Analogical Proportions and Betweenness

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Abstract

Analogical proportions are quaternary relations of the form "$a : b :: c : d$", read as "a is to b as c is to d", that are prominently used for analogical reasoning. The special case of continuous analogical proportions "$a : b :: b : d$", where the second and third argument are identified, naturally leads to a (ternary) betweenness relation. In case the inputs of the relations are concepts, continuous analogical proportions prove to be relevant for ideas of concept blending from cognitive science. In this paper we embark on a study on the relation between these two fundamental structures focusing on the question how to define analogical proportions based on betweenness relations. We show how to define prominent analogical proportions discussed in the literature with minimal assumptions on the betweenness relation. Furthermore we show how to extend the results if the definitions are required to be inverse in the sense that the continuous version of the defined analogical proportion is the betweenness relation. Last we give a first result on decomposability, showing that with a sufficiently strong betweenness relation an analogical proportion is definable that can be naturally decomposed in a binary relation denoted by "::" and a binary function denoted by ":".

Keywords

Analogical Reasoning, Continuous Analogical Proportion, Betweenness

1. Introduction

Analogical reasoning has found and continues to attract the interest of researchers in the intersection of philosophy, logic, cognitive science, and computer science. Especially in the artificial intelligence (AI) community various practical as well as theoretical challenges of analogical reasoning were tackled [1]. Analogical reasoning can be described as reasoning over so-called *analogical proportions*, i.e., quaternary relations "$a : b :: c : d$" with the intended reading "a is to b as c is to d". Roughly, analogical reasoning consists of deriving one of the arguments "a, b, c, d" given the others. For example, from "uncle:aunt :: man:d" one would infer that "d" must be woman. Analogical reasoning can also be used for deriving properties of instances. For example, when s and t share property P and s has property Q, it can be derived that t has property Q (the so-called *analogical jump*) [2].

Analogical proportions have been approached and analyzed from various angles and with various mathematical and logical tools [3]. This paper focuses on an equally simple and general approach, namely an axiomatic treatment of analogical proportions as described by Prade and Richard [3]. Treating "$a : b :: c : d$" as a quaternary relation, the authors describe three basic axioms in First Order Logic (FOL) that any analogical proportion should fulfill. Already from these axioms further relevant properties of analogical proportions can be deduced.

However, the set of possible analogical proportions (fulfilling the basic axioms) is large and quite diverse. And so the question arises how to tame this large set of models. Usually, analogical proportions are used in some specific reasoning setting. Hence, from a practical point of view, the inputs a, b, c, d of analogical proportions are known to have a specific structure, i.e., are elements of a specific domain. For example, Prade and Richard [3] consider the case where the domain is the set of Boolean values $\{1, 0\}$. Already in this case, a bunch of models of analogical proportions, concretely: eight models, can be identified. (We will describe these models in the section on preliminaries). The authors further develop their approach to handle the domain of vectors of Boolean values which allows a fine-grained description of domain objects by a bunch of (Boolean) features, each feature being represented by a component of the vector. And even more, the authors show that if the features are continuous or if confidence values are associated with them, then analogical proportions over vectors of real numbers (in [0,1]) can be handled.

In this work we take a different approach to taming the plethora of models of analogical proportions: We assume that the domain is equipped with a (ternary) betweenness relation. The motivation to do so is that we can naturally bootstrap an analogical proportion to reach a simple betweenness relation $Btw(a, b, d)$ by identifying the second and third argument of an analogical proportion: $Btw(a, b, d) = a : b :: b : d$. Prade and Richard [3] call this relation a *continuous analogical proportion*. So, instead of arbitrarily choosing a specific domain on which to consider analogical proportions we work with a domain equipped with a general structure in form of

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betweenness. The commitment to betweenness is not due to a specific application scenario. Rather it can be justified on grounds of the analogical proportions themselves. In fact, this form of bootstrapping can be understood as a transformation known as forgetting in category theory.

Another reason to consider betweenness for analogical proportions is that both structures are considered quite often in geometrical settings. In particular, many analogical proportions such as those induced by knowledge-graph embeddings are defined in a geometric space. A typical representative is the classical approach TransE [4], where analogical reasoning amounts to applying a translation in a vector space.

This paper contributes with foundational considerations on the relation between two theoretically fundamental and practically relevant structures mentioned above: analogical proportions and betweenness relations. In all our considerations we are interested in defining some given analogical proportion based on a betweenness relation. In the first part of the paper (after the preliminaries) we work with a very weak notion of betweenness and then answer the question how to define a given analogical proportion with it. In that first part of the paper there are no restrictions on the allowed constructions for defining an analogical proportion from the betweenness relation. As a result of the fact that the presumed betweenness relation is quite a weak notion of order, it is possible to define nearly any weak analogical proportion—independently of the structure of the space and further assumptions on the betweenness relation. In particular, we discuss how to define each (of appropriate generalizations; see preliminaries) of the conditions in order to satisfy the eight models of analogical proportion of Prade and Richard [3]. In fact, we can show that the chosen betweenness relation is as weak as possible: it relies only on properties that are necessary to ensure the definability of the given analogical proportion.

In the second part of the paper we again consider the definability of analogical proportions based on betweenness, but now we require the construction used in the definition to act as an inverse of the forgetting operation on analogical proportions. That is, the betweenness relation $B(a, b, d)$ is now required to be identical to the continuous analogical proportion $a : b :: b : d$ induced by the given analogical proportion $a : b :: c : d$. So in the second half of the paper we describe how to define analogical proportions— on the same foundations—but leading to a generalization of continuous analogical proportions.

In the last part of the paper (before the related work and the conclusion) we make even further restrictions on the analogical proportion induced by a betweenness relation, namely requiring the analogical proportion to allow for a compositional semantics in the sense explicated in formal linguistics [5]. The motivation for this restriction is due to the observation that the notation of analogical proportions suggests a decomposition of the analogical proportions into a binary relation :: and two occurrences of a binary function :. The question we tackle is whether analogical proportions that are defined via betweenness relations can be read in this way $a : b :: c : d$, i.e., whether we can give ":" and "::" natural interpretations such that the analogical proportion holds iff $a : b :: c : d$. We give a positive answer to this problem for an analogical proportion for the special case where the betweenness relation is the one induced by the euclidean metric.

At the bottom line, the introduction of betweenness to the considerations on analogical proportions helps to grasp the axioms of analogical proportions also in an algebraic sense. Furthermore, the properties of complex analogical proportions can be determined by reducing them to their properties regarding betweenness. Last but not least, betweenness lays the foundation for the definition of analogical proportions exactly with the desired strength by combining the constraints of the desired axioms of the proportion.

The rest of the paper is structured as follows: Section 2 contains preliminaries on the basic axioms of analogical proportions, the eight models of analogical proportions in the Boolean case and the basic betweenness axioms. In Section 3, for each of the conditions on analogical proportions, an interpretation based on betweenness is presented, leading to the definition of analogical proportions for the eight models for arbitrary domains. After that, in Section 4, an interpretation of the analogical proportion with the help of betweenness relations is shown that has the continuous analogical proportion as a special case. Section 5 considers a different definition of an analogical proportion, though still based on betweenness, and it is shown how a decomposition of this proportion is possible. The paper ends with a discussion of related work in Section 6 and a conclusion in Section 7.

