Structural analysis of cubic graphs based on 5-cycle clusters

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Abstract

A 5-*cycle cluster* is a connected subgraph of a cubic graph, where each edge belongs to some 5-cycle. It turns out, that 5-cycle clusters provide a very useful and efficient tool for the structural analysis of cubic graphs, mostly with respect to colourings problems. In this work, we develop necessary theory regarding 5-cycle clusters and describe algorithms for generating 5-cycle clusters and analysing them in cubic graphs. Finally, we present applications of our methods in the structural analysis of all snarks, that is cubic graphs with no 3-edge colouring, up to order 36 and in generation of uniquely 3-edge-colourable cubic graphs.

Keywords

cubic graphs, snarks, 5-cycle clusters, uniquely 3-edge colourable, structural analysis

1. Introduction

In 1880, as one of the first attempts to prove the four colour theorem, Tait [1] proved that each planar graph is 4-vertex-colourable if and only if each 2-connected planar cubic graph is 3-edge-colourable. Later on, cubic 2-connected graphs that are not 3-edge-colourable were named *snarks* [2]. The class of snarks appeared in the study of other famous conjectures regarding colourings, flows and cycle covers, like the Tutte's 5-flow conjecture [3], Seymour's [4] and Szekeresz's [5] cycle double cover conjecture or the Berge-Fulkreson conjecture [6, 7]. One can easily prove that all of them are true for 3-edge-colourable cubic graphs. Thus any potential counterexample lies in the family of snarks.

Although the essential property of snarks is the absence of their 3-edge-colouring, many authors put stronger requirements on snarks to avoid some "trivial" cases. Typical nontriviality criteria consist of higher *girth*, that is the length of a shortest cycle, and cyclic edge-connectivity, where a cubic graph G is *cyclically* k*-edge-connected* if G contains no edge-cut consisting of fewer than k edges that separates two cycles of G. According to perhaps the most common requirements, we call a snark *nontrivial* if it is cyclically 4-edge-connected and has girth at least 5.

Beside the absence of a 3-edge-colouring, the number of all possible 3-edge-colourings for a given cubic graph is also studied. Cubic graphs that have up to a permutation of colours only one 3-edge-colouring, also called *uniquely* 3*-edge-colourable*, are connected to a potential minimum counterexample for the cycle double cover conjecture [8]. Uniquely k-edge-colourable graphs, a more general notion, were studied by Greenwell and Kronk [9], and characterised by Thomasson [10] except the case $k = 3$.

Moreover, Kászonyi [11] studied, for a given snark G and its edge e, the value denoted by $\psi(G, e)$ which is the number of 3-edge-colourings of the cubic graph $G \sim e$ obtained from G by removing the edge e and suppressing the resulting two vertices of degree 2. According to one of his results [11] (see also [12]), if e and f are edges of the same 5-cycle in a snark G, then $\psi(G, e) = \psi(G, f)$. This result leads us to study connected subgraphs of snarks where each edge lies on some 5-cycle which are called 5*-cycle clusters*. Thus, the value $\psi(G, e)$ is equal for all the edges e of a 5-cycle cluster in a snark G.

In this paper, we show how the 5-cycle clusters can be used for structural analysis of cubic graphs, especially snarks and uniquely 3-edge-colourable cubic graphs. Firstly in Section 2, we describe an algorithm using which we generated all 5-cycle clusters up to order 20. Then in Section 3 we analyse structural and colouring properties of small 5-cycle clusters with emphasis on those contained in the Petersen graph. In Section 4 we describe algorithms for finding and identifying 5-cycle clusters in a given cubic graph. Finally, in Section 5, we present applications of our results.

At the end of this section, we clarify some notions we shall use. By a graph we always mean a graph without loops and parallel edges. The distance between the vertices u and v in a graph G, denoted by $\mathrm{dist}_G(u, v)$ is the length (that is number of edges) of a shortest u-v-path. The distance of two edges e and f is defined as a minimal value of $\mathrm{dist}_G(u, v)$ where u and v are some end points of e and f, respectively.

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2. Generation of 5-cycle clusters

Firstly, since we want to analyse 5-cycle clusters in cubic graphs, we need to know all possible 5-cycle clusters up to a certain order and, eventually, also certain girth. We developed an algorithm to generate all such 5-cycle clusters.

Here, the union of graphs $G_1 \cup G_2$ is the graph G with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. If we consider each 5-cycle as a graph, then we can express every 5-cycle cluster as a union of some 5-cycles.

