Reconstruction of Lattice-Valued Functions by Integral Transforms

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Abstract

Fuzzy transforms provide a powerful tool for reconstructing functions from compressed values called the components of fuzzy transform. Lower and upper fuzzy transforms were introduced for residuated lattice-valued functions, and it has been shown that their composition results in either upper or lower approximation of the original function depending on the order of the types of fuzzy transforms in the composition. Currently, a generalization of lower and upper fuzzy transforms was proposed using Sugeno-like integrals and fuzzy kernel relations imitating the standard integral transforms as Fourier or Laplace transforms. The paper presents preliminary results showing that a composition of two integral transforms can approximate an original function similarly as in the case of fuzzy transforms. In addition, we demonstrate that reconstruction based on integral transformations can filter outliers in a lattice-valued function.

Keywords

Integral transform, fuzzy transform, lattice-valued functions, fuzzy kernel relation

1. Introduction

Integral transforms are mathematical operators that produce a new function $g(y)$ by integrating the product of an existing function $f(x)$ and an integral kernel function $K(x, y)$ between suitable limits. Recall the Fourier and Laplace transforms as the most important examples of integral transforms whose applications can be found in solving (partial) differential equations, algebraic equations, signal and image processing, spectral analysis of stochastic processes (see, e.g., [1, 2, 3]).

In fuzzy set theory and fuzzy logic, the values of a function usually belong to an algebra of truth values as a residuated lattice and its special variants as the BL-algebra, MV-algebra, IMTL-algebra (see, e.g., [4, 5]). In 2006, Perfilieva introduced in [6] an upper fuzzy (F-)transform $F^{\uparrow}$ and a lower fuzzy (F-)transform $F^{\downarrow}$ for lattice-valued functions using which an original function can be reconstructed, specifically compositions of $F^{\downarrow}$ and $F^{\uparrow}$ approximate the original function from above or below, see Fig. 1a. To better understand the lower and upper approximation, we recall the definition of lower and upper F-transforms. Let L be a complete residuated lattice. Denote $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ the sets of all fuzzy subsets of non-empty sets X and Y, respectively, and let $K : X \times Y \to L$ be a fuzzy relation such that the system of fuzzy sets $K_y(x) = K(x, y)$

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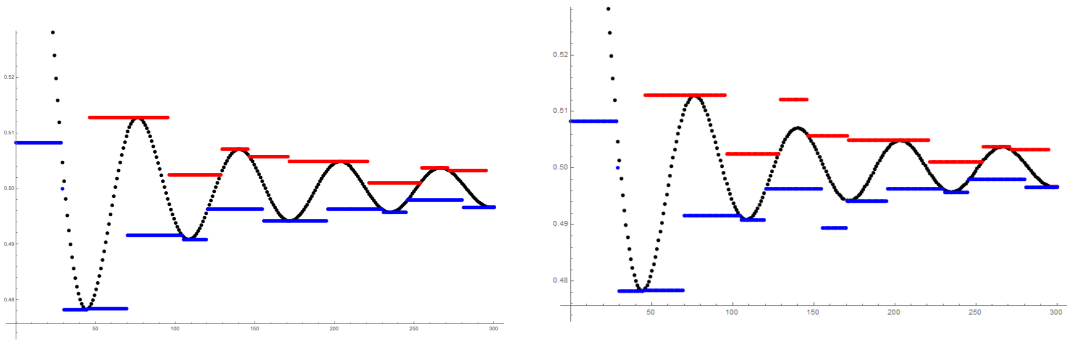


(a) discrete function with no biased function values (b) a discrete function with two biased function values

