A Goodman-style Betweenness Relation on Orthoframes

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Abstract

Understanding inductive and deductive reasoning requires in some or other form the representation of concepts. A basic assumption underlying Gärdenfors' linguistic-cognitive framework of conceptual spaces as well as the machine-learning based framework of knowledge-graph embeddings is that concepts and reasoning over them are geometrical. The main ingredient of Gärdenfors' conceptual spaces is a ternary betweenness relation which is the basis for defining concepts as convex (= betweenness-closed) sets. Though many interesting phenomena of cognitive reasoning can be explained in the framework of conceptual spaces, it is at least not obvious how to use betweenness for other, more logico-formal aspects of reasoning that, e.g., require defining logical operators. In particular, for the logical operator of negation other mathematical structures such as the orthoframes of Goldblatt have proven more useful. In this paper, we provide first ideas and results on the connection between conceptual spaces and orthoframes. The main technical result of this paper concerns the definition of a betweenness relation within an orthoframe. The construction is an adaptation of Goodman's mereology-based betweenness relation over so-called qualia to a set-theoretic betweenness relation based on an orthoframe.

Keywords

conceptual space, orthonegation, geometry

1. Introduction

Enriching sub-symbolic by conceptual information or understanding reasoning in general requires the representation of concepts as well as operators on them, such as conjunction, and relations between them, such as subsumption. One possible representation of this kind is the framework of conceptual spaces introduced by Gärdenfors [1]. His claim is that concepts can be represented geometrically as (sets of) convex regions in a space, whereas subsymbolic information is represented as points in this space. This enables to combine sub-symbolic information, e.g., as used in neural networks, with symbolic information.

In conceptual spaces, (logical) conjunction of concepts is easily defined as intersection, however, it is not straight-forward to define other logical operators. According to Gärdenfors, the definition of negation is particularly challenging [1]. Concepts in conceptual spaces are defined as convex sets. The basis for this definition is a notion of betweenness: convex sets are those sets containing with each pair of elements also all elements in between them. Thus, the

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question arises, whether it is possible to define a representation which sticks to the advantages of a conceptual space by ensuring convexity but can also represent logical operations that preserve convexity, e.g., a negation operator such that negated concepts are convex.

One such representation allowing to define negation is that of orthoframes as introduced by Goldblatt [2]. Orthoframes are structures with a domain and a binary relation that is irreflexive and symmetric. Hence, formally orthoframes correspond to symmetric loopless graphs. Orthoframes are immensely important structures, not only because they can be used to define an orthonegation, a generalization of classical Boolean negation. But additionally, as shown by Goldblatt [2, 3], orthoframes are able to represent any kind of orthonegation.

So, how does one combine these two frameworks—the one focusing on a (ternary) betweenness relation, the other on a (binary) orthogonality relation—in order to get the best of both worlds? There has been some relevant work dealing with the problem of defining one relation based on the other. For example, in Appendix B of [4] one can find a survey of betweenness relations defined in the context of partial orders and [5] gives a definition of betweenness based on reflexive antisymmetric relations. But until now, as far as we can tell, no approach tackled the interconnection of betweenness with binary relations in the context of negation and orthogonality relations.

This paper tries to fill this gap with some first ideas on understanding the interconnection of betweenness and orthogonality. Technically, we define a betweenness relation based on a construction going back to Goodman [6]. Goodman's betweenness notion is couched in mereological terms (terms developed in a theory of parts and wholes [7]) and is defined for qualia. Qualia are abstract phenomenal qualities, describing a sensory impression of a property under a condition, e.g., the color of an object under a specific illumination. Our key idea for adapting Goodman's approach comes from the observation that his definition of the betweenness relies on a binary relation called matching, which is intended to reflect "similarity" of qualia. We adapt his construction to a set-theoretical setting, replacing matching by the induced relation of non-orthogonality in an orthoframe. This observation leads to our main technical result, the definition of a betweenness-relation for orthoframes that enables the representation of convex concepts and their negations which are also convex.