Proposition 1. *Let $H = \bigcup_{1 \leq i \leq k} C_i$ be a 5-cycle cluster with girth g consisting of 5-cycles $C_1, C_2, \ldots, C_k$ for some $k \geq 2$. Then there exists $j \in \{1, 2, \ldots, k\}$ such that $H' = \bigcup_{1 \leq i \leq k,\, i \neq j} C_i$ is a 5-cycle cluster with girth at least g and, if $H' \neq H$, H can be obtained form H' by*

(i) adding a u-v-path of length $5 - d$ for some degree 2 vertices u and v in H' that are connected by a path of length $d \in \{1, 2, 3, 4\}$; or

(ii) adding an edge ux and a v-y-path of length 2 for some degree 2 vertices u, v, x, and y of H' such that $uv, xy \in E(H')$; or

(iii) adding edges ux and vy for some degree 2 vertices u, v, x, and y of H' such that $uv \in E(H')$, and x and y are connected by a path of length 2.

Consider the graph G on the vertex set $V(G) = \{C_1, C_2, \ldots, C_k\}$, where the cycles C_i and C_j are connected by an edge if and only if C_i and C_j share an edge in H. Since H is a 5-cycle cluster, G is connected and also, G contains a vertex C_j that is not an articulation (for instance, an end of a longest path). Thus the subgraph $H' = \bigcup_{1 \leq i \leq k,\, i \neq j} C_i$ is also connected, hence it is a 5-cycle cluster. Clearly, the girth of H' cannot decrease with respect to H. In the end, a straightforward case analysis of the edges of C_j that are not contained in H' leads to cases (i), (ii) and (iii).

Based on Proposition 1, we developed Algorithm 1 that generates all 5-cycle clusters starting from the 5-cycle by recursively adding paths according to cases (i), (ii) and (iii). We store the generated 5-cycle clusters in a set S together with their canonical representation sparse6 provided by nauty [13] to prevent generating isomorphic 5-cycle clusters, and also in a priority queue q so we can recursively construct new clusters always from a 5-cycle cluster with the smallest order.

Using Algorithm 1 we generated all 91, 827 5-cycle clusters up to order 20.

Algorithm 1: Algorithm for generating all 5-cycle clusters with girth at least g up to order n

```
input  : n, g
output : A set S of all pairwise non-isomorphic
         clusters of 5-cycles up to order n with
         girth at least g
q ← priority queue containing only C_5;
S ← {};
while q is not empty do
    G ← q.pop();
    foreach {u, v} ⊆ V_2 do
        foreach d ∈ {1, 2, 3, 4}, there is a
          u-v-path of length d do
            if d = 4 ∧ uv ∉ E(G) then
                H = G ∪ {uv}
            else
                H ← G ∪
                  {ux_1, x_1x_2, ..., x_{3-d}x_{4-d}, x_{4-d}v};
            try_adding(H);
            if d ∈ {1, 2} then
                foreach {x, y} ⊆ V_2 − {u, v} do
                    if d = 1 then
                        try_adding(
                          G ∪ {xu, yz, zv});
                        try_adding(
                          G ∪ {xz, zu, yv});
                    if d = 2 then
                        try_adding(G ∪ {xu, yv});

Procedure try_adding(H)
    if |H| ≤ n ∧ girth(H) ≥ g then
        if S contains no isomorphic copy of H
          then
            S.add(H);
            q.add(H);
```

3. Small 5-cycles clusters

In this section, we describe most commonly used 5-cycle clusters and their colouring properties. For this purpose, it is useful to allow dangling edges, that is edges incident with only one vertex. From now on, we will consider that each vertex of a 5-cycle cluster is incident with three edges, one of them may be a dangling edge. This is formally comprehended in the notion of the multipole, introduced in [14], which we now define along with other notions needed to describe colouring properties.

Multipoles and colourings

A *multipole* M consists of a vertex set $V(M)$ and an edge set $E(M)$. Each edge has two ends which may, or may not, be incident with a vertex. An edge whose ends are incident with two distinct vertices is called a *link*. If only one end of an edge is incident with a vertex, then the edge is a *dangling edge* or *semiedge*. Other types of edges do not appear in this paper. The set of all semiedges is denoted by $S(M)$. The *order* $|M|$ of a multipole M is the number of its vertices. Note that we only consider *cubic multipoles* where each vertex is incident with three edge ends.

It is often convenient to partition the set $S(M)$ into pairwise disjoint sets $S_1, S_2 \dots, S_n$ called *connectors*. A multipole M with n connectors $S_1, S_2, \dots, S_n$ such that $|S_i| = k_i$ for $i \in \{1, 2, \dots, n\}$ is denoted by $M(S_1, S_2, \dots, S_n)$ and called a $(k_1, k_2, \dots, k_n)$*-pole*, or simply a k_1-pole if $n = 1$.

A *colouring* of a multipole M is a mapping which assigns to each edge a non-zero element from the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ in such a way that for each vertex $v \in V(G)$ the three edges incident to v have assigned pairwise distinct colours (or equivalently, colours with zero sum). For brevity, we denote the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ by $0 = (0, 0)$, $1 = (0, 1)$, $2 = (1, 0)$, and $3 = (1, 1)$. One can easily observe that for a cubic graph G, a mapping $\varphi\colon E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 - \{0\}$ is a colouring if and only if φ is a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow for any orientation of the edges of G.