2. Preliminaries

The basic axioms B_{ax} of analogical proportions according to Prade and Richard [3] are the following:

$$\forall a \forall b \ (a : b :: a : b) \qquad \text{(reflexivity)}$$

$$\forall a \forall b \forall c \forall d (a : b :: c : d \rightarrow c : d :: a : b) \quad \text{(symmetry)}$$

$$\forall a \forall b \forall c \forall d (a : b :: c : d \rightarrow a : c :: b : d) \qquad \text{(central permutation)}$$

The authors consider (in the first place) the domain of Boolean values $\{0, 1\}$. In that case $a : b :: c : d$ represents a Boolean function $f(a, b, c, d)$ of arity four. As Prade and Richard [3] show, there are eight different Boolean functions that fulfill the three axioms of analogical proportions. These functions form a Boolean algebra given in Fig. 1. We will identify a Boolean function f on propositional variables a, b, c, d with the set of truth assignments over $\{0,1\}^4$ which are mapped to 1 and write those assignments as bit vectors of length four. Following the naming convention of Prade and Richard the lattice elements are the following:

$$\begin{aligned}
\Omega_0 &= \{0000, 1111, 0101, 1010, 0011, 1100\} \\
Kl &= \{0000, 1111, 0101, 1010, 0011, 1100\} \cup \\
&\quad\{0110, 1001\} \\
M_3 &= \{0000, 1111, 0101, 1010, 0011, 1100\} \cup \\
&\quad\{1110, 1101, 1011, 0111\} \\
M_4 &= \{0000, 1111, 0101, 1010, 0011, 1100\} \cup \\
&\quad\{0001, 0010, 0100, 1000\} \\
M_5 &= M_3 \cup \{0110, 1001\} = M_3 \cup Kl \\
M_6 &= M_4 \cup \{0110, 1001\} = M_4 \cup Kl \\
M_7 &= M_3 \cup \{0001, 0010, 0100, 1000\} \\
&= M_3 \cup M_4 \\
\Omega &= \{0,1\}^4 = \{0000, 0001, 0010, \ldots, 1111\}
\end{aligned}$$

The lattice structure (actually it is even a Boolean algebraic structure) is cited here on the left of Fig. 1.

Next to the basic axioms of analogical proportions, we will also consider the following axioms, the first of which is also mentioned by Prade and Richard [3] (there it is called *unicity*, here we prefer the term *universal anti-neutrality* to clarify the connection to other related axioms). The other ones are our additions. The intention of these axioms is to give an individual axiomatization of each of the eight analogical proportions and thereby pave the way for considering analogical proportions for arbitrary domains. In the following, all variables are assumed to be (implicitly) universally quantified.

$$a : a :: b : c \to b = c \qquad \text{(universal anti-neutrality)}$$

$$a : b :: b : a \qquad \text{(ratio symmetry)}$$

$$a : b :: b : a \to a = b \qquad \text{(ratio anti-symmetry)}$$

$$a : a :: b : c \qquad \text{(universal neutrality)}$$

$$0 : 0 :: a : b \qquad \text{(0-neutrality)}$$

$$0 : 0 :: a : b \to a = b \qquad \text{(0-anti-neutrality)}$$

$$1 : 1 :: a : b \qquad \text{(1-neutrality)}$$

$$1 : 1 :: a : b \to a = b \qquad \text{(1-anti-neutrality)}$$

$$a : b :: c : d \qquad \text{(universality)}$$

Prade and Richard [3] consider the case of analogical proportions where the domain X of the quantifiers range over $X = \{0,1\}$ and the generalized case where X are n-bit vectors. For betweenness considerations the domain X is going to be a general structure and so we generalize the eight models to arbitrary domains. Hence, in Fig. 1 we present axiomatizations of the generalization of the eight models w.r.t. the (new) axioms introduced above. It is readily checked that these axiomatizations uniquely identify the eight models over the domain X—so that, indeed, they can be considered as properties of generalizations of the eight models to arbitrary domains. The easy part is to check (just by going through 4-bit assignments) that the eight models fulfill the corresponding axioms. That those axiom sets exclude each other can be verified by checking each pair of axiom sets $F_m, F_{m'}$ and noting that there is always a pair of complementary axioms (over domains with at least two elements). For example, in case of F_{Kl} and F_{Ω_0} we have the axioms of ratio symmetry and ratio antisymmetry.

Continuous analogical proportions as introduced by Prade and Richard [3] are defined as analogical proportions where the second and third arguments are identified. So, given an analogical proportion $R(a, b, c, d) = a : b :: c : d$ the ternary relation Btw_R is defined as

$$Btw_R(a, b, d) \text{ iff } R(a, b, b, d)$$

Continuous analogical proportions can be considered as betweenness relations. The whole point of this paper is to investigate the relationship between (properties of) analogical proportions and betweenness relations.

There are several axioms of different strength defining betweenness. In the following, the most basic ones are presented. As it will turn out later on, not all of the axioms have to be fulfilled for the intended definitions of analogical proportions. ($a \neq b \neq c$ denotes in the following that a, b and c are pairwise unequal)

Definition 1. *The axioms we consider are those discussed, e.g, by Huntington and Kline [6] and [7].*

$$\textit{If } Btw(a, b, c), \textit{ then } a, b, c \textit{ are distinct} \qquad \text{(B0)}$$

$$\textit{If } Btw(a, b, c), \textit{ then } Btw(c, b, a) \qquad \text{(B1)}$$

$$\textit{If } Btw(a, b, c) \textit{ and } a \neq b \neq c, \textit{ then not } Btw(b, a, c) \qquad \text{(B2)}$$

$$\textit{If } Btw(a, b, c) \textit{ and } Btw(b, c, d) \textit{ and } b \neq c, \textit{ then } Btw(a, b, d) \qquad \text{(B3)}$$

$$\textit{If } Btw(a, b, d) \textit{ and } Btw(b, c, d), \textit{ then } Btw(a, b, c) \qquad \text{(B4)}$$

When (B0) is valid, the ternary betweenness relation Btw is called *open betweenness* [8] or sometimes also *strict betweenness*. Without (B0), the betweenness is called *closed betweenness*. We will focus in the following on closed betweenness. (B1) expresses commutativity of betweenness w.r.t. the outer points. (B2) expresses the constraint that if a point is between two points it cannot have one of those two points in between itself and the other point. (B1) and (B2) are the two fundamental axioms of betweenness which should be fulfilled. (B3) is sometimes called *outer transitivity* and (B4) *inner transitivity*.