The rest of the paper is structured as follows: In Section 2, the idea of conceptual spaces introduced by Gärdenfors is discussed and his notion of betweenness is introduced. Whereas in Section 2 negation is considered in a conceptual-space point of view, in Section 3, orthoframes and the notion of orthonegation are introduced. There, also an example for an orthonegation defined via betweenness is given. In Section 4, the approach of Goodman is introduced and connected to the betweenness-relation of Gärdenfors. The main theoretical result of the paper can be found in Section 5, where the notion of betweenness introduced by Goodman is adapted to orthoframes. The paper ends with a short conclusion.

2. Conceptual Spaces and Betweenness

Gärdenfors' conceptual spaces (see the textbook [1] and the summary in [8], on which this short section is partially based) are the main elements of a cognition oriented semantics of language which tries to fill the gap between sub-conceptual semantics on a sub-symbolic level as that of neural networks and the higher symbolic level of logical languages.

The basic assumption underlying the general framework of conceptual spaces is that the nature of concepts and the nature of reasoning with concepts is geometrical. Conceptual spaces are geometrical and as such are not only relevant for spatial (and temporal) reasoning, as argued in [9], but generally for representing and reasoning with concepts. The same assumption on the geometrical nature of concepts is at the core of knowledge-graph embedding (KGE) [10]. In KGEs one also has some domain X with dimensions, but usually these dimensions do not come with a prescribed meaning (they are latent), and the space X has usually a richer mathematical structure in that it is a Hilbert space.

The ingredients of conceptual spaces can be illustrated with Gärdenfors' example of the color concept. The color of an object is determined by three quality dimensions, chromaticness (saturation), brightness, and hue. The first two of these are isomorphic to $\mathbb{R}$ with the usual ordering while the latter is isomorphic to a circle in the plane. Other examples of quality dimensions are force, height, width. This aspect of quality dimensions marks already a significant difference to the vector-spaces used in knowledge graph embeddings (KGEs), as in KGEs the dimension do not have a pre-defined meaning but are latent entities "gaining their meaning" during the embedding process.

Defining the color of an object calls for assigning it to a point in a conic-shaped space induced by the three quality dimensions. One cannot assign a value on one dimension without giving a value on the other dimensions. Gärdenfors says that this set of quality dimensions is *integral*. The complementary notion is that of *separability*. The space made up of a set of integral quality dimensions that are separable from all other quality dimensions is called a *domain*. Note that according to this definition, a domain is identified by its dimensions.

The most important component of conceptual spaces is the betweenness relation. A betweenness relation $btw(x, y, z)$ is a ternary relation with the intended meaning that point y is between points x and z. This notion generalizes that of a metric d, which induces a betweenness relation btw_d by setting $btw_d(x, y, z)$ iff $d(x, z) = d(x, y) + d(y, z)$.

In our following considerations we are going to deal with this bare bone of conceptual spaces. Thus, we will be dealing mainly with structures of the form (X, btw) where X is an arbitrary set, also called domain, and a ternary relation btw fulfilling some properties expressed by axioms. (So we abstract from the fact that there are dimensions with a predefined meaning.) We are going to work with Gärdenfors' list of axioms for the betweenness relation btw.

Definition 1. *Betweenness axioms according to Gärdenfors are the following:*

$$\textit{If } btw(a, b, c)\textit{, then } a, b, c \textit{ are distinct} \tag{B0}$$

$$\textit{If } btw(a, b, c)\textit{, then } btw(c, b, a) \tag{B1}$$

$$\textit{If } btw(a, b, c)\textit{, then not } btw(b, a, c) \tag{B2}$$

$$\textit{If } btw(a, b, c) \textit{ and } btw(b, c, d)\textit{, then } btw(a, b, d) \tag{B3}$$

$$\textit{If } btw(a, b, d) \textit{ and } btw(b, c, d)\textit{, then } btw(a, b, c) \tag{B4}$$

Axiom (B0) constrains the ternary relation btw to what is sometimes called *open betweenness* [4]. (B1) expresses commutativity of betweenness w.r.t. the outer points. (B2) expresses that if a point is between two points it cannot have one of the points in between itself and the other point. (B3) is sometimes called *outer transitivity* and (B4) *inner transitivity.*

We stick to the talk of "betweenness axioms according to Gärdenfors" though, of course, the axioms mentioned above are part of the folklore definitions of betweenness (see, e.g., [5]). Actually, Gärdenfors refers the reader to [11] for the notion of betweenness. But note that [11] treats betweenness as a primitive notion next to those of lines and points. The axioms (B0)–(B4) Gärdenfors mentions are only those that do not refer to lines and points. But there are additional theorems of geometry (as developed in [11]) that constrain the betweenness relation even further. One such property is stated as axiom (B5) below.