For the purpose of the following definitions, we assume that the set of semiedges of a multipole M is linearly ordered. The *type of a colouring* φ of M is the lexicographically smallest sequence $c_1, c_2, \dots, c_k$ that can be obtained from $\varphi(e_1), \varphi(e_2), \dots, \varphi(e_k)$ by permuting colours. The *colouring set* of a multipole M is the set containing the colouring types of each colouring of M. A k-pole M is called *colour-open* if there exists a 3-edge-colourable k-pole N such that $\mathrm{col}(M) \cap \mathrm{col}(N) = \emptyset$; otherwise M is called *colour-closed*.

The following well-known result restrict sequences that can occur as colouring types.

[Parity lemma [15]] Let M be a k-pole and let k_1, k_2, and k_3 be the numbers of semiedges coloured by 1, 2 and 3, respectively. Then

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

For instance, the possible colouring types for a 4-pole are 1111, 1122, 1212, and 1221.

Petersen 5-cycle clusters

Now, we are prepared to describe the structure and colouring properties of small 5-cycle clusters. For the purpose of the structural analysis of snarks, colour-open 5-cycle clusters are most relevant. Although a colour-closed multipole M can also occur in a snark, it is possible only when M is complemented with some multipole that already admits no colouring.

We focus on those 5-cycle clusters that occur in the smallest snark – the Petersen graph. Note that the Petersen graph is a 5-cycle cluster on its own. In what follows, we present a complete list of all 5-cycle clusters contained in the Petersen graph which, as we verified, also coincides with a list of all colour-open 5-cycle clusters up to order 10. This set of 5-cycle clusters is also considered in [16] and it is sufficient to describe the structure of all snarks up to order 36.

Pentagon

The *pentagon* $\mathbf{P}$ is the smallest 5-cycle cluster. It consists of a single cycle of length 5 together with 5 dangling edges which in the order corresponding to the 5-cycle form its unique connector (see Figure 1). It can be constructed from the Petersen graph by removing any 5-cycle. The colouring set of the pentagon is

$$\mathrm{col}(\mathbf{P}) = \{12333, 12223, 11123, 11231, 12311\},$$

so the colour, which appears three times, is always assigned to three dangling edges incident with three consecutive vertices on the pentagon, respectively. The pentagon is the only 5-cycle cluster with 5 vertices. It has 5 edges and 5 dangling edges.

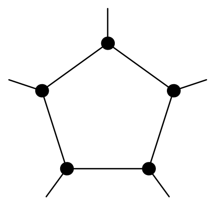

Figure 1: Pentagon

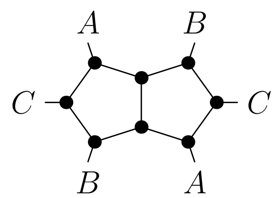

Figure 2: Double pentagon

Double pentagon

The *double pentagon* $\mathbf{dP}$ is a 5-cycle cluster containing two 5-cycles sharing an edge. It can be obtained from the Petersen graph by removing two adjacent vertices u and v and severing an edge e at distance 2 from uv. The natural distribution of semiedges turns it into a $(2, 2, 2)$-pole $\mathbf{dP}(A, B, C)$, shown in Figure 2, where the connectors A and B correspond to the vertices u and v respectively, and C corresponds to the edge e. By parity lemma, it admits no colouring, where the colours in the connector A are the same and the colours in B and also in C are different. However, not all of the remaining colourings satisfying parity lemma are admissible for $\mathbf{dP}$. Precisely, its colouring set is

$$\mathrm{col}(\mathbf{dP}) = \{111111, 111122, 112211, 121121, 121211, 121222, 121323, 122133, 122221, 122313, 123132, 123231, 123312\}.$$

The double pentagon is the unique 5-cycle cluster with 8 vertices, 9 edges and 6 dangling edges.

Dyad

The *dyad* $\mathbf{D}$ (or *Petersen negator*) is a 5-cycle cluster consisting of two 5-cycles sharing a path of length 2. It can be constructed by removing a path uwv of length 2 from the Petersen graph. The natural distribution of semiedges into connectors makes it a $(2,2,1)$-pole $\mathbf{D}(I,O,R)$, with 2-connectors I and O containing the dangling edges formerly incident with u and v respectively, and the 1-connector R containing the only dangling edge formerly incident with w (see Figure 3). For each colouring of dyad, the sum of the colours in exactly one of the connectors I or O is zero. Thus its colouring set is

$$\mathrm{col}(\mathbf{D}) = \{11123, 11213, 11231, 12311, 12322, 12333\}.$$

The dyad is the unique 5-cycle cluster with 7 vertices, 8 edges, and 5 semiedges.

