Circumscription in DL-Lite: Progress Report

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Abstract
Circumscription is a prominent approach to bring non-monotonicity to Description Logics (DLs), but unfortunately, it usually displays very high computational complexity of reasoning. Many works have studied circumscribed DLs, but most of them focus on expressive DLs containing $\mathcal{ALC}$, and the results for low-complexity DLs are limited. This paper summarises some recent progress in characterizing the computational complexity of reasoning in circumscribed DL-Lite. We perform a two-dimensional analysis, considering different languages of the DL-Lite family, and varying how concepts and roles are treated. In addition to classical circumscription, we consider the recently studied pointwise circumscription, which shows better complexity, in some cases, and remains decidable in the presence of minimized roles.

Keywords
Circumscription, Non-monotonic reasoning, DL-Lite, Lightweight DLs, Computational Complexity

1. Introduction

As decidable fragments of first-order logic, Description Logics (DLs) are intrinsically monotonic and do not allow to perform *commonsense reasoning*. To overcome this limitation, several non-monotonic extensions of DLs have been proposed [1, 2, 3, 4, 5]. A prominent research line here is given by circumscribed DLs [6, 7, 8, 9, 10].

Circumscription, as originally introduced by McCarthy to formalize commonsense reasoning [11], extends first-order logic with the ability to minimize predicate extensions. Later it was extended to allow fixed and varying predicates [12]; a so-called *circumscription pattern* specifies how predicates are partitioned into these three types. The non-monotonic extensions of DLs based on circumscription are very expressive, and the complexity of reasoning increases accordingly, up to undecidability (a common issue affecting many non-monotonic logics). The computational complexity of reasoning in expressive circumscribed DLs has been largely classified in [6, 8]. Moreover, [9, 10, 8] deal with the low-complexity logics $\mathcal{EL}$ and *DL-Lite*; early preliminary results for *DL-Lite* can be found in [13].

In this paper, we focus on the DL-Lite family and address three gaps that remain in the above rich set of results. First, we provide decidability and even tractability results for circumscribed knowledge bases with *minimized and fixed roles and nonempty TBoxes*. Previous decidability results concerned: (i) $\mathcal{ALCQO}$ with minimized roles but empty TBoxes (reasoning with nonempty TBoxes is undecidable) [6], and (ii) $\textit{DL-Lite}^{\mathcal{H}}_{Bool}$ and $\textit{DL-Lite}^{\mathcal{F}}_{Bool}$ with fixed roles [8].

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Second, we extend the analysis of reasoning *in the DL-Lite family* by addressing concept satisfiability, subsumption, and instance checking (with implications on conjunctive query answering). We refine and extend the lower complexity bounds for query answering in *DL-Lite* provided in [9], as well as the recent results in [10], where—unlike our results—lower complexity bounds rely on priorities, and roles can be neither minimized nor fixed. Similarly, the results of [7] for *DL-Lite* extended with *defeasible inclusions* do not overlap with ours. Defeasible inclusions can be encoded in circumscription (by replacing each $C \sqsubseteq_n D$ with $C \sqcap \neg Ab \sqsubseteq D$ for a minimized fresh concept name Ab) but this falls beyond *DL-Lite* and *DL-Lite*$_{Horn}$.

Third, we consider *pointwise circumscription* in the *DL-Lite* family. Pointwise circumscribed DLs were introduced in [14], and the complexity of reasoning has been characterized for $\mathcal{ALCIO}$ [15]. If roles can be minimized or fixed and nesting of quantifiers is disallowed, then reasoning in pointwise circumscribed $\mathcal{ALCIO}$ is decidable in NExp-time. This is notable since, under the same assumptions, reasoning under global circumscription is undecidable already for $\mathcal{ALC}$ [6]. In pointwise circumscription, the *single global* minimality check of classical circumscription is replaced by *multiple local* minimality checks at all domain elements and their immediate neighborhood. The semantics of [15] is closely related to pointwise circumscription as introduced by Lifschitz for first-order logic [16], with some important differences: varying predicates are not allowed to be reconfigured across the model, and the minimization at a single tuple is replaced by multiple minimizations at one domain element and its direct neighborhood.

We perform a two-dimensional analysis considering the generality of circumscription patterns and the expressiveness of the DL language. Our main contributions are summarized as follows:

- Under the assumption that no predicate is varying, concept satisfiability remains tractable in *DL-Lite*$^{\mathcal{H}}$ under global and pointwise circumscription. Tractability is not preserved in *DL-Lite*$_{Horn}$ and Σ_2^p-hardness holds for *DL-Lite*$_{Bool}$.
- Reasoning is intractable already in *DL-Lite* if predicates are allowed to vary. We provide a Σ_2^p upper bound for concept satisfiability in *DL-Lite*$^{\mathcal{H}}$ under global circumscription and an NP upper bound for *DL-Lite* under pointwise circumscription, under some syntactic restrictions.
- We show that if no restrictions on the circumscription pattern are assumed, concept satisfiability is undecidable in *DL-Lite*$^{\mathcal{H}}_{Bool}$ under global circumscription. The undecidability is shown via a reduction from the domino problem, and it builds a KB in which varying roles are subsumed by minimized ones. We note that the latter falls into a fragment where pointwise circumscription is decidable in NExp-time.