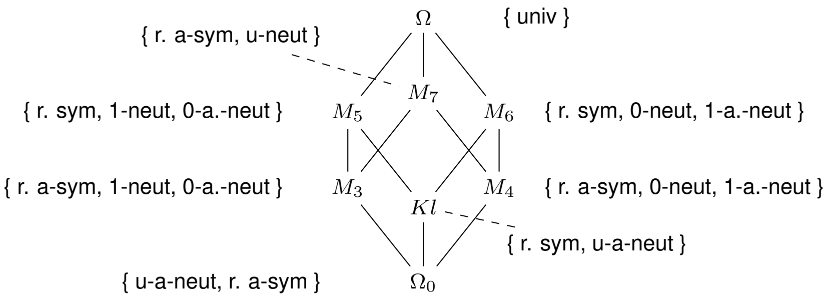

Figure 1: The Boolean algebra of analogical proportions m over $\{0, 1\}$ and its axiomatizations $B_{ax} \cup F_m$. In the figure we only give F_m, it being understood that B_{ax} has always to be taken into account.

3. Eight Models via Betweenness

The eight axiomatizations given in Fig. 1 are motivated by the Boolean case. Now, we take a different perspective on these axiomatizations by assuming a betweenness relation which is a sufficiently weak and thus general notion. It allows for a definition of the several axioms as specific as possible in the sense that each restriction should be minimal, thus should represent exactly the given axioms for a model. The goal of this section is to define analogical proportions based on betweenness for each of the eight models, i.e., to give for each model a definition fulfilling the three basic axioms of analogical proportions and the additional axioms mentioned in Fig. 1.

Assume an arbitrary domain X equipped with a betweenness relation is given and the analogical proportions to be defined are determined on element-level, thus each input is an element of X. As X can be an arbitrary space equipped with a betweenness relation, the following definitions are widely applicable.

First of all, we discuss how to define in general an analogical proportion from a betweenness relation. However, for a better understanding of the discussion, it is useful to think of a concrete betweenness relation such as Euclidean betweenness which is defined with the Euclidean distance d as follows.

$$Btw_d(a, b, c) \text{ iff } d(a, c) = d(a, b) + d(b, c),$$

According to this definition, b is in between of a and c if it is on the line segment connecting a and c.

A first idea for defining an analogical proportion $R(a, b, c, d)$ is to enforce the inputs a, b, c, d to be comparable w.r.t. the betweenness relation by, e.g., enforcing $Btw(a, b, c)$. But already here it becomes obvious that such a definition is too strong to save as a general template for arbitrary models of analogical proportion because it would enforce a, b and c (and depending on the other constraints also d) to be on a line.

So instead, for a general notion, we consider only betweenness constraints that contain at most two elements out of $\{a, b, c, d\}$. As (B1) and (B2) can be considered as fulfilled by the betweenness relation (as these are the most basic betweenness axioms), it remains to consider a definition based on arbitrary combinations of $Btw(x, u, y)$ and $Btw(v, x, y)$ for some $x, y \in \{a, b, c, d\}$ and $u, v \in X$. Obviously, it is also possible to consider more complex connections—which we saved for future work.

Especially for analogical proportions the interplay of the elements is of importance. Hence we focus here on a definition based on $Btw(x, u, y)$ for some $x, y \in \{a, b, c, d\}$ and $u \in X$ chosen arbitrarily.

Before going into detail regarding the different models, we first discuss how definitions of analogical proportions based on these betweenness relations could look like in general. An analogical proportion must fulfill the basic axioms, independently of its construction. Especially symmetry and central permutation enforce restrictions on the construction, as they change the order of elements in the analogical proportion. Thus, postulated conditions need some sort of symmetric behavior to cover the changed order. We will capture this with the notion of a substitution of subformulas. For a FOL formula F we denote by $F[G/H, J/K]$ the result of substituting in parallel all occurrences of subformula G (if contained in F) with subformula H and all occurrences of subformula J (if contained in F) with K. This notion is generalized canonically to the case of more than two subformulas being substituted in parallel. Here and in the following, for simplicity, we use the following shortcut $((a, b), (c, d))$:

$$((a, b), (c, d)) \quad \text{iff} \quad \text{there is } u, v \in X \text{ with } Btw(a, u, b) \text{ and } Btw(c, v, d).$$

With these notions, a first result on a sufficient and necessary condition for quaternary relations fulfilling symmetry and central permutation can stated.

Proposition 1. *Assume a quaternary relation* R ($R(a, b, c, d)$) *is defined with a first order logic formula* F_{ap} *over* Btw *containing atoms of the form* $Btw(x, u, y)$

with distinct $x, y \in \{a, b, c, d\}, u \in X$. Then R fulfills symmetry and central permutation iff:

1. *for all distinct $w, x, y, z \in \{a, b, c, d\}$: for any subformula $C(i, k)$ of F_{ap} of the form $Btw(i, j, k)$ or $Btw(k, j, i)$ with a variable j it holds that $F_{ap} \models F_{ap}[C(w, x)/C(y, z), C(y, z)/C(w, x)]$ and*

2. *for any subformula $C(i, k)$ of F_{ap} of the form $Btw(i, j, k)$ or $Btw(k, j, i)$ with a variable j it holds that $F_{ap} \models F_{ap}[C(a, b)/C(a, c), C(c, d)/C(b, d), C(a, c)/C(a, b), C(b, d)/C(c, d)]$*

Proof. $\rightarrow$: (1.) (i) Assume $R(a, b, c, d)$ is defined based on an arbitrary constraint on $Btw(a, u, b)$ but not on $Btw(c, v, d)$ for arbitrary $u, v \in X$. Then with symmetry it follows $R(c, d, a, b)$ and thus, the constraint needs to be valid on $Btw(c, v, d)$ (for arbitrary $v \in X$), a contradiction. (ii) Assume $R(a, b, c, d)$ based on an arbitrary constraint on $Btw(a, u, c)$ but not on $Btw(b, v, d)$. With central permutation, symmetry and central permutation follows: $R(a, b, c, d)$,$R(a, c, b, d)$, $R(b, d, a, c)$, $R(b, a, d, c)$ and thus $Btw(b, v, d)$ need to be constrained, a contradiction. (iii) For a constraint on $Btw(a, u, d)$ and all other cases, analogous arguments apply. (2.) Needs to be the case because of central permutation.

$\leftarrow$: Symmetry is valid, as $R(a, b, c, d)$ leads to constraints on $((a, b), (c, d))$ and $((a, c), (b, d))$ and independently constraints on $((a, d), (b, c))$. With (B1) and (1.) it follows that the constraints can be reformulated to constraints on $((c, d), (a, b))$ and $((c, a), (d, b))$ and constraints on $((c, b), (a, d))$ and thus $R(a, b, c, d)$ iff $R(c, d, a, b)$. Central permutation can be shown analogously. ☐

As the axioms for the (eight) models of analogical proportion are motivated by the binary case, it is necessary to make some adaptation before using it in the general case. In particular, two questions arise: (i) how can 0 and 1 be defined and (ii) when are two elements considered equal, e.g., when should $a : b :: c : d$ be interpreted as $a : a :: c : d$ to apply the specific conditions?

The first question is only relevant for some of the conditions and thus considered later on. So we consider now the second question. As we assume that a general space X and a betweenness relation on it, but that no other information is provided, the idea is to interpret two elements of X to be the same (or similar) if they behave the same regarding the betweenness relation. In setting up the conditions on compared elements as before (now for similarity) this can be restricted to those mentioned in the inputs. For example, a is considered similar to b if both behave the same regarding c and d, i.e., intuitively, if the elements share the properties relevant for exactly this analogical proportion.