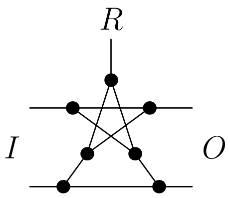

Figure 3: Dyad

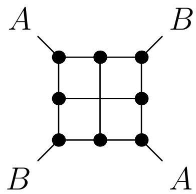

Figure 4: Isochromatic

Isochromatic

The *isochromatic* $\mathbf{I}$ is a 5-cycle cluster consisting of an 8-cycle $v_1v_2\dots v_8$ with two additional edges v_1v_5 and v_3v_7. It can be constructed from the Petersen graph by removing an edge uv together with its end vertices. Isochromatic has two connectors containing the semiedges formerly incident with u and v, respectively. Its colouring set is

$$\mathrm{col}(\mathbf{I}) = \{1111, 1122\},$$

so the edges from the same connector have always the same colour. It is the unique 5-cycle cluster having 8 vertices, 10 edges, 4 semiedges and girth 5. (There are also two other 5-cycle clusters with 8 vertices and 4 semiedges but they have girth 4.)

Triad

The *triad* $\mathbf{T}$ is a 5-cycle cluster formed by three 5-cycles C_1, C_2, and C_3 such that C_1 and C_2 have exactly one edge in common while C_3 contains the common edge of C_1 and C_2 and one additional edge of each C_1 and C_2. It can be constructed from the Petersen graph by removing one vertex and severing an edge not incident with it. The natural distribution of semiedges into connectors turns it into a $(2,3)$-pole $\mathbf{T}(B,C)$, shown in Figure 5, where the connector B corresponds to the severed edge and the connector C corresponds to the removed vertex. The sum of the colours in both connectors is non-zero for each colouring of $\mathbf{T}$, so

$$\mathrm{col}(\mathbf{T}) = \\ = \{12333, 12113, 12223, 12131, 12232, 12311, 12322\}.$$

Together with the triad, there are three 5-cycle clusters having 9 vertices, 11 edges and 5 semiedges. One of them has girth 4, the other two have girth 5 (see Figure 9 for the other one). The triad is distinguishable by the fact that it contains two pairs of dangling edges at distance 1.

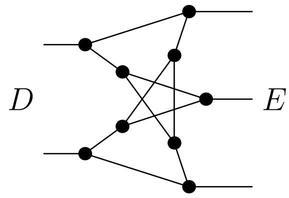

Figure 5: Triad

Heterochromatic

The *heterochromatic* is a 5-cycle cluster that arises from the Petersen graph by severing two nonadjacent edges, leaving a natural distribution of its 4 semiedges into 2 connectors containing the two dangling edges that arose by severing the same edge. Two nonadjacent edges in the Petersen graph can be at distance 1 or 2. Depending on this there are two nonisomorphic heterochromatics which are called *heterochromatic 1* and *heterochromatic 2* according to the distance of the severed edges (see Figure 6). We denote them $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. Each colouring of a heterochromatic assigns different colours to the edges from the same connector, so

$$\mathrm{col}(\mathbf{H}_1) = \mathrm{col}(\mathbf{H}_2) = \{1212, 1221\}.$$

There are four 5-cycle clusters with 10 vertices, 13 edges, 4 dangling edges and girth 5, amongst them there are $\mathbf{H}_1$ and $\mathbf{H}_2$ (see Figure 10 for the remaining two). The distinguishing property of them is that $\mathbf{H}_1$ is the only one having a pair of dangling edges at distance 1 and $\mathbf{H}_2$ is the only one with distance 2 or 4 between any two dangling edges.

Triple pentagon

The *triple pentagon* $\mathbf{tP}$ is a 5-cycle cluster consisting of three 5-cycles, each pair having two edges in common. It

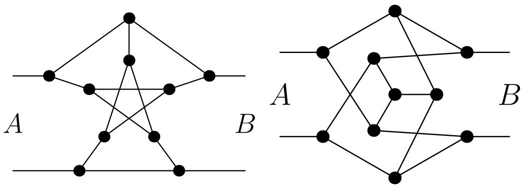

Figure 6: Heterochromatic 1 (left) and 2 (right)

can be constructed from the Petersen graph by severing three pairwise non-adjacent edges lying on a 6-cycle in an alternating order, making the triple pentagon a $(2,2,2)$-pole $\mathbf{tP}(A,B,C)$ as depicted in Figure 7. Note that the three severed edges cannot be extended to a perfect matching of the Petersen graph. Due to parity lemma, $\mathbf{tP}$ admits no colouring where in each connector, its two semiedges have the same colour. Like the double pentagon, this is not sufficient to describe the colouring set of $\mathbf{tP}$ which is exactly

$$\mathrm{col}(\mathbf{tP}) = \{111122, 111212, 112121, 112211, 112222, 112233, 121112, 121211, 121222, 121332, 122212, 122313, 122331, 123231\}.$$

The triple pentagon has 10 vertices, 12 edges and 6 dangling edges.