2. Preliminaries

We recall some DLs of the *DL-Lite* family [17]. We consider countably infinite, pairwise disjoint sets N_C of concept names and N_R of role names. *Roles* R are either a role name $r \in N_R$ or its inverse r^-; if $R = r^-$ is an inverse role, then R^- denotes r. *Basic concepts* take the form A or $\exists R$, with $A \in N_C$ a concept name and R a role. In basic *DL-Lite*, *concept inclusions* take the form $B_1 \sqsubseteq B_2$ or $B_1 \sqsubseteq \neg B_2$ with B_1 and B_2 basic concepts. We consider also the variants *DL-Lite*$_{Horn}$ and *DL-Lite*$_{Bool}$. The former allows concept inclusions of the form $B_1 \sqcap \ldots \sqcap B_n \sqsubseteq B$ and $B_1 \sqcap \ldots \sqcap B_n \sqsubseteq \neg B$ where B_i and B are basic concepts, while in the latter we may write concept inclusions $C \sqsubseteq D$, where C and D are arbitrary Boolean combinations of basic concepts built using $\sqcap, \sqcup$ and $\neg$. Let $\mathcal{L}$ be one of *DL-Lite*, *DL-Lite*$_{Horn}$ or *DL-Lite*$_{Bool}$. An $\mathcal{L}$ TBox $\mathcal{T}$ is a finite set of $\mathcal{L}$ concept inclusions. We extend these three logics with *role inclusions* of the form $R \sqsubseteq S$, where R and S are roles. An $\mathcal{L}^{\mathcal{H}}$ TBox is a finite set of $\mathcal{L}$ concept inclusions and role inclusions. An $\mathcal{L}^{(\mathcal{H})}$ knowledge base (KB) $\mathcal{K}$ is a pair consisting of an ABox $\mathcal{A}$ and an $\mathcal{L}^{(\mathcal{H})}$ TBox. The latter is defined as a finite set of assertions of the forms $A(c)$ and $r(c, d)$, where $A \in N_C$, $B \in N_R$, and c, d are *individuals* from a countably infinite alphabet N_I, disjoint from N_C and N_R. The semantics of these DLs, and in particular the notion of a *model* are defined as usual; please see [17]. We write $\mathcal{I} \models \Gamma$ if $\mathcal{I}$ is a model of the TBox or KB Γ, and use $M(\Gamma)$ to denote the set of models of Γ.

2.1. Circumscribed DLs

We recall the notion of *circumscription* for DLs. Following [6], we denote circumscription patterns as triples $\mathcal{P} = (M, V, F)$, where M, V, and F are mutually disjoint sets partitioning the predicates in $\mathcal{K}$, respectively standing for *minimized, varying*, and *fixed* predicates[1]. If $\mathcal{K}$ is a KB and $\mathcal{P} = (M, V, F)$ a circumscription pattern such that M, V and F partition the signature of $\mathcal{K}$, we say that "$\mathcal{K}$ is circumscribed with the pattern $\mathcal{P}$", in symbols $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$.

Definition 1. *Let $\mathcal{P} = (M, V, F)$ be a circumscription pattern, and assume a pair of interpretations $\mathcal{I}, \mathcal{J}$. We write $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ if the following conditions are satisfied:*

(i) $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$ *and* $a^{\mathcal{I}} = a^{\mathcal{J}}$ *for all individuals* a,

(ii) $Q^{\mathcal{I}} \subseteq Q^{\mathcal{J}}$ *for all* $Q \in M$, *and*

(iii) $Q^{\mathcal{I}} = Q^{\mathcal{J}}$ *for all* $Q \in F$.

We write $\mathcal{I} \prec_{\mathcal{P}} \mathcal{J}$, *if* $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ *and* $Q^{\mathcal{I}} \subset Q^{\mathcal{J}}$ *for some* $Q \in M$.

Definition 2. *An interpretation $\mathcal{I}$ is a* minimal model *of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, *in symbols* $\mathcal{I} \models \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$, *if* $\mathcal{I} \models \mathcal{K}$ *and there is no interpretation* $\mathcal{J}$ *s.t.* $\mathcal{J} \models \mathcal{K}$ *and* $\mathcal{J} \prec_{\mathcal{P}} \mathcal{I}$. *We use* $MM(\mathcal{K}, \mathcal{P})$ *to denote the set of minimal models of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$.

Now we recall *pointwise circumscription* [15]. We can see that Definition 1 does not restrict in any way how the extension of Q may differ in $\mathcal{I}$ and $\mathcal{J}$. It quantifies *globally* over all subsets of $Q^{\mathcal{J}}$. We call this semantics *global circumscription.* In *pointwise circumscription*, we use a more cautious comparability relation $\sim^{\bullet}$ between interpretations, which may differ only *locally* on the concepts and roles satisfied by one domain element.