Formally, a is the same as b regarding the set $\{a, b, c, d\}$ iff for all $x \in X$ the following assertion holds: $(Btw(a, x, c) \leftrightarrow Btw(b, x, c))$ and $(Btw(a, x, d) \leftrightarrow Btw(b, x, d))$. This leads to the definition of the condition y-same$((a, b), (c, d))$ stating that (at least) one element of $\{a, b\}$ is equivalent to (at least) one element of $\{c, d\}$ (regarding the set $\{a, b, c, d\}$):

y-same$((a, b), (c, d))$ iff
y-all$((x, v), (y, v))$ & y-all$((x, w), (y, w))$
for distinct $v, x \in \{a, b\}, w, y \in \{c, d\}$,

with y-all$((a, b), (c, d))$ stating intuitively that all elements in between a and b should be also in between c and d (and vice versa):

y-all$((a, b), (c, d))$ iff $\forall x(Btw(a, x, b) \leftrightarrow Btw(c, x, d))$

Note that for the case of Euclidean betweenness this condition reduces to the fact that two elements are equivalent if they are at the same point in the vector space.

We aim at a general construction of analogical proportions from a betweenness relation that is easy to grasp and that works also for analogical proportions outside the set of eight models. Hence we define fragments of an analogical proportion based on betweenness where each fragment fulfills exactly the desired constraint. These constraints can then be combined to create each of the eight models, but can also be arbitrarily combined with each other or other constraints and thus can serve as basis for an arbitrary analogical proportion based on betweenness that fulfills the desired constraints.

In general there are two types of constraints, existential constraints and filter constraints. Having a bunch of potential conditions to be used in the definition, the conditions of the first type are connected via a disjunction, thus not interfering with the other constraints, whereas constraints of the latter type are connected via conjunction to existing constraints, as they need to restrict each analogical proportion where they are applicable.

In the following, the axioms of analogical proportions are considered case by case. In this context, we introduce a (weak) new betweenness axiom (B0') stating that in case of closed betweenness the only element between x and x itself is x again.

$$\forall x, y \in X(Btw(x, x, x) \ \& \ Btw(x, y, x) \rightarrow x = y) \tag{B0'}$$

We start by considering each of the basic axioms. In fact, out of these only reflexivity needs to be considered, because the other two are already fulfilled when using the construction principles mentioned in Proposition 1. Though, it would have been possible to create conditions which are minimal in the sense that they allow exactly for the condition mentioned, but in the following, it has been assumed that symmetry and central permutation are such basic axioms that they could (and should) be incorporated into the definition of the constraints. Thus, not only the constraints are fulfilled but also all permutations of it regarding symmetry and central permutation.

Reflexivity is represented based on betweenness straightforwardly, stating that the analogical proportion must be valid in case of $a : b :: a : b$ (and permutations of it).

Proposition 2. *Reflexivity ($\forall a \forall b(a : b :: a : b)$) is valid iff there is an arbitrary (possibly empty) constraint C such that the following equivalence holds: $a : b :: c : d$ iff $C \vee$ (y-same$((a, d), (b, c))$ & y-all$((a, d), (b, c))$).*

Proof. $\rightarrow$: Let $a : b :: a : b$ be valid. Then y-same$((a, b), (b, a))$ and y-all$((a, b), (b, a))$ are trivially fulfilled.

$\leftarrow$: Let y-same$((a, d), (b, c))$ and y-all$((a, d), (b, c))$ and let C be empty. Then $a = b$, $a = c$, $d = b$ or $d = c$ are possible. Let $a = c$, the other cases follow with Proposition 1, central permutation and symmetry. Then y-all$((a, d), (b, a))$ and thus $\forall x : Btw(a, x, d) \leftrightarrow Btw(b, x, a) \leftrightarrow Btw(a, x, b)$ (with (B1)). Thus, by definition of equality of elements, $b = d$. □

The next constraint considers ratio symmetry, i.e., $\forall a, b(a : b :: b : a)$. It states a symmetry condition on the analogical proportion, namely that the ratio of a and b is the same as the ratio of b and a.

This leads to a definition of the constraint similar to the case of reflexivity, only considering different elements.

Proposition 3. *Ratio symmetry ($\forall a \forall b(a : b :: b : a)$) is valid iff there is an arbitrary (possibly empty) constraint C such that the following equivalence holds: $a : b :: c : d$ iff $C \vee$ (y-same$((a, b), (c, d))$ & y-same$((a, c), (b, d))$ & y-all$((a, b), (c, d))$ & y-all$((a, c), (b, d))$).*

Proof. Proof similar to the proof of Proposition 2. □

The direct opposite to ratio symmetry is ratio-antisymmetry, stating that the ratio of a and b is in no case the same as the ratio of b and a (except $a = b$).

Proposition 4. *Ratio-antisymmetry ($\forall a \forall b(a : b :: b : a) \rightarrow a = b$) is valid iff there is an arbitrary (possibly tautological) constraint C such that the following equivalence holds: $a : b :: c : d$ iff C & (y-same$((a, b), (c, d))$ & y-same$((a, c), (b, d))$ & y-all$((a, b), (c, d))$ & y-all$((a, c), (b, d)) \rightarrow$ y-all$((a, d), (b, c))$).*

Proof. Proof similar to the proof of Proposition 3. □

The next restriction is universal neutrality, stating that the ratio of an element to itself is similar to the ratio of arbitrary elements. This can be satisfied in terms of betweenness by stating that the analogical proportion $a : b :: c : d$ must be fulfilled when $a = b$.

Proposition 5. *Universal neutrality ($\forall a \forall b \forall c(a : a :: b : c)$) is valid iff there is an arbitrary (possibly empty) constraint C such that the following equivalence holds: $a : b :: c : d$ iff $C \vee$ y-same$((a, d), (b, c))$.*

Proof. $\leftarrow$: Let y-same$((a, d), (b, c))$. Then $a = b$ or $a = c$ or $d = b$ or $d = c$. With Proposition 1 and by applying central permutation and symmetry, this leads to the case that $(a : a :: c : d)$ is valid.

$\rightarrow$: Let $a : a :: b : c$ be valid and C be empty. Then, y-same$((a, c), (a, b))$ is valid. □

In the following, different types of neutrality and anti-neutrality are considered. In contrast to the other conditions, these conditions are not only based on arbitrary elements but also on elements of type 0 and 1. As the model Kl can be represented as conjunction of M_5 and M_6, it is necessary that the restriction of 0-anti-neutrality and 1-anti-neutrality behave in combination as (general) anti-neutrality. Thus, it is not sufficient to define one 0- and one 1-element but to partition the space into 0- and 1-elements. As M_3 and M_5 are analogue to M_4 and M_6, 0 and 1 could be defined interchangeably. Therefore, in the following we assume that all elements of type 0 are universal in the sense that they have a betweenness relation to any other element, thus for all $x \in X$, there is $y \in X$ with $Btw(0, y, x)$. Elements are of type 1 if they do not fulfill this condition. As for the validity of an analogical proportion only the elements contained in this proportion are directly considered, an element x has type 0 regarding a, b, c, d (for $a : b :: c : d$), if $x \in \{a, b, c, d\}$ and x is universal regarding $\{a, b, c, d\}$. As 0 and 1 depict no special element but a type of element, it is possible to have $0_x : 0_y :: a : b$, thus two different elements of type 0. On such an analogical proportion, e.g., the condition of 0-neutrality would not be applicable.