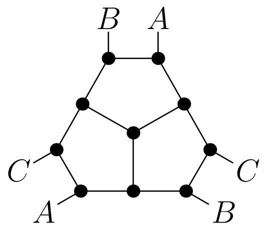

Figure 7: Triple pentagon

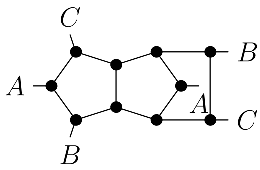

Figure 8: Tricell

Tricell

The *tricell* $\mathbf{tC}$ is a 5-cycle cluster containing three 5-cycles C_1, C_2 and C_3, where C_1 and C_2 share one edge, C_2 and C_3 share two edges, and C_1 and C_3 are disjoint. Like the triple pentagon, it arises from the Petersen graph by severing three pairwise non-adjacent edges, however, in this case these edges do not lie on a 6-cycle and can be extended to a perfect matching of the Petersen graph. The 3-cell has a natural representation as a $(2,2,2)$-pole $\mathbf{tC}(A,B,C)$ as shown in Figure 8. Since it was also constructed form the Petersen graph by severing three edges like $\mathbf{tP}$, we get the same restrictions on colourings like for $\mathbf{tP}$. However, there are some differences in the colouring set, precisely

$$\mathrm{col}(\mathbf{tC}) = \{111212, 111221, 112121, 121112, 121121, 121211, 121233, 121332, 122111, 122133, 122313, 122331, 123132, 123312, 123321\}.$$

The only 5-cycle clusters with 10 vertices, 12 edges and 6 dangling edges are the triple pentagon and the tricell. In contrast to the triple pentagon, the tricell has one dangling edge whose distance to each other dangling edge is at least 2.

Non-petersen 5-cycle clusters

There are 86 5-cycle clusters of order up to 10. We summarised their numbers divided by order and girth in Table 1. Nine of them are colour-open and contained in the Petersen graph, so there remain 77 colour-closed 5-cycle clusters.

order	5	6	7	8	9	10
$g=3$	1	3	3	7	13	12
$g=4$	0	1	2	6	9	14
$g=5$	1	0	1	2	3	8

Table 1
Distribution of the 5-cycle clusters up to order 10 according to order and girth g.

Here, we describe only 5-cycle clusters that are most relevant for our work – that is those with girth 5, because 5-cycle clusters with girth less than 5 can not appear in nontrivial snarks. Also all Petersen 5-cycle clusters have girth 5, so if we analyse snarks that may contain also colour-closed multipoles, we need to distinguish between colour-open (Petersen) and colour-closed 5-cycle clusters. We mentioned these 5-cycle clusters in previous subsection.

Among the 5-cycle clusters with girth 5 of order 9, there is triad which shares the same number of dangling edges with the 5-cycle cluster depicted in Figure 9. The remaining third 5-cycle cluster is the 3-pole obtained from the Petersen graph by removing one vertex. Of order 10, there are eight 5-cycle clusters: two heterochromatics, the triple pentagon, the tricell, the Petersen graph, two 4-poles shown in Figure 10, and the 2-pole obtained from the Petersen graph by severing an edge.

Number of colourings of 5-cycle clusters

Besides the colouring set of 5-cycle clusters, it is useful to know also number of colourings of the small 5-cycle clusters. The pentagon, double pentagon, dyad and triad all admit only one colouring for each of their possible types (up to permutation of colours). The triple pentagon

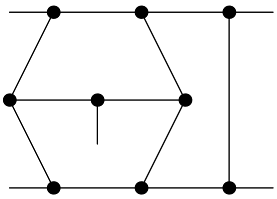

Figure 9: The non-petersen 5-cycle cluster with order 9, girth 5 and 5 dangling edges.

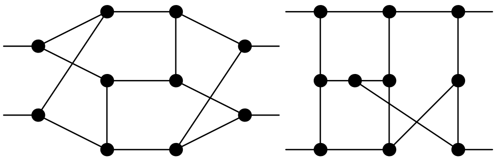

Figure 10: Two non-petersen 5-cycle clusters with order 10, girth 5 and 4 dangling edges.

has two colourings of types 111122, 112211, 112222, and 112233, and one colouring for the remaining types from col($\mathbf{tP}$). Similarly, the tricell admits two colourings of type 111221 and one for the remaining types from col($\mathbf{tC}$).