The betweenness relation is the basis for defining convex sets Y within a domain X. A set Y is *convex* iff for all pairs of points contained in Y one has also all points lying in between them in Y, formally: for all $x, y, z \in X$: if $x, z \in Y$ and $btw(x, y, z)$, then also $y \in Y$. In a way, convex sets are well-shaped and as such are candidates for the mental pendants of concepts. Gärdenfors defines a *(natural) concept* to be a set of convex regions in possibly multiple domains.

What are the reasons to define concepts on the basis of convex regions? The main reason is that of "cognitive economy" [1, p. 70], as learning and reasoning with concepts on convex regions seems to demand less cognitive capacities, or, more technically, less resources such as space (memory) and time. Technical underpinnings of this observation can be found in different areas in which convex regions seem to ease the computation [12, 13, 14].

Nice at it is from the philosophical or cognitive-linguistic point of view, the conceptual-space approach has problems in defining linguistic or logical constructors for combining concepts. The only real concept constructor defined in [1] is a form of combination that bears similarities to concept conjunction, but is a bit more complicated [1, p.122], as it tries to explain adjective noun combinations such as "small elephant". Nonetheless, restricting the combinations to simple noun-noun combinations with "and" may allow the interpretation by intersection of convex regions. Because the intersection of convex regions is again a convex region, the operator would be well-defined. In so far, the approach of conceptual spaces is more promising than, say, support vector machines. These lead to representations of concepts by half-spaces, which are not closed under intersection. (See also the discussion in our forthcoming [15]).

But according to Gärdenfors, the main problem within the conceptual-space framework is the handling of negation and quantifiers [1, p.202]. One can readily add disjunction as being as difficult because disjunction can be defined via De Morgan through conjunction and negation.

When talking about negation, Gärdenfors probably has a more sophisticated, cognitively and linguistically founded operator in mind—which must have lead him to the conviction that handling negation is hard. In the next section we are going to tackle the point of negation from a logical or rather lattice theoretical point of view and discuss in how far defining negation in structures with a betweenness relation is possible.

3. The Case of Negation and Orthoframes

Following roughly the wording of the title in Gabbay's paper [16], we would like to know "What is negation in conceptual spaces in 2022"? A similar question is of importance for KGEs: "What is negation in embedding approaches in 2022"? Because also for KGEs the question of the kinds of supported negations is still not settled in a satisfying manner. Some ideas—couched in geometrical terms—have been described [8, 17, 18, 19]. In particular, the approach of [19] pushes the limits of expressivity: whereas [8, 17, 18] do not allow for full negation of concepts to be represented, [19] defines negation for a model of cones that uses polarity.

We propose that the theory of orthologics and orthoframes according to [2] can take the role of a foundational geometric framework that at least can be considered as a useful completion of KGEs and conceptual spaces in particular w.r.t. negation. Here we want to give an informal argument for our claim.

Orthologics are a generalization of propositional logic that provide a full notion of negation called orthonegation and (using De Morgan) also full disjunction. Orthonegation exhibits not all properties of Boolean negation but at least it fulfills antitonicity (contraposition), the intuitionistic absurdity principle (anything follows from a sentence stating A and its negation) and allows for double negation elimination. Now, as Goldblatt showed, any orthologic is characterizable by a very basic class of structures, called orthoframes, that is as simple and basic as conceptual structures. Formally, an *orthoframe* $(X, \perp)$ consists of a domain X and a binary *orthogonality relation* $\perp \subseteq X \times X$, i.e., a relation that is irreflexive and symmetric. In other words, an orthoframe is nothing else than a symmetric graph without self-loops. Of course, when Goldblatt invented the notion of an orthoframe he had in mind (also) examples of a more geometrical kind, e.g., X being a vector space and $\perp$ standing for the usual orthogonality relation that holds between two vectors when they have an angle of 90 degrees.

