172, 8976188, 121223668, 1665558544, \ldots,$

see [22], as A264607.

Table 5: The first columns of 3-$Pascal$ triangle right part.

1									
	1		1						
4		3		2		1			
	12		10		6		3		1
44		40		31		20		10	
	155		135		101		65		35
580		546		456		336		216	
	2128		1918		1554		1128		720
8092		7728		6728		5328		3823	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\binom{2n}{3n}_3$	$\binom{2n-1}{3n-1}_3$	$\binom{2n}{3n+1}_3$	$\binom{2n-1}{3n}_3$	$\binom{2n}{3n+2}_3$	$\binom{2n-1}{3n+1}_3$	$\binom{2n}{3n+3}_3$	$\binom{2n-1}{3n+1}_3$	$\binom{2n}{3n+3}_3$	$\ldots$

Through using 5-Pascal triangle, the first values of 5-Catalan numbers, are

$1, 1, 6, 111, 2666, 70146, 1949156, 56267133, 1670963202, 50720602314, \ldots,$

see [22], as A272391.

The following theorem gives the generalization of Formulae (10) and (11), respectively.

Table 6: Generating of 3-$Catalan$ numbers by definition.

n	$\binom{2n}{3n}_3$	$\binom{2n}{3n+1}_3$	$C_{n,3} = \binom{2n}{3n}_3 - \binom{2n}{3n+1}_3$
0	1	0	1
1	4	3	1
2	44	40	4
3	580	546	34
4	8092	7728	364
5	116304	112035	4269
6	1703636	1650792	52844
7	25288120	24608948	679172
8	379061020	370084832	8976188
9	5724954544	5603730876	121223668
10	86981744944	85316186400	1665558544

Theorem 3.1 *We have,*

$$C_{n,s} = \binom{2n-1}{sn}_s - \binom{2n-1}{sn+1}_s, \ n \geq 1. \tag{13}$$

$$C_{n+1,s} = \binom{2n}{sn}_s - \binom{2n}{sn+(s+1)}_s, \ n \geq 0. \tag{14}$$

Proof 3.1 *By using Formula (5), we get,* $C_{n,s} = \binom{2n}{sn}_s - \binom{2n}{sn+1}_s = \sum_{m=0}^{s}\binom{2n-1}{sn-m}_s - \sum_{m=0}^{s}\binom{2n-1}{sn+1-m}_s = \binom{2n-1}{sn-s}_s - \binom{2n-1}{sn+1}_s$. *Formula (4) gives,* $\binom{2n-1}{sn-s}_s = \binom{2n-1}{sn}_s$, *then we find,* $C_{n,s} = \binom{2n-1}{sn}_s - \binom{2n-1}{sn+1}_s$.

To get Formula (14), first we calculate $C_{n+1,s}$, *by using Formula (12), then we follow the same proof of Formula (13), by applying Formula (5) twice.*

As a future work, we want to find a combinatorial interpretation of s-Catalan numbers, especially, by using up-oriented paths on hypergrids.

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