The isochromatic and both the heterochromatics admit two colouring for each of its types. Interestingly, we found no colour-open 5-cycle cluster with 4 dangling edges and an odd number of colourings for some type (note that no colouring for some type is also an even number). We state it in the following proposition which we verified using a computer for all 5-cycle clusters with 4 dangling edges up to order 20 which we generated using Algorithm 1.

Proposition 2. *If M is a colour-open 5-cycle cluster with 4 dangling edges and at most 20 vertices, then it has an even number of colourings for each colouring type.* □

4. Algorithm for finding clusters

The first step in finding 5-cycle clusters is to find all 5-cycles in a given cubic graph G. We represent a 5-cycle of G as a sequence of 5 vertices. Since the automorphism group of the 5-cycle, which is generated by rotations and reflections, has order 10, each 5-cycle of G can be represented using 10 different sequences. Thus, assuming that the vertex set of G is a subset of integers, we restrict ourselves to finding only sequences $(v_1, v_2, v_3, v_4, v_5)$ in the form, where v_1 is minimal and $v_3 < v_4$. This representation is unique for every 5-cycle since only one rotation leads to minimal v_1 and only one reflection leads to $v_3 < v_4$.

For each vertex v of G, we find the array N_1 of neighbours of v and the array N_2 of the vertices that are at distance 2 from v (that is neighbours of the vertices from N_1). We put in both N_1 and N_2 only vertices with numbers higher than v. This ensures that each found 5-tuple for a 5-cycle starts with a vertex with the smallest number. Together with the requirement $a < b$ we can conclude that the found 5-tuple is a canonical representation of a 5-cycle. Thus, Algorithm 2 finds each 5-cycle exactly once. The time complexity of Algorithm 2 is clearly $O(|V(G)|)$.

Algorithm 2: Algorithm for finding all 5-cycles in a cubic graph

foreach $v \in V(G)$ **do**
  $N_1 = (u \in N(v);\ u > v)$;
  $N_2 = (w \in N(u);\ u \in N_1 \wedge w > v)$;
  foreach $\{a, b\} \subseteq N_2,\ a < b$ **do**
    if $ab \in E(G)$ **then**
      result.add($(v, parent(a), a, b, parent(b))$);

Now we describe our algorithm for finding all maximal 5-cycle clusters in a given cubic graph.

1. Find all 5-cycles in G using Algorithm 2.
2. Determine maximal 5-cycle clusters using union-find algorithm.
3. Determine the type of each cluster according to the properties described in Section 3.
4. Determine the connections between clusters and remaining vertices of G.

In the third step, we firstly determine the order, number of dangling edges and girth of a cluster (if we do not know that we have a cluster contained in some nontrivial snark, hence with girth 5). If there is more than one suitable 5-cycle cluster, we perform additional checks to distinguish it. For that purpose and also for the purpose of further identification, we compute distances between every pair of dangling edges.

The last step is optional – sometimes it is sufficient to know only which clusters are contained in a cubic graph. For each cluster, we identify which of its dangling edges belong to which connector. Then we create a multigraph, were we contract each 5-cycle cluster M in G to a single vertex and label the edge ends leaving M according to their connectors.

We illustrate the last two steps on an example. Assume that we have a 5-cycle cluster M with 10 vertices, 4 dangling edges and girth 5. To characterise this cluster, further checks are needed. Say that we find two dangling edges e and f at distance 1. Now we know that M is the heterochromatic 1. In the step 4, we find unique dangling edges e' and f' at distance 4 from e and f, respectively.

Then the connectors of M are then $A = \{e, e'\}$ and $B = \{f, f'\}$.

An example of the output

At the end, we illustrate the output of our algorithm on one snark which we denote G_{34} of order 34. Its adjacency list together with the output is shown in Figure 11. The output firstly lists for each maximal 5-cycle cluster M of G_{34} denotation and name of M, and then for each semiedge e of M a line consisting of the connector of e, the number of the vertex incident with e and the identifier of the other end of e in G_{34}. Secondly, for each vertex v contained in no maximal 5-cycle cluster, the output contains one line listing the identifiers of the neighbours of v. The identifier of a vertex v (or equivalently edge end) is its number or the denotation of the 5-cycle cluster containing v together with the connector the semiedge incident with v is in. We see that G_{34} consists of four 5-cycle clusters: two dyads D_0 and D_2, two triads T_1 and T_3, and two vertices 10 and 11 that are contained in no 5-cycle cluster.