Now, Goldblatt's "concepts" in an orthoframe are defined as $\perp$-closed sets: Define $Y^{\perp} = \{x \in X \mid \text{for all } y \in Y : x \perp y\}$. A set Y is $\perp$-closed iff $Y^{\perp\perp} = Y$. The class of all orthogonality-closed sets makes up an ortholattice, i.e., a lattice with $(\cdot)^{\perp}$ as ortholattice complement. In this paper we call the orthocomplement orthonegation. Applied to $\perp$-closed concepts, the three properties of orthonegation mentioned above are explicated formally as follows: $Y_1 \subseteq Y_2$ entails ${Y_2}^{\perp} \subseteq {Y_1}^{\perp}$ (antitonicity); $Y^{\perp\perp} = Y$ (double negation elimination); and $\emptyset = Y \cap Y^{\perp}$ (intuitionistic absurdity).

Part of our research program is the working hypothesis that under further constraints on an orthoframe rich betweenness relations can be defined that fulfill all of Gärdenfors' betweenness axioms mentioned above and even other axioms that have been discussed as potential betweenness axioms [20, 21]. In this paper we show that at least we can use an adaptation of Goodman's notion of betweenness that fulfills all axioms except for axiom (B3).

Before proving this in the next section, we close this section with a simple example illustrating the other direction regarding the interconnection of orthoframes and betweenness: given a betweenness relation on X, we can define an orthogonality relation on the same space X. The starting point is the observation that a conceptual space, read as a structure (X, btw) with a betweenness relation btw already comes equipped with a means to define an orthoframe (and hence negation). Take any an arbitrary and henceforth fixed element $0 \in X$. Then define an orthogonality relation for all $x, y \neq 0$ as follows:

$$x \perp_0 y \text{ iff } btw(x, 0, y) \tag{1}$$

The relation is indeed an orthogonality relation: it is irreflexive due to (B0) and symmetric due to (B1). So, we have a means to define a an orthogonality relation even in conceptual spaces where betweenness does not necessarily fulfill all axioms of Gärdenfors. Now, based on $\perp_0$ we can consider the $\perp_0$-closed subsets as concepts on which a form of negation is defined, namely orthonegation of the ortholattice induced by $\perp_0$. So, why did Gärdenfors think defining negation in a conceptual space is a challenging task? The first point was mentioned above: he probably had an advanced, cognitively and linguistically justified notion of negation in mind. But even for orthonegation there might be a reason to consider defining an appropriate one based on betweenness challenging. As we show below, it might be the case that the class of $\perp_0$-closed subsets is not sufficiently rich.

To get it right from the beginning: If we do not make further restrictions on the betweenness relation other than those expressed by (B1)-(B4) (or even just (B0) and (B1)), then $\perp_0$ may give a rich set of concepts. But if we consider further natural restrictions on betweenness, the set of $\perp_0$-closed sets becomes too simple. By "natural restriction" we mean properties that follow from considering betweenness in absolute geometry [11]. Absolute geometry is the geometry developed by Euclid on the basis of the notions of points, lines and planes (and the incidence relation)—but without the parallel axiom. Now, we consider one of the axioms of absolute geometry (see [11, axiom I3, page 21]) that states that for two distinct points a, b there is maximally one line containing them. We transfer this axiom into a property for the betweenness relation as expressed by (B5).

$$btw(y, z_1, z_2) \text{ or } btw(y, z_2, z_1) \text{ or } z_1 = z_2 \text{ if } btw(x, y, z_1) \text{ and } btw(x, y, z_2) \tag{B5}$$

So assume that btw fulfills (B0)–(B5). Note that we can define notions of line and half-line on the basis of a betweenness notion as follows [11]. Let $a, b \in X$ be arbitrary, then define the line $L(a, b)$ through a, b as

$$L(a, b) := \{x \mid x = a \text{ or } x = b \text{ or } btw(a, x, b) \text{ or } btw(x, a, b) \text{ or } btw(a, b, x)\} \tag{2}$$

Given a line L and points a, b on L, one has two disjoint open half-lines $HL(a, b)$ and $HL^*(a, b)$ starting at a such that $L = \{a\} \cup HL(a, b) \cup HL^*(a, b)$. These can be defined as follows:

$$HL(a, b) = \{x \mid btw(a, x, b) \text{ or } btw(a, b, x) \text{ or } x = b\} \tag{3}$$
$$HL^*(a, b) = \{x \mid btw(x, a, b)\} \tag{4}$$

The property expressed in (B5) just ensures that the notion of line is a proper one. Now, the $\perp_0$-closed sets are nothing else than half-lines starting at 0.