Definition 3. *Assume a pair of interpretations* $\mathcal{I}, \mathcal{J}$ *with* $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$ *and* $a^{\mathcal{I}} = a^{\mathcal{J}}$, *for all* $a \in N_I$. *We write* $\mathcal{I} \sim^{\bullet} \mathcal{J}$ *if there exists* $e \in \Delta^{\mathcal{I}}$ *such that:*

(i) $A^{\mathcal{I}} \setminus \{e\} = A^{\mathcal{J}} \setminus \{e\}$ *for all concept names* A, *and*

(ii) $r^{\mathcal{I}} \cap (\Delta \times \Delta) = r^{\mathcal{J}} \cap (\Delta \times \Delta)$ *for all role names* r, *where* $\Delta = \Delta^{\mathcal{I}} \setminus \{e\}$.

[1] We do not assume *priorities* over the minimized predicates

Definition 4. *Assume a circumscription pattern* $\mathcal{P}$ *and a pair of interpretations* $\mathcal{I}, \mathcal{J}$*. We write* $\mathcal{I} \preceq^{\bullet}_{\mathcal{P}} \mathcal{J}$*, if* $\mathcal{I} \preceq_{\mathcal{P}} \mathcal{J}$ *and* $\mathcal{I} \sim^{\bullet} \mathcal{J}$*. We write* $\mathcal{I} \prec^{\bullet}_{\mathcal{P}} \mathcal{J}$*, if* $\mathcal{I} \preceq^{\bullet}_{\mathcal{P}} \mathcal{J}$ *and* $Q^{\mathcal{I}} \subset Q^{\mathcal{J}}$ *for some* $Q \in M$*.*

Definition 5. *An interpretation* $\mathcal{I}$ *is a* pointwise minimal model *of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$*, in symbols* $\mathcal{I} \models^{\bullet} \mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$*, if* $\mathcal{I} \models \mathcal{K}$ *and there is no interpretation* $\mathcal{J}$ *s.t.* $\mathcal{J} \models \mathcal{K}$ *and* $\mathcal{J} \prec^{\bullet}_{\mathcal{P}} \mathcal{I}$*. We use* $PMM(\mathcal{K}, \mathcal{P})$ *to denote the set of pointwise minimal models of* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$*.*

The reasoning tasks of concept satisfiability, subsumption, and instance checking are adapted to global circumscription and pointwise circumscription. Assume a globally (resp. pointwise) circumscribed knowledge base $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$:

- A concept C is satisfiable w.r.t. $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ if there exists $\mathcal{I} \in MM(\mathcal{K}, \mathcal{P})$ (resp. $PMM(\mathcal{K}, \mathcal{P})$) such that $C^{\mathcal{I}} \neq \emptyset$.
- For all concepts C and D, C is subsumed by D w.r.t. $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, for all $\mathcal{I} \in MM(\mathcal{K}, \mathcal{P})$ (resp. $PMM(\mathcal{K}, \mathcal{P})$). In symbols, $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \models^{(\bullet)} C \sqsubseteq D$.
- Given any individual a and any concept C, a is an instance of C w.r.t. $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ for all $\mathcal{I} \in MM(\mathcal{K}, \mathcal{P})$ (resp. $PMM(\mathcal{K}, \mathcal{P})$). In symbols, $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K}) \models^{(\bullet)} C(a)$.

In expressive DLs the reasoning tasks above can be interreduced [6], but in DL-Lite fragments the reductions are trickier. In [7] the authors provide reductions between reasoning tasks for DL-Lite$^{\mathcal{H}}$, but they rely on *varying predicates* and do not extend to all the settings we consider here.

Example 1. *Consider the following simple example:* Students normally do not work; Ann is a student. *We expect to conclude that Ann does not work. If later it is stated that Ann works, then the previous conclusion should be retracted without deriving any contradiction. This example can be encoded as an instance checking problem in DL-Lite*$_{Bool}$ *with both global and pointwise circumscription, using one abnormality predicate* $\mathrm{Ab}_{\mathrm{Student}}$*:*

$$\mathrm{Student} \sqcap \exists \mathrm{hasJob} \sqsubseteq \mathrm{Ab}_{\mathrm{Student}} \quad \textit{(a working student is an abnormal student)}$$
$$\mathrm{Student}(\mathrm{Ann})\,.$$

If $\mathrm{Ab}_{\mathrm{Student}}$ *is minimized and the other predicates are allowed to vary, then the above knowledge base entails* $\neg\exists\mathrm{hasJob}(\mathrm{Ann})$ *(as expected), under both semantics. After adding the assertion* $\exists\mathrm{hasJob}(\mathrm{Ann})$*, the resulting knowledge base is consistent under both semantics and* $\neg\exists\mathrm{hasJob}(\mathrm{Ann})$ *is not derivable any longer.*

Example 2. *Circumscribed DL-Lite*$_{core}$ *allows us to do some reasoning under the closed world assumption. For instance, given the following knowledge base where all predicates are minimized,*

$$\mathrm{Student} \sqsubseteq \mathrm{Person} \qquad \mathrm{Student} \sqsubseteq \exists \mathrm{hasStudentCard} \qquad \mathrm{Person}(\mathrm{Bob})$$

it can be concluded that $\neg\mathrm{Student}(\mathrm{Bob})$ *and* $\neg\exists\mathrm{hasStudentCard}(\mathrm{Bob})$*, under both semantics.*

The next example illustrates the power of circumscription by showing that it may introduce concept unions and functional roles (two features that in general increase complexity).