Figure 1: Upper approximation (red) and lower approximation (blue) given by F-transforms

for $y \in Y$ forms a fuzzy partition of X.[1] Then an *upper F-transform* $F_K^{\uparrow} : \mathcal{F}(X) \to \mathcal{F}(Y)$ and a *lower F-transform* $F_K^{\downarrow} : \mathcal{F}(X) \to \mathcal{F}(Y)$ of a function $f : X \to L$ with respect to a fuzzy partition determined by K are defined as

$$F_K^{\uparrow}(f)(y) = \bigvee_{x \in X} K(x,y) \otimes f(x) \text{ and } F_K^{\downarrow}(f)(y) = \bigwedge_{x \in X} K(x,y) \to f(x), \tag{1}$$

respectively. To be more specific, the previous formulas define the so-called *direct* F-transforms. Since the *inverse upper* and *lower* F-transforms from $\mathcal{F}(Y)$ to $\mathcal{F}(X)$ can be defined by the same formulas in (1) for a fuzzy partition of Y, we use the same notation and only change K to determine a fuzzy partition on Y. The upper and lower approximations of a function presented in Fig. 1a are consequences of the following theoretical result:

$$F_{K^{-1}}^{\uparrow} \circ F_K^{\downarrow}(f)(x) \leq f(x) \leq F_{K^{-1}}^{\downarrow} \circ F_K^{\uparrow}(f)(x), \quad x \in X, \tag{2}$$

where $K^{-1} : Y \times X \to L$ is given by $K^{-1}(y,x) = K(x,y)$. It is worth noting that lattice-valued F-transforms are closely related to the lattice-valued operators in mathematical morphology, namely, fuzzy dilations and erosions, as was demonstrated in [8]. In Fig. 1b, the compositions of upper and lower F-transforms are applied to approximate a "corupted" function, specifically the same function as in Fig. 1a but with two biases. The result is not surprising because of (2) and only demonstrates that the lattice-valued F-transforms based reconstruction cannot be used as a filter similarly as the real-valued F-transform or Fourier and some other integral transforms.

In a current article [9], we demonstrated that the lower and upper F-transforms can be naturally introduced as two types of integral transforms for a Sugeno-like integral for lattice-valued functions [10, 11]. Namely, for a fuzzy measure space $(X, \mathcal{F}, \mu)$, a fuzzy relation $K : X \times Y \to L$ called the integral kernel and a function $f : X \to L$, we proposed two integral transforms given as:

$$F_{(K,\mu)}^{\otimes}(f)(y) = \int_X^{\otimes} K(x,y) \otimes f(x)\, d\mu \text{ and } F_{(K,\mu)}^{\to}(f)(y) = \int_X^{\otimes} K(x,y) \to f(x)\, d\mu. \tag{3}$$

[1] A system of fuzzy subsets of X is a fuzzy partition if the cores of fuzzy sets are non-empty and form a classical partition of X. For details, see [6]. In addition, it has been shown that a fuzzy partition can be equivalently replaced by a fuzzy relation [7] which is used in this contribution.

For the precise definition of concepts, see Sections 2 and 3. Assuming that the fuzzy relation (integral kernel) K determines a fuzzy partition X and $\mathcal{F} = \mathcal{P}(X)$ is the powerset of X, it is easy to show that $F^{\otimes}_{(K,\mu^{\top})} = F^{\uparrow}_{K}$ holds for a trivial measure $\mu^{\top}$ given by $\mu^{\top}(A) = 1$ for any $A \in \mathcal{F}$ such that $A \neq \emptyset$, and $F^{\rightarrow}_{(K,\mu^{\perp})} = F^{\downarrow}_{K}$ holds for another trivial measure $\mu^{\perp}$ given by $\mu^{\perp}(A) = 0$ for any $A \in \mathcal{F}$ such that $A \neq X$. Note that $\mu^{\perp}$ ($\mu^{\top}$) is the least (highest) fuzzy measure on the measurable space $(X, \mathcal{F})$.

Since the above-introduced integral transforms generalize the upper and lower F-transforms, a natural question is whether their compositions can approximate the original functions. This article aims on this problem, and we demonstrate that

$$F^{\rightarrow}_{(K^{-1},\mu')} \circ F^{\otimes}_{(K,\mu)}(f)(x) \approx f(x), \quad x \in X,$$

for a suitable setting of the fuzzy measure μ', and similarly for the opposite composition. In addition, we show that the reconstruction based on integral transforms can filter out biased function values.