Proposition 1. *If btw fulfills $(B1) - (B5)$ and $\perp_0$ is defined according to (1), then for all sets Y in the orthoframe $(X, \perp_0)$: Y is $\perp_0$-closed iff it is the empty set or or the complete space X or a half-line starting at 0.*

The proof is omitted due to space restrictions. As an example for a betweenness-relation fulfilling this restriction a two-dimensional space and the betweenness-relation btw_d based on the Euclidean distance can be considered.

In fact, the $\perp_0$-closed sets are also closed w.r.t. the betweenness relation, i.e., they are concepts according to Gärdenfors. But though we have a proper notion of negation the resulting set of "concepts" is not sufficiently rich: being half-lines they are of dimension 1 and the conjunction of two half-lines does not lead to interesting other concepts, only half-lines again or the empty set.

4. Goodman's Original Approach Based on Matching

In note 20 of his book on conceptual spaces [1] Gärdenfors remarks that Nelson Goodman [6] establishes a notion of betweenness based on his notion of matching. Nelson Goodman's approach is relevant for the topic of this paper, because matching is a binary, symmetric and reflexive relation. Now, a similarity notion $\sim$ can be defined as the negation of an orthogonality relation $\perp$ as follows:

$$x \sim y \text{ iff not } x \perp y \tag{5}$$

According to this definition, $\sim$ is a symmetric and reflexive relation, hence it can take the role of Goodman's matching. Gärdenfors notes that Goodman meant to show that he can establish many properties of the betweenness relations, which suggests that not all of (B0)–(B4) are fulfilled. In fact, in Goodman's book we found only proofs showing that (B0)-(B2) are fulfilled.

The ideas of Nelson Goodman concern a betweenness relation of so-called qualia, abstract phenomenal qualities, describing a sensory impression of a property under a condition, e.g., the color of an object under a specific illumination. This constraint is in fact not a problem, as his general construction can be used also for non-qualia (as he himself states and as we show below with our adaptation). The theory of qualia is developed in a mereological system (a theory of parts and wholes) with a mereological sum operator (see, e.g., [7] for a modern, book length treatment and [22] for recent advances in the intersection of geometry and mereology). We adapt Goodman's approach to a set theoretical setting (replacing sum by union). Though mereology has its merits and a long tradition in epistemology, metaphysics, and ontology research there are technical reasons [23] why it has not found the same acceptance as set theory—at least by mathematicians.

Goodman defined his notion of betweenness only for finite domains. This is the other point in which our adaptation of Goodman's construction deviates from the original approach: we consider also infinite (and even more: dense and continuous) spaces as they are used in general by embedding approaches.

The relevant betweenness relation of Goodman is called "betwixtness" (an old English word for betweenness) and reads as follows [6, Definition D10.02, p. 219]:

$$x/y/z \quad \text{iff} \quad M(x,y) \ \& \ M(y,z) \ \& \ M(x,z) \ \& \ G(x\dagger z, x\dagger y) \ \& \ G(x\dagger z, y\dagger z) \tag{6}$$

$M(x,y)$ stands for Goodman's reflexive and symmetric notion of matching. This notion seems to be a primitive notion, i.e., to be undefined. The definition Goodman gives for matching [6, Definition DA-2, p. 205] relies on the notion of allying "A". This notion in turn is defined on the basis of M. So M is given only "implicitly" defined by some axioms [6, pp. 209 ff]. He has a mereological notion of symmetric difference denoted by the dagger † (p. 211) and a binary notion of aggregatively greater than $G(x,y)$. [6, Definition D8.021, p. 182] . This one is based on a primitive notion of "is of equal aggregate size" Z. (Translated to set theory Z just means:

has the same cardinality). Definition D8.021 reads: $G(x, y)$ iff there is a proper part t of x such that t is equal in aggregate size to y. And this seems to be the main place where Goodman assumes a finite domain: Because, if x where infinite, then one could find a proper part (a proper subset of x) that is of equal size as x. But then x would be greater than itself.