0: 18 12 14	17: 24 33 15
1: 20 5 6	18: 0 25 10
2: 16 32 11	19: 8 9 25
3: 8 5 21	20: 1 13 21
4: 32 28 15	21: 3 20 7
5: 1 3 15	22: 28 12 31
6: 1 10 7	23: 9 26 31
7: 16 21 6	24: 17 14 25
8: 19 3 14	25: 24 18 19
9: 19 13 23	26: 27 11 23
10: 18 11 6	27: 26 12 30
11: 2 26 10	28: 4 29 22
12: 0 27 22	29: 16 33 28
13: 9 20 30	30: 27 13 31
14: 0 8 24	31: 30 22 23
15: 17 4 5	32: 33 2 4
16: 2 29 7	33: 32 17 29

```
D0 (dyad):
B(3): D2-B
A(5): T1-D
B(6): 10
A(7): T1-D
R(20): T3-E
T1 (triad):
E(2): 11
D(15): D0-A
D(16): D0-A
E(17): D2-R
E(28): T3-E
D2 (dyad):
A(0): T3-D
B(8): D0-B
B(18): 10
A(19): T3-D
R(24): T1-E
T3 (triad):
D(9): D2-A
D(12): D2-A
E(13): D0-R
E(22): T1-E
E(26): 11
10: D2-B 11 D0-B
11: T1-E T3-E 10
```

Figure 11: An adjacency list of a snark of order 34 (left) and output of our program (right).

Following this output one can easily draw G_{34} schematically as depicted in Figure 12. This drawing does not hold complete information – we would obtained the same multigraph if we, for instance, replaced the edges 5-15 and 7-16 with 5-16 and 7-15. However, this representation of G_{34} still holds enough information to prove, using the properties of dyads and triads from Section 3, that G_{34} is not 3-edge-colourable. Indeed, suppose to the contrary that G_{34} is 3-edge-colourable. Then the two colours in the connector A of D_0 are different, since they come from the triad T_1. Thus the colours in B are the same. Analogously, the two colours in the connector B of D_2 are the same, but then the vertex 10 is incident with two edges of the same colour – a contradiction.

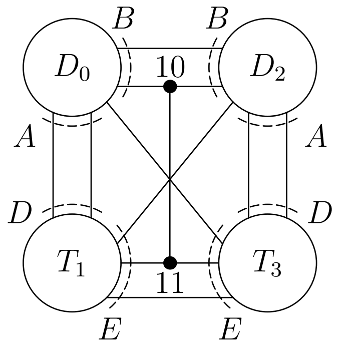

Figure 12: A schematic drawing of the snark G_{34}

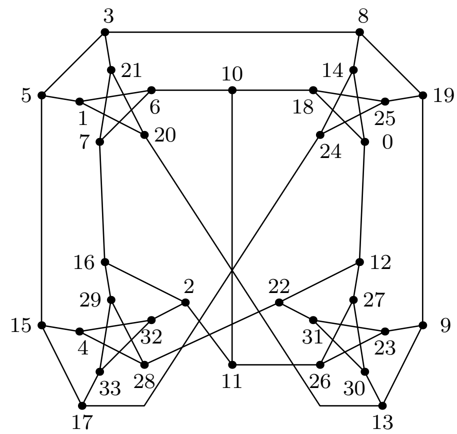

Figure 13: A drawing of the snark G_{34}

5. Applications

Finally, in this section we describe how we used analysis of 5-cycle clusters in our research.

Structural analysis of snarks

In 2013, Brinkmann et al. [17] generated all nontrivial snarks up to order 36. They revealed that vast majority of snarks, at least up to order 36, has girth 5, so they can be analysed using 5-cycle clusters.

Using this approach we described all snarks up to order 36. We analysed the structure of all critical cyclically 5-edge-connected snarks, where a snark is *critical* if removal of any two adjacent vertices produces a 3-edge-colourable graph. Then we described a set of simple operations using which we are able to construct all the remaining snarks.

Moreover, we generalised the structure of these small snarks to several infinite families, where we use instead of the 5-cycle clusters larger multipoles with similar colouring properties. Those multipoles, that are not necessary 5-cycle clusters, are constructed from other snarks in the same way that the corresponding Petersen clusters are constructed from the Petersen graph.

We illustrate these results on the following example. Consider the infinite family of cubic graphs $\mathcal{F}$ constructed as follows: Take three 5-poles N, T_1 and T_2, where N is obtained from some snark by removing a path of length 2, and each of T_1 and T_2 is constructed from some snark (not necessary distinct) by severing an edge and removing a vertex. We then connect N, T_1 and T_2 as depicted in Figure 14. Note that if N, T_1 and T_2 are all constructed from the Petersen graph, we obtain a dyad and two triads.

We proved that N has similar colouring properties like the dyad and T_1 and T_2 have similar colouring properties like the triad, precisely $\mathrm{col}(N) \subseteq \mathrm{col}(\mathbf{D})$ and $\mathrm{col}(T_1), \mathrm{col}(T_2) \subseteq \mathrm{col}(\mathbf{T})$. This implies that all graphs in $\mathcal{F}$ are not 3-edge-colourable. Out of 2110 critical cyclically 5-edge-connected snarks, we found out that 1718 of them are contained in the class $\mathcal{F}$. Although in most cases not every one of the 5-poles N, T_1 and T_2 is a 5-cycle cluster, in each of those snarks, at least two of N, T_1 and T_2 are 5-cycle clusters due to the order not exceeding 36. Thanks to this we were able to identify them using 5-cycle clusters.