Universal neutrality considered in Proposition 5 accounts for the general case, considering all elements. Beside that, there are models incorporating 0-neutrality and 1-neutrality, thus neutrality based on a special type of elements.

First, 0-neutrality is considered. This adds to the general case the restriction of $((a, d), (b, c))$, stating that a and b have a betweenness relation with c and d thus not only being equal but also of type 0.

Proposition 6. *0-neutrality ($\forall a \forall b(0 : 0 :: a : b)$) is valid iff there is an arbitrary (possibly empty) constraint C such that the following equivalence holds: $a : b :: c : d$ iff $C \vee (((a, d), (b, c))$ & y-same$((a, d), (b, c)))$.*

Proof. The general case follows with Proposition 5, it remains to show that the restriction on elements of type 0 is valid.

$\rightarrow$: Let $0 : 0 :: a : b$. Then $((0, d), (0, c))$ by definition of 0.

$\leftarrow$: Let C be empty and $((a, d), (b, c))$ and y-same$((a, d), (b, c))$. Then with y-same, $a = b$ (analogue for the other cases). With $((a, d), (a, c))$ and (B0') follows $Btw(a, x_1, d)$, $Btw(a, x_2, c)$ and $Btw(a, x_3, a)$ for $x_1, x_2, x_3 \in X$. Thus, a (and b) are of type 0 and thus $0 : 0 :: c : d$. ☐

This can be done analogously for 1-neutrality.

Proposition 7. 1*-neutrality (*$\forall a \forall b(1 : 1 :: a : b)$*) is valid iff there is an arbitrary (possibly empty) constraint* C *such that the following equivalence holds:* $a : b :: c : d$ *iff* $C \vee (\neg((a, d), (b, c))$ & *y-same*$((a, d), (b, c)))$.

Proof. Proof similar to the proof of Proposition 6. ☐

The opposite of universal neutrality is universal anti-neutrality: $\forall a \forall b \forall c$: $(a : a :: b : c \rightarrow b = c)$. This means intuitively that the ratio of an element to itself can not be similar to the ratio of different elements, thus that two elements are either the same or distinct enough to be not comparable with equal elements.

Proposition 8. *Universal anti-neutrality (*$\forall a \forall b \forall c$*:* $(a : a :: b : c \rightarrow b = c)$*) is valid iff there is an arbitrary (possibly tautological) constraint* C *such that the following equivalence holds:* $a : b :: c : d$ *iff* C & $($*y-same*$((a, d), (b, c)) \rightarrow$ *y-all*$((a, d), (b, c)))$.

Proof. $\rightarrow$: Let $a : a :: b : c$. Then y-same$((a, c), (a, b))$. y-all$((a, c), (a, b))$ is only valid if $b = c$.

$\leftarrow$: Let y-same$((a, d), (b, c))$ be not valid. Then not $a : a :: b : c$. Let y-same$((a, d), (b, c))$. Let $a = b$ (other cases analog). Then y-all$((a, d), (a, c))$ only if $c = d$. Thus universal anti-neutrality is fulfilled. ☐

This can be again restricted to 0 and 1, following the same principles as for neutrality:

Proposition 9. 0*-anti-neutrality (*$\forall a \forall b(0 : 0 :: a : b \rightarrow a = b)$*) is valid iff there is an arbitrary (possibly tautological) constraint* C *such that the following equivalence holds:* $a : b :: c : d$ *iff* C & $(((a, d), (b, c))$ & *y-same*$((a, d), (b, c)) \rightarrow$ *y-all*$((a, d), (b, c)))$.

Proof. Proof similar to the proof of Proposition 6. ☐

Proposition 10. 1*-anti-neutrality (*$\forall a, b(1 : 1 :: a : b \rightarrow a = b)$*) is valid iff there is an arbitrary (possibly tautological) constraint* C *such that the following equivalence holds:* $a : b :: c : d$ *iff* C & $(\neg((a, d), (b, c))$ & *y-same*$((a, d), (b, c)) \rightarrow$ *y-all*$((a, d), (b, c)))$.

Proof. Proof similar to the proof of Proposition 6. ☐

Having these fragments for all of the conditions, the domain independent generalizations of the eight Boolean models can be easily constructed. This construction is illustrated here for the model Ω_0 but of course analogous constructions for all other models are possible.

Definition 2. *The* minimal analogical proportion R_{Ω_0} *based on a general betweenness relation* Btw *is defined as follows:*

$$
\begin{aligned}
&R_{\Omega_0}(a, b, c, d) \textit{ iff}\\
&(\textit{y-same}((a, b), (c, d)) \;\&\\
&\textit{y-same}((a, c), (b, d)) \;\&\\
&\textit{y-all}((a, b), (c, d)) \;\&\\
&\textit{y-all}((a, c), (b, d)) \rightarrow \textit{y-all}((a, d), (b, c))) \;\&\\
&(\textit{y-same}((a, d), (b, c)) \rightarrow \textit{y-all}((a, d), (b, c)))
\end{aligned}
$$

Due to the construction the following (intended) proposition is easily proved.

Proposition 11. R_{Ω_0} *fulfills universal anti-neutrality and ratio anti-symmetry.*

When considering the definitions of betweenness mentioned above, it turns out that all of them are quite weak. For most of the use cases they allow for too many analogical proportions. That is true even for the model Ω_0, the most restricted model of the eight. However, they show that with betweenness it is possible to define an analogical proportion fulfilling the respective axioms without introducing any unnecessary restrictions. These can be used as basis and can be, e.g., combined with other axioms to define a analogical proportion that fits to the given use-case.

A stronger analogical proportion based on betweenness is demonstrated in the next section. There an analogical proportion having the continuous analogical proportion as special case is defined.

4. A Basic Analogical Proportion over a Geometrical Space

Continuous analogical proportions as introduced by Prade and Richard [3] are defined as analogical proportions where the second and third arguments are identified. They interpret the analogical proportion $R(a, b, b, d)$ directly as betweenness relation. To recap:

$$Btw(a, b, d) \text{ iff } R(a, b, b, d).$$

As $R(a, a, a, a)$ needs to be valid based on reflexivity, it is necessary that the betweenness relation fulfills

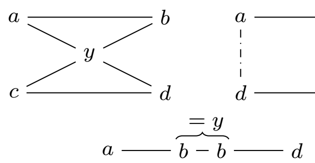

Figure 2: Examples of the use of R_{cap} in the case of Euclidean betweenness; a, b, c, d, y are points in a two-dimensional space, connected via lines (thus via betweenness relations)