According to Goodman [6, p. 219], his betweenness relation fulfills the following axioms (with the corresponding enumeration in Goodman's book given in brackets): (B0) (Goodman: 10.25), (B1) (Goodman: 10.26), (B2) (Goodman: 10.27). There is no proof of (B3), which our adaptation does not fulfill too, and of (B4), which in case of our adaptation is provable.

(B3) is problematic due to several reasons. One problem is that each pair of x, y, z must match. And this cannot be guaranteed: whereas each pair in $\{a, b, c\}$ and in $\{b, c, d\}$ may match, a and d may not match anymore. However, even when matching of each pair is assumed, (B3) is not necessarily fulfilled, as, e.g., the creation of circles in similarity (meaning $a \sim b, b \sim c, c \sim d, d \sim a$) could lead to the case that $btw(a, b, c)$ and $btw(b, c, d)$, but $btw(d, a, b)$ instead of $btw(a, b, d)$.

Despite that (B3) and (B4) are strongly related, (B4) is (in this context) simpler to establish than (B3). (B4) represents the inner transitivity, thus the elements a and d restrict the subset which needs to be considered. In contrast, (B3) represents outer transitivity, hence (B3) needs to be fulfilled for a relation $btw(a, b, c)$ for each d with $btw(b, c, d)$, thus for an unrestricted region.

5. Goodman-style Betweenness on Orthoframes

Let $(X, \perp)$ be an orthoframe and $\sim$ be the negation of $\perp$ according to Eq. (5). Let $A \triangle B = A \setminus B \cup B \setminus A$ denote the symmetric difference of two sets A, B. We define a corresponding notion of symmetric difference $x \oplus y$ for elements $x, y \in X$ of an orthoframe as follows:

$$x \oplus y = \{z \in X \mid z \sim x\} \triangle \{z \in X \mid z \sim y\} \tag{7}$$

So $x \oplus y$ denotes the set of elements z which are similar to one of $\{x, y\}$ but not the other.

Now, our adaptation of Goodman's betweenness relation is given in the following definition.

Definition 2. *The* Goodman-style betweenness relation based on subset-inclusion *over an orthoframe* $(X, \perp)$ *is defined as follows:*

$$b(x, y, z) \quad \textit{iff}\; x \sim y \;\&\; y \sim z \;\&\; x \sim z \;\&\; x \oplus y \subsetneq x \oplus z \;\&\; y \oplus z \subsetneq x \oplus z \tag{8}$$

This mimics Goodman's construction but replaces the greater-than relation G with proper set inclusion. The reason is that we—in contrast to Goodman—do not assume that X is finite.

Which of the betweenness axioms (Bi) above are fulfilled? We show that (B0), (B1), (B2), and (B4) are fulfilled and construct a counterexample for (B3).

In the proof of (B4) we need the following lemma.

Lemma 1. *If* $x \oplus y \subseteq x \oplus z$ *holds, then:* $x \oplus y \subsetneq x \oplus z$ *is the case iff* $y \oplus z \neq \emptyset$.

Proof. Assume $x \oplus y \subsetneq x \oplus z$, then there is $v \in x \oplus z$ and $v \notin x \oplus y$. If $v \sim x$ and not $v \sim z$, then we must have $v \sim y$ and hence $v \in z \oplus y$. If not $v \sim x$ and $v \sim z$, then $v \sim y$ cannot be the case and we again have $v \in z \oplus y$. Assume now that $y \oplus z \neq \emptyset$, say with $v \in z \oplus y$, i.e., $v \sim z$ iff not $v \sim y$. We have to show that $x \oplus y \subsetneq x \oplus z$ holds. If not, then we have $x \oplus y = x \oplus z$. On the one hand $v \in x \oplus z$ iff ($v \sim x$ iff not $v \sim z$). On the other hand $v \in y \oplus z$ iff ($v \sim x$ iff not $v \sim y$). So we have ($v \sim x$ iff not $v \sim z$) iff ($v \sim x$ iff not $v \sim y$). Contradiction. □

Proposition 2. *The Goodman-style betweenness relation according to Equation* (8) *fulfills (B0), (B1), (B2), and (B4) but in general not (B3).*

Proof.