The article has the following structure. The next section recalls the basic concepts used in the article. The third section is devoted to integral transforms and their elementary properties. The fourth section introduces a reconstruction of lattice-valued functions using the composition of appropriate integral transforms. The last section is a conclusion.

2. Preliminaries

Truth value algebras We assume that the algebra of truth values is a *complete residuated lattice*, i.e., an algebra $L = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, where 0 is the least element and 1 is the greatest element of L, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes 1 = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{4}$$

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering, i.e., $a \leq b$ if $a \wedge b = a$ for $a, b \in L$. The operations $\otimes$ and $\rightarrow$ are called the *multiplication* and *residuum*, respectively. For details, we refer to [4].

Example 2.1. It is easy to prove that the algebra

$$L_T = \langle [0, 1], \min, \max, T, \rightarrow_T, 0, 1\rangle,$$

where T is a left continuous t-norm (see, e.g., [12]) and $a \rightarrow_T b = \bigvee\{c \in [0, 1] \mid T(a, c) \leq b\}$ defines the residuum, is a complete residuated lattice.

Fuzzy sets Let L be a complete residuated lattice, and let X be a non-empty set. A function $A : X \rightarrow L$ is called a *fuzzy subset in* X. The set of all fuzzy sets on X is denoted by $\mathcal{F}(X)$. A fuzzy set A on X is called *crisp* if $A(x) \in \{0, 1\}$ for any $x \in X$. The symbol $\emptyset$ denotes the empty fuzzy set on X, i.e., $\emptyset(x) = 0$ for any $x \in X$. The set of all crisp fuzzy sets on X (i.e., the power set of X) is denoted by $\mathcal{P}(X)$. A constant fuzzy set A on X (denoted as a_X) satisfies $A(x) = a$ for any $x \in X$, where $a \in L$. The sets $\mathrm{Supp}(A) = \{x \mid x \in X \ \& \ A(x) > 0\}$ and $\mathrm{Core}(A) = \{x \mid x \in X \ \& \ A(x) = 1\}$ are called the *support* and the *core* of a fuzzy set A, respectively. A fuzzy set A is called *normal* if $\mathrm{Core}(A) \neq \emptyset$.

Fuzzy measure spaces Let X be a non-empty set. A subset $\mathcal{F}$ of $\mathcal{P}(X)$ is an *algebra of sets on* X provided that.

(A1) $X \in \mathcal{F}$,

(A2) if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$,

(A3) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

It is easy to see that if $\mathcal{F}$ is an algebra of sets, then the intersection of finite number of sets belongs to $\mathcal{F}$. A pair $(X, \mathcal{F})$ is called a *measurable space* (on X) if $\mathcal{F}$ is an algebra (σ-algebra) of sets on X. Let $(X, \mathcal{F})$ be a measurable space and $A \in \mathcal{F}(X)$. We say that A is $\mathcal{F}$*-measurable* if $A \in \mathcal{F}$. Obviously, the sets $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are algebras of fuzzy sets on X.

A map $\mu : \mathcal{F} \to L$ is called a *fuzzy measure* on a measurable space $(X, \mathcal{F})$ if

(i) $\mu(\emptyset) = 0$ and $\mu(X) = 1$,

(ii) if $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

A triplet $(X, \mathcal{F}, \mu)$ is called a *fuzzy measure space* whenever $(X, \mathcal{F})$ is a measurable space and μ is a fuzzy measure on $(X, \mathcal{F})$. For details, we refer to [13]. Let μ be a fuzzy measure on $(X, \mathcal{F})$. We say that a map $\mu^c : \mathcal{F} \to L$ is *conjugate to* μ if $\mu^c(A) = \mu(X \setminus A) \to \perp$ for any $A \in \mathcal{F}$, where $X \setminus A$ is the complement of A in X and $\to$ is the residuum of L (cf., [10]).