Example 3. *Assume the following TBox:*

$$\text{Company\_Owner} \sqsubseteq \text{Has\_Income} \qquad \text{Employee} \sqsubseteq \text{Has\_Income}$$

and the pattern $M = \{\text{Has\_Income}\}$ *and* $F = \{\text{Company\_Owner}, \text{Employee}\}$*. We can derive that* $\text{Circ}_{\mathcal{P}}(\mathcal{T}) \models \text{Has\_Income} \sqsubseteq \text{Company\_Owner} \sqcup \text{Employee}$*. Moreover, if we add:*

$$\text{Company\_Owner(Bob)} \quad \text{Company\_Owner} \sqsubseteq \exists\text{associated\_VAT}$$

where $\text{associated\_VAT} \in \text{M}$*, one can derive that there is a unique VAT number associated to* Bob*.*

Global Circumscription vs Pointwise Circumscription. Pointwise circumscription is a sound approximation of circumscription, as all minimal models are also pointwise minimal. The converse is often true. In fact, we provide a sufficient condition in Proposition 1. However, there are cases where they differ, as pointwise minimal models need not be globally minimal.

Example 4. *Consider the DL-Lite*$_{core}$ *TBox* $\mathcal{T} = \{A \sqsubseteq \exists R, \exists R^- \sqsubseteq A\}$ *and with the circumscription pattern* $\mathcal{P} = (M, \emptyset, \emptyset)$ *such that* $M = \{A, R\}$*.The interpretation* $\mathcal{I} = \{e, d\}$ *with* $A^{\mathcal{I}} = \{e, d\}$ *and* $R^{\mathcal{I}} = \{(e, d), (d, e)\}$ *is a pointwise minimal model. However,* $\mathcal{I} \notin MM(\mathcal{T}, \mathcal{P})$*.*

3. Complexity Results

Bonatti et al. [6] provide an extensive study of the computational complexity of expressive circumscribed DLs. Analogously to the propositional case [18, 19, 20], the complexity is significantly affected by the choice of the circumscription patterns. We will see that this is also the case for *DL-Lite*. In the following sections, we study the complexity of reasoning in three DLs in the DL-Lite family [17] under three different forms of circumscription patterns.

Definition 6. *Given a circumscription pattern* $\mathcal{P} = (M, V, F)$*, we say that*

(i) $\mathcal{P}$ *is a* basic pattern *if* $V = \emptyset$*,*

(ii) $\mathcal{P}$ *is a* role-varying pattern *if roles are only allowed to vary, i.e.* $(M \cup F) \cap N_R = \emptyset$*,*

(iii) $\mathcal{P}$ *is a* general pattern *if no restrictions are imposed.*

Table 1 summarizes our complexity results. They are stated for concept satisfiability. For basic patterns, we can transfer the lower bounds to the complement of subsumption and instance checking (as concept unsatisfiability easily reduces to subsumption and to instance checking), but without varying predicates, we do not have the converse reductions, thus the P-time upper bound is given only for concept satisfiability. In contrast, both lower and upper bounds in columns 2,3, 5 and 6 apply to the complement of subsumption and instance checking [7].

Before we delve deeper into the effect of different kinds of circumscription patters, we first observe that we can infer some lower bounds from propositional logic. In particular, we can rephrase in our setting some results proved in [18] for propositional logic under the *extended close world assumption* (ECWA), and in [19] for circumscribed propositional formulas. This can be done using only concept names, and does not require roles. It is not hard to see that in such a setting global and pointwise circumscription coincide, and we get the following.

	Global Circumscription			Pointwise Circumscription		
	basic	**varying roles**	**general**	**basic**	**varying roles**	**general**
DL-Lite$^{(\mathcal{H})}$	NL-c	$\geq$ NP and $\leq \Sigma_2^p$	?	$\leq$ P	$\geq$ NP and $\leq$ NP*	$\leq$ NExp‡
DL-Lite$^{(\mathcal{H})}_{Horn}$	$\geq$ NP	$\geq$ NP and $\leq \Sigma_2^{p\diamond}$	?	$\geq$ NP	$\geq$ NP	$\leq$ NExp‡
DL-Lite$^{(\mathcal{H})}_{Bool}$	$\geq \Sigma_2^p$	$\geq \Sigma_2^p$ and $\leq$ NExp§	undecidable†	$\geq \Sigma_2^p$	$\geq \Sigma_2^p$	$\leq$ NExp‡

Table 1
Complexity of concept satisfiability in circumscribed *DL-Lite*. $\cdot^{\diamond,\S}$ Follows from [10]. $\cdot^\dagger$ Undecidability is shown for *DL-Lite*$^{\mathcal{H}}_{Bool}$, the remaining lower bounds hold even without role inclusions. $\cdot^\ddagger$ If varying roles are subsumed by minimized or fixed roles, it follows from [15]. $\cdot^\star$ Under some syntactic restrictions.

Theorem 1. *Concept satisfiability under both global and pointwise circumscription is:*

- *NP-hard for DL-Lite*$_{Horn}$*, even with* $V = \emptyset$*,*
- Σ_2^p*-hard for DL-Lite*$_{Bool}$*, even with* $V = \emptyset$*.*
- *NP-hard for DL-Lite if* $V \neq \emptyset$*.*

Subsumption and instance checking are hard for the complementary class.