$Btw(a, a, a)$ for all $a \in X$. Continuous analogical proportions seem to be a good basis for an analogical proportion, as they are widely used also in practical learning approaches, e.g., for enlarging a data set with new examples [9]. Now the question arises whether it is possible to define an analogical proportion based on betweenness for the general case of $a : b :: c : d$, having the continuous analogical proportion as a special case. To state it differently: How does an analogical proportion $R_{cap}(a, b, c, d)$ need to look like such that $R_{cap}(a, b, b, d)$ iff $Btw(a, b, d)$? Assume in the following an arbitrary betweenness relation fulfilling (B1), (B2) and (B0'). Then, a possible definition would be

$$R_{cap}(a, b, c, d) \text{ iff}$$
$$\text{y-ex}((a, d), (b, c)) \vee \text{y-all}((a, d), (b, c)),$$

where y-ex$((a, d), (b, c))$ states intuitively that there must be at least one element in between a and d which is also in between b and c (and vice versa):

$$\text{y-ex}((a, d), (b, c)) \text{ iff}$$
$$\exists x \in X(Btw(a, x, d) \;\&\; Btw(b, x, c)).$$

For an intuitive understanding, consider again Euclidean betweenness in a two-dimensional space. The graph on the upper left of Fig. 2 illustrates the intuition behind y-ex$((a, d), (b, c))$. Basically, it states that the line segment ab and the line segment cd must be in a specific orientation to each other, as the lines ad and bc have to intersect. The upper right graph is a counterexample. At the bottom of Fig. 2 the special case of the continuous analogical proportion can be seen. If $b = c$, then b plays the role of the element existing in between of both a and d and of b and c $(= b)$ and thus leads for Euclidean betweenness to $Btw(a, b, d)$.

The disjunct y-all$((a, d), (b, c))$ is necessary for the general case to ensure reflexivity. It is not necessary in the Euclidean case, because there is always an element in between each two other elements and thus also y-ex$((a, b), (b, a))$ is valid.

First, we show that R_{cap} is a correct analogical proportion in the sense that it fulfills the basic axioms.

Proposition 12. *R_{cap} fulfills the basic axioms B_{ax}.*

Proof. Symmetry and central permutation are fulfilled as shown in Proposition 1. Thus, it remains to show reflexivity. Consider $R_{cap}(a, b, a, b)$, thus y-ex$((a, b), (b, a))$ or y-all$((a, b), (b, a))$. If there is $x \in X$ with $Btw(a, x, b)$, then with (B1) also $Btw(b, x, a)$ and thus y-ex$((a, b), (b, a))$. If there is no such $x \in X$, then there is also no $x \in X$ with $Btw(b, x, a)$ (with (B1)) and thus y-all$((a, b), (b, a))$. □

After having showed that R_{cap} is a correct analogical proportion fulfilling the basic axioms, we proceed by showing that the continuous analogical proportion is in fact a special case.

Proposition 13. *$R_{cap}(a, b, b, d)$ iff $Btw(a, b, d)$*

Proof. $\rightarrow$: $R_{cap}(a, b, b, d)$ iff y-ex$((a, d), (b, b))$ or y-all$((a, d), (b, b))$ and thus (i) there is $x \in X$ with $Btw(a, x, d), Btw(b, x, b)$. Because of (B0'), $x = b$ and thus $Btw(a, b, d)$. (ii) for all $x \in X$: $Btw(a, x, d)$ iff $Btw(b, x, b)$, but again there is exactly one x fulfilling it, namely $x = b$ and thus $Btw(a, b, d)$.

$\leftarrow$: Let $Btw(a, b, d)$ be valid. $Btw(b, b, b)$ is valid by definition and thus y-ex$((a, d), (b, b))$. □

5. Decompositionality

Whereas in Section 3 the definition of an analogical proportion based on a restriction as minimal as possible has been examined, in this section, an analogical proportion based on a stronger restriction is presented, namely (when considering again the Euclidean betweenness) the restriction that $a : b :: c : d$ can be only valid if a, b, c and d are on the same line and the line *bc* is part of ad or vice versa. In the general case, the restriction is as follows:

$$R_0^{Btw}(a, b, c, d) \quad \text{iff} \quad Btw(a, b, d) \;\&\; Btw(a, c, d) \text{ or } Btw(b, a, c) \;\&\; Btw(b, d, c).$$

The analogical proportion defined in this way fulfills symmetry and central permutation. To gain the missing reflexivity, the betweenness is enforced to fulfill a specific constraint, namely,

$$\text{for all } a, b\text{: } Btw(a, a, b) \qquad \text{(B6)}$$

This definition can be brought to practice by considering a special type of betweenness relation, namely one which allows for the definition of a line in the usual sense (including, but not limited to the Euclidean betweenness and a line in the Euclidean sense). On the basis of such a betweenness relation we can define a notion of a line, collinearity etc. For two points a, b let $\langle ab \rangle$ denote the closed section between a, b on the line going through a, b

and containing a, b. Formally, we can define the closed section on the basis of the betweenness relation and set theoretical operations as follows:

$$\langle ab \rangle = \{x \in X \mid Btw(a, x, b)\} \cup \{a, b\}$$

For this making up a line in the usual sense, axioms (B0') and (B1) to (B4) are necessary (see [10]) and additionally the axiom (B7)[6]:

$$\text{If } Btw(a, x, b) \text{ and } Btw(a, y, b) \text{ (and } a \neq b \neq x \neq y), \text{ then } Btw(a, x, y) \text{ or } Btw(a, y, x) \tag{B7}$$

Note that though we use the interval notation we do not have an orientation so that $\langle ab \rangle = \langle ba \rangle$. In the following one may first think of a Euclidean space with betweenness induced by the Euclidean distance.

We define a quaternary relation $R_1^{Btw}(a, b, c, d)$ that says the section $\langle ad \rangle$ contains $\langle bc \rangle$ or vice versa. In other words: There is a line on which all of a, b, c, d occur and either the section from b to c occur in between of a, d or vice versa.

$$R_1^{Btw}(a, b, c, d) \text{ iff } \langle ad \rangle \subseteq \langle bc \rangle \text{ or } \langle bc \rangle \subseteq \langle ad \rangle$$

This is in fact a special case of the analogical proportion R_0^{Btw}.

Proposition 14. *If Btw fulfills (B1)–(B4) and (B7), then for all a, b, c, d the following equivalence holds: $R_0^{Btw}(a, b, c, d)$ iff $R_1^{Btw}(a, b, c, d)$*

Proof. $\rightarrow$: Let $R_0^{Btw}(a, b, c, d)$ for arbitrary a, b, c, d and thus $Btw(a, b, d)$ and $Btw(a, c, d)$ or $Btw(b, a, c)$ and $Btw(b, d, c)$. Let $Btw(a, b, d)$ and $Btw(a, c, d)$ be valid. If $a \neq b \neq c \neq d$, then with (B7) follows $Btw(a, b, c)$ or $Btw(a, c, b)$ and with (B3) follows $Btw(b, c, d)$ or $Btw(c, b, d)$. This is possible for all elements in between a and d and thus $\langle bc \rangle \subseteq \langle ad \rangle$. If there are no such elements not equaling a, b, c, d, then the condition is trivially fulfilled. Analog for $Btw(b, a, c)$ and $Btw(b, d, c)$ being valid.