Example 2.2. Let L_{T} be an algebra from Ex. 2.1, where T is a continuous t-norm. Let $X = \{x_1, \ldots, x_n\}$ be a finite non-empty set, and let $\mathcal{F}$ be an arbitrary algebra. A *relative* fuzzy measure μ^r on $(X, \mathcal{F})$ can be given as

$$\mu^r(A) = \frac{|A|}{|X|}$$

for all $A \in \mathcal{F}$, where $|A|$ and $|X|$ denote the cardinality of A and X, respectively. Let $\varphi : L \to L$ be a monotonically non-decreasing map with $\varphi(0) = 0$ and $\varphi(1) = 1$. The relative measure μ^r can be generalized as a fuzzy measure μ^r_φ on $(X, \mathcal{F})$ given by $\mu^r_\varphi(A) = \varphi(\mu^r(A))$ for any $A \in \mathcal{F}$.

Multiplication based fuzzy integral The integrated functions are fuzzy sets on X and are denoted by f, g etc. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and let $f : X \to L$. The $\otimes$*-fuzzy integral of* f *on* X is given by

$$\int_X^{\otimes} f \, d\mu = \bigvee_{A \in \mathcal{F}} \mu(A) \otimes \left(\bigwedge_{x \in A} f(x) \right). \tag{5}$$

It should be noted that the previous definition of $\otimes$-fuzzy integral was proposed in [10] and coincides with the definition in [14] whenever $\otimes$ distributes over $\bigwedge$ in the algebra of truth values (e.g. an MV-algebra).

3. Integral transforms

We say that a fuzzy relation $K : X \times Y \to L$ is *normal* in the second argument if $\mathrm{Core}(K_y) \neq \emptyset$ for any $y \in Y$, where $K_y(\cdot) = K(\cdot, y)$. Recall that a crucial condition for a fuzzy relation K in the definition of lower and upper F-transforms (see, (1)) is that the family of fuzzy sets K_y forms a fuzzy partition. This condition seems to be unnecessarily strict for introducing integral transforms. Therefore, we propose the following more general definition.

Definition 3.1. A fuzzy relation $K : X \times Y \to L$ which is normal in the second argument is said to be an *integral kernel*.

The next definition generalizes the upper and lower F-transforms and unifies the definitions of integral transforms provided in (3).

Definition 3.2. Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, let $K : X \times Y \to L$ be an integral kernel, and let $\odot \in \{\otimes, \to\}$. A map $F^{\odot}_{(K,\mu)} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by

$$F^{\odot}_{(K,\mu)}(f)(y) = \int_X^{\otimes} K(x, y) \odot f(x)\, d\mu, \tag{6}$$

is called a $(K, \mu, \odot)$-*integral transform.*

The following theorem provides a summary of elementary properties of integral transforms for lattice-valued functions (see, [9]).

Theorem 3.1. *Let $\odot \in \{\otimes, \to\}$. For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have*

(i) $F^{\odot}_{(K,\mu)}(f) \leq F^{\odot}_{(K,\mu)}(g)$ *if* $f \leq g$,

(ii) $F^{\odot}_{(K,\mu)}(f \cap g) \leq F^{\odot}_{(K,\mu)}(f) \wedge F^{\odot}_{(K,\mu)}(g)$,

(iii) $F^{\odot}_{(K,\mu)}(f) \vee F^{\odot}_{(K,\mu)}(g) \leq F^{\odot}_{(K,\mu)}(f \cup g)$,

(iv) $a \otimes F^{\odot}_{(K,\mu)}(f) \leq F^{\odot}_{(K,\mu)}(a \otimes f)$,

(v) $F^{\odot}_{(K,\mu)}(a \to f) \leq a \to F^{\odot}_{(K,\mu)}(f)$.