All the NP and Σ_2^p hardness results in Table 1 follow from this theorem. We devote the rest of the paper to proving the remaining bounds: the membership in NL and P for basic patterns, the membership in NP and Σ_2^p for varying roles, and the undecidability for general patterns.

3.1. Basic Patterns

In this section, we show that, under the assumption that no predicate is allowed to vary, concept satisfiability is tractable for *DL-Lite*$^{\mathcal{H}}$, under both global and pointwise circumscription. This is remarkable, especially since we allow for roles to be minimized or fixed. For expressive DLs in $\mathcal{ALCIOQ}$ the latter assumption easily leads to undecidability [6], while Bonatti et al. [7] proved that reasoning in $\mathcal{EL}$ with defeasible inclusions is undecidable if roles are fixed.

We first observe that the satisfiability of negated concepts is classical in both globally circumscribed and pointwise circumscribed *DL-Lite*$^{\mathcal{H}}$, *DL-Lite*$^{\mathcal{H}}_{Horn}$, and *DL-Lite*$^{\mathcal{H}}_{Bool}$. The latter follows from a result in [9] (Theorem 5). This does not extend to positive concepts. However, in *DL-Lite*$^{\mathcal{H}}$ we can still obtain a tractability result relying on the fact that in a minimal model every object in the extension of a minimized predicate must be justified by an assertion in the ABox or a fixed predicate.

In what follows we assume w.l.o.g. that the given circumscribed KB $\mathcal{K}$ does not contain inclusions of the form $\top \sqsubseteq C$ with $C \notin F$. If such inclusions are present, then we can remove them by introducing a fresh concept name Top, adding the inclusion $\top \sqsubseteq Top$ and replacing any $\top \sqsubseteq C \in \mathcal{K}$ with $Top \sqsubseteq C$, and extending the circumscription pattern $\mathcal{P}$ to $\mathcal{P}' = (M, V, F \cup \{Top\})$; these steps preserve both global and pointwise semantics.

We define the *dependency graph* of a *DL-Lite*$^{\mathcal{H}}$ KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ as the directed graph $DG(\mathcal{K}) = (V_\mathcal{K}, E_\mathcal{K})$ where $V_\mathcal{K}$ is given by all concept names and role names occurring in $\mathcal{K}$, and $E_\mathcal{K}$

contains the pairs (P, Q) for which there exists $\alpha \in \mathcal{T}$ such that (1) P occurs on the left-hand side of α, and (2) Q occurs on the right-hand side of α and is not under negation.

Theorem 2. *Assume a DL-Lite$^{\mathcal{H}}$ KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ and a basic pattern $\mathcal{P}$. A concept B is satisfiable w.r.t* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ *iff $\mathcal{K}$ has a classical model $\mathcal{I}$ with $Q^{\mathcal{I}} \neq \emptyset$ for some predicate Q such that:*
(a) there exists a path in $DG(\mathcal{K})$ from Q to B if $B \in N_C$, and to R if B is of the form $\exists R$, and
(b) Q is a fixed predicate or occurs in $\mathcal{A}$.

Since consistency checking in *DL-Lite*$^{\mathcal{H}}$ is complete for NL [17], and the conditions of Theorem 2 can be checked non-deterministically in logarithmic space, we get:

Theorem 3. *In DL-Lite$^{\mathcal{H}}$ concept satisfiability w.r.t. circumscribed KBs where $V = \emptyset$ is NL-complete.*

The hardness follows from classical reasoning in *DL-Lite*$^{\mathcal{H}}$, which corresponds to circumscribed *DL-Lite*$^{\mathcal{H}}$ with all predicates fixed. Under pointwise circumscription, the (if) direction of Theorem 2 holds too, but the (only if) may fail. Recall Example 4. Given the dependency graph of $\mathcal{T}$ and the concept A, one can easily observe that neither (a) nor (b) in Theorem 2 is satisfied. Intuitively, A is self-supported, using a cycle involving A in the dependency graph of $\mathcal{T}$. A characterization very similar to Theorem 2 for pointwise circumscription can be achieved by accommodating such cycles. To do this, we use a more sophisticated dependency graph.

Definition 7 (Dependency graph revisited)**.** *(Re)define the* dependency graph $DG(\mathcal{K}) = (V_{\mathcal{K}}, E_{\mathcal{K}})$ *of a DL-Lite$^{\mathcal{H}}$ KB $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ as follows. The set of vertices $V_{\mathcal{K}}$ is the least set containing:*

- *all positive concepts C such that for some D, either $(C \sqsubseteq D) \in \mathcal{T}$ or $(D \sqsubseteq C) \in \mathcal{T}$;*
- *all the concepts P such that for some a $P(a) \in \mathcal{A}$*
- *all the concept $\exists P$ and $\exists P^-$ such that for some a, b, $P(a, b) \in \mathcal{A}$.*
- *all nominals $\{a\}$ such that a occurs in $\mathcal{K}$.*

The set of edges $E_{\mathcal{K}}$, labelled with the symbols in $\{i, c\}$, is the least set containing:

- *all (C, i, D) such that there exists $(C \sqsubseteq D) \in \mathcal{T}$;*
- *all $(\{a\}, i, C)$ such that $C(a) \in \mathcal{A}$;*
- *all $(\{a\}, i, \exists R)$ such that for some b, $R(a, b) \in \mathcal{A}$;*
- *all $(\{a\}, i, \exists R^-)$ such that for some b, $R(b, a) \in \mathcal{A}$;*
- *all $(\exists R, i, \exists S)$ and $(\exists R^-, i, \exists S^-)$ such that for $R \sqsubseteq S \in \mathcal{T}$;*
- *all $(\exists R, c, \exists R^-)$ such that for some D, $(D \sqsubseteq \exists R) \in \mathcal{T}$.*

Edges labelled with c are called cross edges*, while the rest (labelled by i) are* inner edges.