$\leftarrow$: Follows trivially out of the definition of a line. □

Proposition 15. *For any ternary betweenness relation fulfilling (B1) and (B6) (commutativity of Btw), the induced relation R_0^{Btw} fulfills the basic axioms of analogical proportions.*

Proof. Reflexivity: $R_0^{Btw}(a, b, a, b)$ holds because it holds if $Btw(a, b, b)$ and $Btw(a, a, b)$. This is the case for all a, b because of (B6).

Symmetry: $R_0^{Btw}(a, b, c, d)$ iff $Btw(a, b, d)$ and $Btw(a, c, d)$ or $Btw(b, a, c)$ and $Btw(b, d, c)$ iff $Btw(b, a, c)$ and $Btw(b, d, c)$ or $Btw(a, b, d)$ and $Btw(a, c, d)$ (and with (B1) iff $Btw(c, a, b)$ and $Btw(c, d, b)$ or $Btw(d, c, a)$ and $Btw(d, b, a)$ iff $R_0^{Btw}(c, d, a, b)$.

Central permutation: $R_0^{Btw}(a, b, c, d)$ iff $Btw(a, b, d)$ and $Btw(a, c, d)$ or $Btw(b, a, c)$ and $Btw(b, d, c)$ iff $Btw(a, c, d)$ and $Btw(a, b, d)$ or $Btw(b, d, c)$ and $Btw(b, a, c)$ iff $R_0^{Btw}(a, c, b, d)$. □

Thus, also R_1^{Btw} fulfills the basic axioms of betweenness.

Prade and Richard use a notation for analogical proportions that suggests a decomposition of the analogical proportions into a binary relation :: and two occurrences of a binary function :. The question is whether $R_1^{Btw}(a, b, c, d)$ can be read in this way $a : b :: c : d$, i.e., whether we can give ":" and "::" natural interpretations such that R_1^{Btw} iff $a : b :: c : d$. Note that this problem is known in the literature on linguistics and model theory also as an extension problem for compositionality of the first form [5, p.16]. A trivial one would be to treat $a : b$ as the pair (a, b) and allow in :: the use of projections operators, so that $a : b :: c : d$ can be reduced to $R(\pi_1(a : b), \pi_2(a : b), \pi_1(c : d), \pi_2(c : d))$. Here we used the operators of left projection (projection on the first argument) and right projection (projection on the second argument) $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.We seek for non-trivial decompositions where $a : b$ fits to the idea of a difference or quotient that can be used to do analogical reasoning. The next proposition gives an affirmative answer. For this we need the notion of orientation. Each line L has exactly one of two orientations [10, Theorem 40] which we call r_L and l_L (for from-left-to-right and from-right-to-left on L). Such an orientation x induces a total order $<_L^x$ and a dual order on the dual orientation x' with $u <^x v$ iff $v <^{x'} u$. Each segment $\langle ab \rangle$ has exactly one line L_{ab} going through it. In particular, each orientation x positions a, b w.r.t $<^x$. Define $sign(a, b)$ as follows

$$sign(a, b) = \begin{cases} +, & \text{iff } a <^l b \\ -, & \text{iff } b <^l a \\ 0, & \text{otherwise} \end{cases}$$

We denote by $[a, b]$ the interval w.r.t the left orientation (where a comes before b reading from left to right). So $sign(a, b)$ tells us whether under the left-to-right orientation the order chosen in the pair (a, b) fits the order of the induced order $<^l$ by left-to-right-orientation. The special case of $a = b$ is treated separately. Now we define on the basis of this operation the following operation:

$$\begin{aligned} : X \times X &\longrightarrow \text{segments} \times \{+, -, 0\} \\ (a, b) &\mapsto (\langle ab \rangle, sign(a, b)) \end{aligned} \tag{1}$$

Note that we can consider $a : b$ really as some form of difference: If for example a comes before b then $\langle ab \rangle = [a, b]$ can be considered as the halfline starting from a to the right minus the half line to the right starting at b. For any two closed proper segments on a line we consider the following eight jointly exhaustive and mutually disjoint relations known from the region connection calculus RCC 8 [11], applied to the 1-dimensional case of a line:

$$\{DC, EC, TPP, NTPP, PO, EQ, TPPi, NTPPi\}$$

Let be given closed proper sections $x = \langle ab \rangle$ and $y = \langle cd \rangle$ define

$xDCy$	iff	$x \cap y = \emptyset$ ("disconnected")
$xECy$	iff	$x \setminus \{a, b\} \cap y \setminus \{c, d\} = \emptyset$
		$x \cap y \neq \emptyset$
		("externally connected")
$xTPPy$	iff	$x \subsetneq y$ and $x \not\subseteq y \setminus \{c, d\}$
		("tangential proper part");
$xNTPPy$	iff	$x \subsetneq y \setminus \{c, d\}$
		and all of a, b, c, d are different.
		(" nontangential proper part")
$xPOy$	iff	$x \setminus \{a, b\} \cap y \setminus \{c, d\} \neq \emptyset$
		and $x \not\subseteq y$ and $y \not\subseteq x$
$xEQy$	iff	$x = y$ ("equals")
$xTPPiy$	iff	$yTPPx$ ("contains tangentially")
$xNTPPiy$	iff	$yNTPPx$
		("contains non-tangentially")

Note that we have defined those relations such that they can be applied also to the case where x or y is improper. In particular, if x or y is improper then it can never be the case that $NTPP$ or $NTPPi$ holds.

We use the usual set theoretical abbreviation to express a relationship out of a given set of relations. For example $\langle ab \rangle \{DC, EC\} \langle cd \rangle$ stands for: $\langle ab \rangle DC \langle cd \rangle$ or $\langle ab \rangle EC \langle cd \rangle$.

The following proposition is going to show that we have to exclude the relations of non-tangential proper part $NTPP$ and its inverse $NTPPi$ between $\langle ab \rangle$ and $\langle cd \rangle$ for the analogical proportion R_1^{Btw} to hold and vice versa.

Proposition 16. $R_1^{Btw}(a, b, c, d)$ *iff*

1. *the lines through* $\langle ab \rangle$ *and* $\langle cd \rangle$ *is the same line* L *and*
2. $sign(a, b) = 0$ *or* $sign(c, d) = 0$ *or* $sign(a, b) = sign(c, d)$ *and*
3. $\langle ab \rangle \{DC, EC, TPP, TPPi, PO, EQ\} \langle cd \rangle$.

Proof. $\rightarrow$: Assume $R_1^{Btw}(a, b, c, d)$. Then a, b, c, d are on the same line. We prove the result considering case-wise. Case 1: $a = b$. Then $sign(a, b) = 0$. Now $R_1^{Btw}(a, b, c, d)$ in this case is $R_1^{Btw}(a, a, c, d)$ which means $\langle ad \rangle \subseteq \langle ac \rangle$ or $\langle ac \rangle \subseteq \langle ad \rangle$ which amounts to saying that the following holds: $\langle ad \rangle \{TPP, TPPi, EQ\} \langle ac \rangle$.