The following theorem shows conditions under which a constant function (fuzzy set) a_X is transformed to a constant function a_Y, i.e., $F^{\odot}_{(K,\mu)}(a_X) = a_Y$.

Theorem 3.2. *Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, let K be an integral kernel, and let $a \in L$.*

(i) If for any $y \in Y$ there exists $A_y \in \mathcal{F}$ such that $A_y \subseteq \mathrm{Core}(K_y)$ and $\mu(A_y) = 1$, then $F^{\otimes}_{(K,\mu)}(a_X) = a_Y$.

(ii) If for any $y \in Y$ and for any $A \in \mathcal{F}$ with $A \subseteq X \setminus \mathrm{Core}(K_y)$ it holds that $\mu(A) \leq a$, then $F^{\rightarrow}_{(K,\mu)}(a_X) = a_Y$.

It is worth noting that the standard real-valued F-transforms as well as lower and upper F-transforms preserve constant functions; therefore, it seems to be reasonable to relate integral kernels and fuzzy measures as the parameters of the integral transforms according to the previous theorem.

Example 3.1. Let μ be a fuzzy measure on $(X, \mathcal{F})$ such that (i) of Theorem 3.2 is satisfied. Then the conjugate fuzzy measure μ^c satisfies (ii) of Theorem 3.2 for any $a \in L$. Indeed, for any $y \in Y$ and $A \in \mathcal{F}$ such that $A \subseteq X \setminus \mathrm{Core}(K_y)$ we have

$$\mu^c(A) = \mu(X \setminus A) \rightarrow 0 = 1 \rightarrow 0 = 0,$$

which immediately follows from the existence of $A_y \subseteq \mathrm{Core}(K_y)$ such that $A_y \subseteq \mathrm{Core}(K_y) \subseteq X \setminus A$, and $1 = \mu(A_y) \leq \mu(X \setminus A)$. Similarly, if μ satisfies (ii) of Theorem 3.2, then the conjugate fuzzy measure μ^c satisfies (i) of Theorem 3.2.

4. Reconstruction by integral transforms

Let $K : X \times Y \rightarrow L$ be an integral kernel such that the fuzzy relation $K^{-1} : Y \times X \rightarrow L$ given by $K^{-1}(y, x) = K(x, y)$ for any $(y, x) \in Y \times X$ is an integral kernel. The fuzzy relation K^{-1} is called the *inverse integral kernel* to K. Let $(X, \mathcal{P}(X), \mu)$ and $(Y, \mathcal{P}(Y), \nu)$ be fuzzy measure spaces. To reconstruct the lattice-valued functions by compositions of integral transforms, we propose two maps $F^{\uparrow}, F^{\downarrow} : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ defined as follows:

a) $F^{\uparrow}(f) = F^{\rightarrow}_{(K^{-1},\nu^c)} \circ F^{\otimes}_{(K,\mu)}(f)$,

b) $F^{\downarrow}(f) = F^{\otimes}_{(K^{-1},\nu)} \circ F^{\rightarrow}_{(K,\mu^c)}(f)$

for any $f \in \mathcal{F}(X)$.

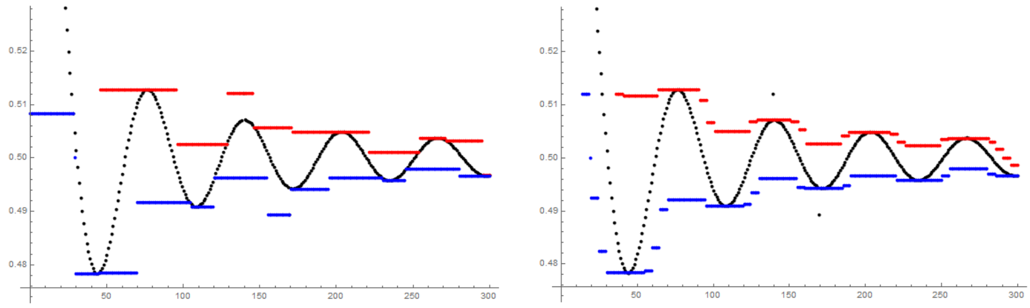
In Fig. 2a, one can see the upper and lower approximation of a discrete function with two biased values given by the standard lattice-valued F-transform. As it is displayed in Fig. 2b, the approximation can be improved by integral transforms, where we used the Łukasiewicz algebra, an integral kernel with overlapped cores (i.e., $\mathrm{Core}(K_{y_1}) \cap \mathrm{Core}(K_{y_2}) \neq \emptyset$ for certain $y_1, y_2 \in Y$), and fuzzy measures μ and ν satisfying (i) and (ii) of Theorem 3.2, respectively.