Definition 8 (Good paths, Cycles)**.** A *path* in $DG(\mathcal{K})$ is a sequence π of edges in $E_{\mathcal{K}}$:

$$(C_0, \ell_1, C_1)(C_1, \ell_2, C_2) \cdots (C_{n-1}, \ell_n, C_n)\,.$$

We say that π is a path from C_0 to C_n. If $C_n = C_0$ then the path is called a *cycle*. A path is *good* if $\ell_i = c$ implies $\ell_{i+1} \neq c$ ($1 \leq i \leq n-1$), and a cycle is *good* if $\ell_i = c$ implies $\ell_{(i \bmod n)+1} \neq c$.

Definition 9 (Inner segments)**.** An *inner segment* of a path $\pi = e_1 \dots e_n$ in $DG(\mathcal{K})$ (where $e_i \in E_{\mathcal{K}}$, for $i = 1, \dots, n$) is a subpath $e_k e_{k+1} \dots e_{k+m}$ of π ($1 \leq k \leq k+m \leq n$) such that for all $i = k, \dots, k+m$, e_i is an inner edge of $DG(\mathcal{K})$. Such an inner segment of π is *maximal* iff it is not a strict subpath of any inner segment of π. An inner segment $e_k \dots e_{k+m}$ of π is *satisfiable* (w.r.t. $\mathcal{K}$) iff there exists $\mathcal{I} \in M(\mathcal{K})$ such that $C_k^{\mathcal{I}} \neq \emptyset$, with $e_k = (C_k, \ell_{k+1}, C_{k+1})$.

This refined notion of dependency graph allows us to distinguish cycles in the interpretation from other cycles in the dependency graph. The notion of *good cycles* ensures that we only consider cycles that pass over more than one object in the interpretation, i.e., passing at least one cross edge, but excluding consecutive symmetric cross edges. The inner segments correspond to sequences of concept implications that may need to hold at one domain element.

Theorem 4. *Assume a DL-Lite*$^{\mathcal{H}}$ *KB* $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ *and a basic pattern* $\mathcal{P}$*. Under pointwise circumscription, a concept* B *is satisfiable w.r.t* $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ *iff at least one of the following hold:*

(a) there exists a path from $\{a\}$ *to* B *in* $DG(\mathcal{K})$ *and* $\mathcal{K}$ *is satisfiable,*

(b) there exists $Q \in V_{\mathcal{K}}$ *with* $Q \in F$*, or* $R \in F$ *if* Q *is of the form* $\exists R^{(-)}$*, such that*

  (1) there exists a path from Q *to* B *in* $DG(\mathcal{K})$*, and*

  (2) there exists $\mathcal{I} \in M(\mathcal{K})$ *such that* $Q^{\mathcal{I}} \neq \emptyset$*;*

(c) there exists a path π *in* $DG_{\mathcal{K}}$ *to* B *from a* good *cycle involving at least one cross edge, and such that all the maximal inner segments in* π *in the cycle are satisfiable w.r.t.* $\mathcal{K}$*.*

Conditions (a) and (b) together check exactly the same conditions as in Theorem 2 (organized and formulated differently due to the modified definition of $DG(\mathcal{K})$). The new condition is (c), which identifies when a concept is satisfiable in a pointwise minimal model but not under global circumscription. Example 4 falls into this category: A satisfies condition (c). Checking the conditions of Theorem 4 is tractable, but we leave its precise complexity open.

Theorem 5. *Concept satisfiability in pointwise circumscribed DL-Lite*$^{\mathcal{H}}$ *with* $V = \emptyset$ *is in P.*

Theorem 2 and Theorem 4 differ only on condition (c), and thus concept satisfiability coincides when $DG(\mathcal{K})$ is acyclic. In fact, both semantics coincide for any reasoning task.

Proposition 1. *Given a KB* $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ *in DL-Lite*$^{\mathcal{H}}$ *and a basic pattern* $\mathcal{P}$*, if* $DG(\mathcal{K})$ *is acyclic then* $PMM(\mathcal{K}, \mathcal{P}) = MM(\mathcal{K}, \mathcal{P})$*.*

3.2. Role-varying Patterns

In this section, we extend circumscription patterns with varying predicates. In particular, we study the complexity under the assumption that *all roles are varying.* In [6] circumscription patterns were restricted in this way to obtain decidability and a NEXP$^{\mathrm{NP}}$ upper bound in $\mathcal{ALCIO}$. We obtain upper bounds that are significantly lower: Σ_2^p for global circumscription in *DL-Lite*$^{\mathcal{H}}$, and NP for pointwise circumscription in plain *DL-Lite* (under some syntactic restrictions on the axioms). The latter is tight by Theorem 1, and we believe that the former may also be so.