Ad (B0): If $btw(x, y, z)$, then x, y, z are distinct. The definition guarantees that x and z must be different. Because otherwise $x \oplus z = \emptyset$ and the empty set cannot have a proper subset. If $x = y$ were the case, then substituting in the definition of $btw(x, y, z)$ the y with x would lead to $x \oplus z \subsetneq x \oplus z$, a contradiction. Analogously the assumption $y = z$ leads to a contradiction.

Ad (B1): If $btw(x, y, z)$, then $btw(z, y, x)$. This follows from the fact that $\oplus$ is symmetric: Let $btw(x, y, z)$, i.e., $x \sim y \;\&\; y \sim z \;\&\; x \sim z \;\&\; x \oplus y \subsetneq x \oplus z \;\&\; y \oplus z \subsetneq x \oplus z$. Then $x \sim y \;\&\; y \sim z \;\&\; x \sim z \;\&\; z \oplus y \subsetneq z \oplus x \;\&\; y \oplus x \subsetneq z \oplus x$, i.e., $btw(z, y, x)$.

Ad (B2): If $btw(x, y, z)$, then not $btw(y, x, z)$. Let $btw(x, y, z)$, i.e., $x \sim y \;\&\; y \sim z \;\&\; x \sim z \;\&\; x \oplus y \subsetneq x \oplus z \;\&\; y \oplus z \subsetneq x \oplus z$, and assume $btw(y, x, z)$. Then the former implies $y \oplus z \subsetneq x \oplus z$, whereas the latter implies $x \oplus z \subsetneq y \oplus z$, a contradiction.

Ad (B3): Not necessarily: If $btw(x, y, z)$ and $btw(y, z, w)$, then $btw(x, y, w)$. Consider the following counterexample: $(X, \sim)$ with $X = \{a, b, c, d, e\}$ and $\sim$ with: $a \sim b, a \sim c, b \sim c, b \sim d, c \sim d, c \sim e$ and $d \sim e$ and assertions $v \sim v$ and if $v \sim w$ also $w \sim v$ for all $v, w \in X$. Then: $a \oplus c = \{d, e\}, a \oplus b = \{d\}, b \oplus c = \{e\}$, hence $btw(a, b, c)$; moreover, $b \oplus d = \{a, e\}, c \oplus d = \{a\}$, hence $btw(b, c, d)$. But $a \sim d$ does not hold, hence $btw(a, b, d)$ does not hold.

Ad (B4): If $btw(x, y, z)$ and $btw(y, w, z)$, then $btw(x, y, w)$. Let $btw(x, y, z)$, i.e.,

$$x \sim y \;\&\; y \sim z \;\&\; x \sim z \;\& \tag{9}$$
$$x \oplus y \subsetneq x \oplus z \;\& \tag{10}$$
$$y \oplus z \subsetneq x \oplus z \tag{11}$$

and let $btw(y, w, z)$, i.e.,

$$w \sim y \;\&\; y \sim z \;\&\; w \sim z \;\& \tag{12}$$
$$y \oplus w \subsetneq y \oplus z \;\& \tag{13}$$
$$w \oplus z \subsetneq y \oplus z \tag{14}$$

We have to show: $bwt(x, y, w)$, i.e., that the following conditions hold:

$$x \sim y \;\&\; y \sim w \;\&\; x \sim w \;\& \tag{15}$$
$$x \oplus y \subsetneq x \oplus w \;\& \tag{16}$$
$$y \oplus w \subsetneq x \oplus w \tag{17}$$