For further details and results of our analysis, we refer the reader to [16].

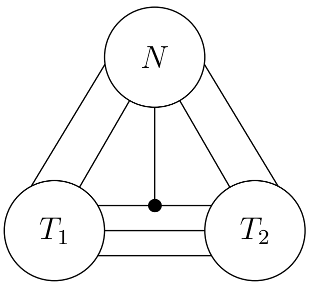

Figure 14: A schematic drawing of snarks contained in the family $\mathcal{F}$

Uniquely 3-edge-colourable graphs

Interestingly, the only know example of a cyclically 4-edge connected uniquely 3-edge colourable cubic graph is the generalized Petersen graph $P(9, 2)$ on 18 vertices [18], although there are infinitely many instances containing triangles [19] or cycle separating 3-edge-cuts [20].

For the purpose of finding more uniquely 3-edge-colourable cubic graph, it is useful to construct multipoles that can be contained in them. We say that a multipole is *possibly uniquely 3-edge-colourable* if it has exactly one colouring for at least one of its types. By Proposition 2 we know that every 5-cycle cluster with 4 dangling edges and at most 20 vertices is not possibly uniquely 3-edge-colourable. Hence it can occur in no uniquely 3-edge-colourable cubic graph.

Thus when one wants to construct a possibly uniquely 3-edge-colourable multipole from a graph G by removing some vertices or severing some edges, one has to ensure that the resulting multipole contains no 5-cycle cluster of order at most 20 and 4 dangling edges. This can be a useful heuristic in finding such multipoles – when we know that a snark G contains some not possibly uniquely 3-edge-colourable multipole M, we know that we need to remove a vertex from M or sever a link of M which narrows down possibilities of possible construction of multipoles form G.

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References

[1] P. G. Tait, Remarks on the colourings of maps, Proc. R. Soc. Edinburgh 10 (1880) 729.

[2] M. Gardner, Mathematical games: Snarks, boojums and other conjectures related to the four- color-map theorem., Sci. Am. 324 (1976) 126–130.

[3] W. T. Tutte, A contribution to the theory of chromatic polynomials, Canad. J. Math. 6 (1954) 80–91.

[4] P. Seymour, On multi-colourings of cubic graphs, and conjectures of fulkerson and tutte, Proc. London Math. Soc. 38 (1979) 423–460.

[5] G. Szekeres, Polyhedral decomposition of cubic graphs, Bull. Aust. Math. Soc. 8 (1973) 367–387.

[6] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, Math. Programming. 1 (1971) 168–194.

[7] G. Mazzuoccolo, The equivalence of two conjectures of berge and fulkerson, J. Graph Theory 68 (2011) 125–128.

[8] C. Q. Zhang, Hamiltonian weights and unique edge-3-colorings of cubic graphs, J. Graph Theory 20 (1995) 91–99.
[9] D. Greenwell, H. Kronk, Uniquely line-colorable graphs, Canad. Math. Bull 16 (1973) 525–529.
[10] A. Thomason, Hamiltonian cycles and uniquely edge colourable graphs, Annals Disc. Math 3 (259–268) 1978.
[11] L. Kászonyi, On the structure of coloring graphs, Ann. Univ. Sci. Budapest Eötvös Sect. Math. 16 (1973) 25–36.
[12] R. C. Bradley, Snarks from a kászonyi perspective, Discrete Appl. Math. 189 (2015) 8–29.
[13] A. P. B. D. McKay, Practical graph isomorphism, ii, J. Symbolic Computation 60 (2013) 94–112.
[14] M. A. Fiol, A boolean algebra approach to the construction of snarks, Graph Theory, Combinatorics and Applications 1 (1991) 493–524.
[15] B. Descartes, Network-colourings, The Mathematical Gazette 32 (1948) 67–69.
[16] M. J. Mazák, J. Rajník, Morphology of small snarks., arXiv:2112.04335, 2021.
[17] G. Brinkmann, J. Goedgebeur, J. Hägglund, K. Markström, Generation and properties of snarks, J. Combin. Theory Ser. B. 103 (2013) 468–488.
[18] S. Fiorini, On the chromatic index of a graph, iii: Uniquely edge colourable graphs„ Quart. J. Math. Oxford Ser. 26 (1975) 129–140.
[19] T. Fowler, Unique coloring of planar graphs, Ph.D. thesis, Georgia Institute of Tech-nology Mathematics Departmenta, 1998.
[20] S.-M. Belcastro, R. Haas, Triangle-free uniquely 3-edge colorable cubic graphs, arXiv:1508.06934, 2015.