Recall the following result from [7, Theorem 4.4], which allows us to eliminate varying concepts: for each concept name $A \in V$, we introduce a fresh varying role R_A and each occurrence of A in $\mathcal{T}$ is replaced with $\exists R_A$.

Theorem 6. *If $\mathcal{L}$ is a DL supporting unqualified existentials, then reasoning w.r.t. (pointwise) circumscribed KBs in $\mathcal{L}$ such that $V \cap N_C \neq \emptyset$ can be reduced to reasoning w.r.t. (pointwise) circumscribed KBs in $\mathcal{L}$ such that $V \subseteq N_R$.*

We assume w.l.o.g. that all concepts are minimized or fixed, while roles are only varying. We show that circumscribed *DL-Lite*$^{\mathcal{H}}$ has the small model property, following the model construction for *DL-Lite*$^{\mathcal{H}}$ with defeasible inclusions used in [7].

Lemma 1. *Assume a KB $\mathcal{K}$ in DL-Lite$^{\mathcal{H}}$ circumscribed with $\mathcal{P} = (M, V, F)$, with $N_R \subseteq V$, and a concept C. If there exists $\mathcal{I} \in MM(\mathcal{K}, \mathcal{P})$ such that $C^{\mathcal{I}} \neq \emptyset$, then there exists $\mathcal{J} \in MM(\mathcal{K})$ such that (i) $|\Delta^{\mathcal{J}}|$ is polynomial in the size of $\mathcal{K}$, and (ii) $C^{\mathcal{J}} \neq \emptyset$.*

Theorem 7. *Concept satisfiability in circumscribed DL-Lite$^{\mathcal{H}}$ with only varying roles is in Σ_p^2. Subsumption and instance checking are in Π_2^p.*

Proof. Assume a concept C_0 and a circumscribed KB $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ in *DL-Lite*$^{\mathcal{H}}$. From Lemma 1, for checking satisfiability of C_0 w.r.t. $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$ if suffices to guess an interpretation whose domain is polynomial in the size of $\mathcal{K}$ and use an NP oracle to check that it is a model of $\mathrm{Circ}_{\mathcal{P}}(\mathcal{K})$. □

NP upper bound for *DL-Lite* We provide in the full version of the paper an NP algorithm for concept satisfiability in pointwise circumscribed *DL-Lite* restricted to axioms of the forms $A \sqsubseteq (\neg)C$ and $C \sqsubseteq (\neg)A$ with A concept name and C a basic concept. That is, we do not allow existentials on both sides of the same axiom. In classical *DL-Lite* this assumption can be done w.l.o.g., but in pointwise circumscribed *DL-Lite* traditional normalization cannot be taken for granted [15]. We use the *mosaic technique*. First, we define so-called *tile types* that intuitively are small model fragments. Similarly to [15], additionally to the local minimality conditions on the tile types, we use special labelings to verify that minimality is preserved when a full model is assembled from tiles. Then we generate a system of *extended inequalities* that has one variable x_t for each tile type t [21]. The solutions to this system of inequalities are functions N assigning to each variable x_t a nonnegative integer number or ω. The inequalities guarantee that, from each solution N, if we take $N(x_t)$ copies of each tile type t we assemble correctly a pointwise minimal model. This is very similar to the technique used for obtaining the NExpTime upper bound in [15]. The key difference here is that, even if the number of variables x_t is exponential in $\mathcal{K}$, the number of inequalities is only polynomial in it. Therefore, one only needs to focus on those solutions where only polynomially many variables are nonzero [22]. This allows us to check that a solution exists in nondeterministic polynomial time in the size of the system [21, 23]. Provided that we can check the minimality of a tile type in polynomial time, which is feasible for *DL-Lite*$^{\mathcal{H}}$ [18], we obtain the desired upper bound.

Theorem 8. *Concept satisfiability in pointwise circumscribed DL-Lite is NP-complete if $N_R \subseteq V$ and no axiom of the form $\exists R \sqsubseteq \exists S$ is allowed.*

We belive that the upper bound above holds for pointwise circumscribed *DL-Lite*$^{\mathcal{H}}$ without this restriction on the axioms, but leave it open for future work.

3.3. General Patterns

In this section, we impose no restriction on the circumscription patterns: concepts and roles may freely participate in the minimized, fixed, and varying predicates. However, we show that already allowing roles to be minimized leads to undecidability in circumscribed $\textit{DL-Lite}^{\mathcal{H}}_{Bool}$.

We reduce the domino problem to the complement of instance checking, relying heavily on minimized and varying roles. We simulate the TBox constructed in [6] (see Lemma 27) for showing the undecidability of circumscribed $\mathcal{ALC}$ with minimized roles. Due to space constraints, we focus on explaining how to simulate the axioms that are not expressible in $\textit{DL-Lite}^{\mathcal{H}}_{Bool}$, and provide the detailed proof in the full version of the paper.