Case 2: $a \neq b$. a) Subcase $c = d$. Then $sign(c, d) = 0$. Now $R_1^{Btw}(a, b, c, d)$ means in this case $R_1^{Btw}(a, b, c, c)$ which means $\langle ac \rangle \subseteq \langle bc \rangle$ or $\langle bc \rangle \subseteq \langle ac \rangle$ which amounts to $\langle ac \rangle \{TPP, TPPi\} \langle ab \rangle$.

b) Subcase $c \neq d$. That $R_1^{Btw}(a, b, c, d)$ means that $\langle ad \rangle \subseteq \langle bc \rangle$ or $\langle bc \rangle \subseteq \langle ad \rangle$. Assume $\langle ad \rangle \subseteq \langle bc \rangle$ (the other disjunct is treated symmetrically). If $sign(a, b) \neq sign(c, d)$ were the case, this would e.g. mean from reading right to left that $a < b$ and $d < c$. But due to $\langle ad \rangle \subseteq \langle bc \rangle$ we must have $b \leq d$ and $d \leq c$, i.e. $b < c$. Now $\langle ad \rangle \subseteq \langle bc \rangle$ also means $b \leq a$, giving a contradiction. So $sign(a, b) = sign(c, d)$ holds. Now, in the left orientation we have intervals $[a, b]$ and $[c, d]$ or intervals $[b, a]$ and $[c, d]$. Consider the first case (the other is treated similarly). We have to treat the case of proper and improper intervals. (i) Case $a = d$ and $b = c$. Here $\langle ad \rangle \subseteq \langle bc \rangle$ becomes $\langle a \rangle \subseteq \langle b \rangle$ which in turn means $a = b$. So $\langle ab \rangle = \langle bd \rangle$ and $\langle cd \rangle = \langle bd \rangle$, so $\langle ab \rangle EQ \langle cd \rangle$.

(ii) Case $d = a$ and $b \neq c$. Then $\langle ad \rangle \subseteq \langle bc \rangle$ means $a \in \langle bc \rangle$ which means $\langle ab \rangle EC \langle cd \rangle$. If $d \neq a$, then $\langle ad \rangle \subseteq \langle bc \rangle$ means that $\langle ab \rangle \{DC, PO\} \langle cd \rangle$. (iii) Case $d \neq a$ and $b = c$ is not possible: $\langle ad \rangle \subseteq \langle bc \rangle$ would mean $\langle ad \rangle \subseteq \langle b \rangle$, so $a = d$, contradiction. (iv) Case $d \neq a$ and $b \neq c$. So all of a, b, c, d are pairwise different. $\langle ad \rangle \subseteq \langle bc \rangle$ means under the orientation with intervals $[a, b]$ and $[c, d]$ that we can have only $\langle ab \rangle DC \langle cd \rangle$.

$\leftarrow$: Assume that all a, b, c, d are different. (The other special cases are treated similarly as above) and all of the condition 1.–3. are true. In particular ab and cd have the same orientation. Between $\langle ab \rangle$ and $\langle cd \rangle$ one of the eight relations must hold. As all a, b, c, d are different the relation cannot be $EQ, EC, TPPi, TPP$. If $R_1^{Btw}(a, b, c, d)$ would not hold then none of the configuration (i)–(viii) could hold which means that also PO, DC would have to be excluded. We are left $NTPP, NTPPi$, but this contradicts our assumption that the third condition holds. Similarly one argues in the case where we have improper segments. $\square$

The proposition in particular shows that we have a non-trivial solution for the extension problem of compositionality: Choose : to be defined according to (1); define the relation :: on left and right each of which is a pair of the form (s, o) where s stands for a closed section and $o \in \{+, -, 0\}$. Last define $(s_1, o_1) :: (s_2, o_2)$ iff s_1 and s_2 make up the same line and $o_1 = 0$ or $o_2 = 0$ or $o_1 = o_2$ and $s_1 \{DC, EC, TPP, PO, EQ\} s_2$.

6. Related Work

Early work on axiomatic treatments of analogies goes back to Yves Lepage [12]. He considers linguistic issues related to analogies of words as in "walk:walked::search:searched". He accounts for similarities between words in the discussion of analogies but betweenness is not mentioned at all. Nonetheless, for future work plan to investigate the relations between his approach and our approach because a recent article [13] shows that there are relevant connections between similarity relations and betweenness relations.

Schockaert et al. [14] incorporate analogical proportions into a description logic framework. They enrich each concept with features and are able to define analogical proportions based on these features. For one variant of analogical proportions described by Schockaert and colleagues [14], some form of betweenness is used. However, the whole approach is based on analogical proportions over the domain of concepts and their features (and thus on sets). In contrast, our approach presumes an arbitrary domain of objects—as long as this domain is equipped with a betweenness relation.

Continuous analogical proportions are widely used also in practical approaches, e.g., for the interpolation of new training examples [9], however, to the best of our knowledge, by now not extended to the general case. Several ways have been proposed to enhance the analogical proportion of the Boolean case to multiple values. On the one hand, there are many practice-oriented approaches such as TransE [4] being focused on a specific (most times geometrical) operation and its usability for classification. On the other hand, there are multiple approaches starting with the Boolean case and generalizing it to the general case. One approach is based on a graded extension of the Boolean case thus allowing values in $[0, 1]$, it results in graded analogical proportions not being true or false but having a truth value in $[0, 1]$[15].

A general form of an analogical proportion based on groups has been proposed by Stroppa and Yvon [16]. This definition depicts a general applicable analogical proportion, having the classical proportion $\frac{a}{b} = \frac{c}{d}$ as special case. However, they do not define, in contrast to our approach, the analogical proportion focused on axioms it needs to fulfill and, though they consider groups (and even more general factorizations), their approach is not applicable to arbitrary spaces equipped with a betweenness relation.

Also related to the approach of this paper is a computational framework [17] on conceptual blending [18]. Conceptual blending is founded in cognition science, in particular, in that part dealing with the mechanisms underlying human creativity in inventing concepts, e.g., inventing the concept of houseboat from concept house and concept boat [17]. The authors stress that concept blending is a non-trivial operation different from simple set operations such as intersection or union. If we consider continuous analogical proportions over the domain of concepts, then concepts in between two others can be considered as their conceptual blending. Also the example discussed by Prade and Richard [3] (the concept of centaur as lying in between the concept of man and horse) indicates at least that work on continuous analogical proportions might profit from work on concept blending and vice versa. Continuous analogical proportions do not directly provide an operator of concept blending, but at least they describe a set of concepts in between two given concepts and hence a set of potential blending concepts, which fits to the idea of a "blended space" [17]. For the sake of completeness, we mention here also work on hyperdimensional computing and, more generally, vector symbolic approaches [19, 20] which provide vector operations corresponding to conceptual blending.

7. Conclusion

The idea of continuous analogical proportions motivated our investigation on the definability of analogical proportions via betweenness relations. As a main result we could show that such a definition is possible in the representative microcosm of the eight models of analogical proportions [3]– and it was even possible to consider a definition being the inverse of the forgetting operation (leading from analogical proportions to continuous analogical proportions).

In future work we plan, first, to widen the perspective from the microcosm of eight models to other analogical proportions, second, to generalize the decomposability questions from Euclidean betweenness to general betweenness relations, and, third, to investigate the consequences of our results for frameworks (such as concept blending) dealing with analogies and betweenness of concepts.

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