Proof of (15): We have trivially $x \sim y$ and $w \sim y$. But we also have $x \sim w$ due to the following: $y \oplus w \subsetneq y \oplus z$ hold. Now, as $y \sim x$ and $z \sim x$ it follows that $x \notin y \oplus z$. Assume for sake of contradiction that not $x \sim w$. As $y \sim x$, this would give us $x \in y \oplus w$, but then x would have to be in $y \oplus z$, leading to a contradiction. Proof of (16): $x \oplus y \subsetneq x \oplus w$. We show first that the $\subseteq$ relation holds: Assume that v exists with $v \in x \oplus y$ and thus $v \in x \oplus z$. Case 1: $v \sim x$ and not $v \sim y$. Thus, not $v \sim z$. We have to show that $v \in x \oplus w$, i.e., not $v \sim w$. If $v \sim w$ were the case, then $v \in w \oplus z \subsetneq y \oplus z$, thus $v \sim y$, a contradiction. Case 2 is analog to case 1: not $v \sim x$ and $v \sim y$. Thus, $v \sim z$. We have to show that $v \in x \oplus w$, i.e., $v \sim w$. If not $v \sim w$ were the case, then $v \in w \oplus z \subsetneq y \oplus z$, thus not $v \sim y$, a contradiction.

Now we are left with showing that $x \oplus y$ is a proper subset of $x \oplus w$ or in other words that $x \oplus w$ is not a subset of $x \oplus y$. Due to Lemma 1, in order to show $x \oplus y \subsetneq x \oplus w$, we have to show that $y \oplus w$ is not empty. But this is due to Lemma 1 and (14).

Proof of (17): We have to show $y \oplus w \subsetneq x \oplus w$. We start again by showing $\subseteq$: Let $v \in y \oplus w$. Case 1: $v \sim y$ and not $v \sim w$. We have to show $v \in x \oplus w$ and thus $v \sim x$. Assume not $v \sim x$. But then (as $v \sim y$) $v \in x \oplus y$. Due to (16) we then have $v \in x \oplus w$, giving us a contradiction. Similar for case 2 having not $v \sim y$ and $v \sim w$. We are left with showing that $x \oplus w$ is not a subset of $y \oplus w$. Due to Lemma 1 we have to show that $y \oplus x$ is not empty which is due to Lemma 1 and (11). $\square$

As Goodman's construction is intended for sensory impressions, the matching operation is assumed to be a similarity relation, which is usually not transitive. And in fact, the betweenness construction according to Goodman does not work out for arbitrary orthogonality relations. Consider an orthogonality relation $\perp$ such that the following holds:

$$\text{For all } x, y, z \in X\text{: If } x \perp z\text{, then } x \perp y \text{ or } y \perp z. \tag{Trans*}$$

This condition is equivalent to the condition stating that $\sim$ is transitive, i.e., as $\sim$ already is assumed to be symmetric and reflexive, that $\sim$ is an equivalence relation. In this case the betweenness relation becomes trivial: if $\perp$ fulfils (Trans*) then no triple of elements stands in btw-relation. The reason is that as $\sim$ becomes transitive, $x \sim y$ and $y \sim z$ and $y \sim z$ would mean that $x \oplus y = y \oplus z = x \oplus z = \emptyset$. But then $x \oplus z$ cannot have any proper subset.

Having an arbitrary orthogonality relation and a betweenness-relation based on it according to Definition 2, it is possible to show that the $\perp$-closed sets are actually convex, meaning that the respective orthoframe can be interpreted as conceptual space.

Proposition 3. *A set Y is convex if it is $\perp$-closed.*

Proof. Assume Y to be $\perp$-closed and let $x, z \in Y$. Assume that Y is not convex. Thus, there must exist a $y \notin Y$ with $btw(x, y, z)$. $y \notin Y$ means that there exists a $v \in Y^{\perp}$ with $v \sim y$. By definition of betweenness, $x \oplus y \subsetneq x \oplus z$. As $x \perp v$, $v \in x \oplus y$, thus $v \in x \oplus z$, thus $v \sim z$. A contradiction, as then $v \notin Y^{\perp}$. $\square$

6. Conclusion

Though conceptual spaces and orthoframes provide apparently different approaches to (representing and reasoning with) concepts, already simple constructions such as that of Goodman [6] uncover interesting relations between them. The investigation reported in this paper is the beginning of ongoing work with an in-depth and systematic treatment of conceptual spaces and orthoframes. Some open research questions are the following: How to deal with negation for conceptual spaces without our simplifying assumptions, for example, how to handle negation for concepts represented as sets of convex sets in own domains? How to find inverse pairs of constructions from betweenness to orthonegation and vice versa? How can these insights be transferred to and exploited by recent KGE approaches?

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