The natural question arises whether the inequalities in (2) holds for $F^{\uparrow}$ and $F^{\downarrow}$. The answer is generally no, but we can determine interesting generalizations of the inequalities as follows.

Definition 4.1. An integral kernel $Q : X \times X \rightarrow L$ is said to be compatible with K and K' ((K, K')-compatible, for short) provided that

$$Q(x, z) \otimes K'(y, z) \leq K(x, y), \quad x, z \in X \text{ and } y \in Y. \tag{7}$$

The following theorem extends the inequalities in (2) for the reconstructions given by integral transforms with respect to compatible integral kernels.



(a) Upper approximation (red) and lower approximation (blue) given by F-transform

(b) $F^{\uparrow}$-reconstruction (red) and $F^{\downarrow}$-reconstruction (blue) given by appropriate integral transforms

Figure 2: Reconstructions of a discrete function with two biased function values.

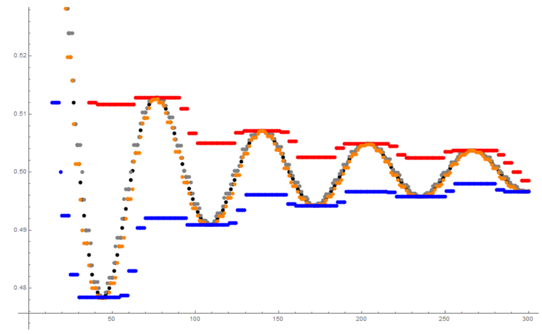


Figure 3: Demonstration of the inequalities (8); the original function f (black), $F^{\uparrow}(f)$ (red) and $F^{\downarrow}(f)$ (blue), $F^{\otimes}_{(Q,\mu)}(f)$ (gray) and $F^{\rightarrow}_{(Q,\nu^c)}(f)$ (orange).

Theorem 4.1. *Let $K : X \times Y \to L$ be an integral kernel such that K^{-1} is the inverse integral kernel, and let $(X, \mathcal{P}(X), \mu)$ and $(Y, \mathcal{P}(Y), \nu)$ be fuzzy measure spaces. Then*

$$F^{\uparrow}(f) \geq F^{\otimes}_{(Q,\mu)}(f) \quad \text{and} \quad F^{\downarrow}(f) \leq F^{\rightarrow}_{(Q,\nu^c)}(f) \tag{8}$$

for any $f \in \mathcal{F}(X)$ and a (K, K^{-1})-compatible integral kernel Q.

The inequalities in (8) are demonstrated in Fig. 3.

5. Conclusion

In this article, we proposed two types of reconstruction of lattice-valued functions given by the compositions of integral transforms, which generalize the upper and lower approximation provided by lattice-valued F-transforms. We demonstrated that new reconstructions can approximate original functions and serve as a filter for biased function values. A deeper analysis of the quality of the reconstructions is a subject of our future research. Finally, we extended the inequalities between upper and lower approximations and the original function using integral transforms with respect to compatible integral kernels. In future research, we also plan to extend our integral transforms and reconstructions to two-dimensional lattice-valued functions and apply them to image processing, such as image reduction/magnification.

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