As usual, the key challenge is to enforce a grid. We use the roles H and V for the horizontal and vertical successor relations. In [6] the authors use an axiom $\top \sqsubseteq N \sqcup (\exists H.\exists V.B \sqcap \exists V.\exists H.\neg B)$ to state that a domain element is either in N or it does not participate in a correct cell of the grid. We can simulate qualified existentials using auxiliary roles in the usual way.

$$\begin{array}{llll} \top \sqsubseteq N \sqcup (B_{hv} \sqcap B_{vh}) & B_{hv} \sqsubseteq \exists H' & B_{vh} \sqsubseteq \exists V' & H' \sqsubseteq H \\ & \exists H'^{-} \sqsubseteq B_{h'} & \exists V'^{-} \sqsubseteq B_{v'} & H'' \sqsubseteq H \\ & B_{h'} \sqsubseteq \exists V'' & B_{v'} \sqsubseteq \exists H'' & V' \sqsubseteq V \\ & \exists V''^{-} \sqsubseteq B & \exists H''^{-} \sqsubseteq \neg B & V'' \sqsubseteq V \end{array}$$

where N, H and V are minimized, while $B, B_{hv}, B_{vh}, B_{h'}, B_{v'}, H', H'', V'$ and V'' are varying.

The key challenge now is to propagate a varying 'error' concept D when an element does not participate in a correct cell of the grid. This is achieved in [6] with the following $\mathcal{ALC}$ axioms:

$$\neg N \sqsubseteq D \qquad D \sqsubseteq \forall H.D \qquad D \sqsubseteq \forall V.D \qquad \exists H.D \sqsubseteq D \qquad \exists V.D \sqsubseteq D$$

We instead simulate the following weaker axioms

$$\neg N \sqsubseteq D \qquad D \sqsubseteq \exists H.D \qquad D \sqsubseteq \exists V.D \qquad \neg D \sqsubseteq \exists H.\neg D \qquad \neg D \sqsubseteq \exists V.\neg D$$

using the following set of axioms, and letting all the fresh roles vary:

$$\begin{array}{lllll} \neg N \sqsubseteq D & D \sqsubseteq \exists E_h \sqcap \exists E_v & & \neg D \sqsubseteq \exists \bar{H} \sqcap \exists \bar{V} & \\ & \exists {E_h}^{-} \sqsubseteq D & E_h \sqsubseteq H & \exists \bar{H}^{-} \sqsubseteq \neg D & \bar{H} \sqsubseteq H \\ & \exists {E_v}^{-} \sqsubseteq D & E_v \sqsubseteq V & \exists \bar{V}^{-} \sqsubseteq \neg D & \bar{V} \sqsubseteq V \end{array}$$

Using the fact that all the varying roles introduced above are subroles of H or V, and that H and V are minimized, we can show that, in a minimal model the following claim holds.

Claim 1. The roles H and V are functional at every domain element d such that $d \in \neg D^{\mathcal{I}}$.

This allows us to obtain a grid and to lift the undecidability in [6] to $\textit{DL-Lite}^{\mathcal{H}}_{Bool}$.

Theorem 9. *Concept satisfiability in circumscribed $\textit{DL-Lite}^{\mathcal{H}}_{Bool}$ is undecidable.*

We underline that the reduction does not carry over to pointwise circumscribed $\textit{DL-Lite}^{\mathcal{H}}_{Bool}$ as it heavily relies on the fact that in the absence of a proper cell the varying concept D is propagated over possibly infinitely many domain elements.

Observe that the TBox above is such that each varying role is subsumed by a minimized one. Under the same assumption, we get a decidability result for pointwise circumscription:

Theorem 10. *Concept satisfiability in pointwise circumscribed $DL\text{-}Lite^{\mathcal{H}}_{Bool}$ is in NExp-time under the assumption that varying roles are subsumed by minimized or fixed ones.*

The result above follows from the more general result for pointwise circumscribed $\mathcal{ALCIO}$ with no nested quantifiers [15], extended with the assumption that varying roles are subsumed by minimized ones. With the latter assumption, the mosaic technique used in [15] can be tuned to accommodate varying roles. (Intuitively, this relies on the fact that varying roles cannot be used for generating new connections).

4. Conclusion

In this paper, we have established several new results for circumscribed logics of the *DL-Lite* family. Remarkably, we have established decidability *with minimized and fixed roles and nonempty TBoxes*, which had only been considered recently in [15] for more expressive logics under pointwise circumscription, and with significantly higher complexity bounds. Several questions remain for further investigation. For instance, the precise complexity of subsumption for $DL\text{-}Lite^{\mathcal{H}}$ with basic patterns is not yet determined, and we believe that it may be harder than concept satisfiability. For the other patterns with variable predicates we already have NP-hardness results that carry over to subsumption, but we lack matching upper bounds. The lower bounds for role-varying patterns for $DL\text{-}Lite_{Horn}$ and $DL\text{-}Lite_{Bool}$ are inherited from the first column of Table 1, so it is not unlikely that they will not be tight. Table 1 shows that we do not know much about *DL-Lite* and general patterns under global circumscription. These settings could very well be undecidable, but it is far from obvious how to prove it.

Reducing the complexity of classical circumscription was a key motivation for its pointwise variant, but it seems that it does not always lead to better complexity in the *DL-Lite* family. It does seem to make things more manageable for $DL\text{-}Lite^{\mathcal{H}}$ and varying roles, where we think that global circumscription is likely to be hard for the second level of the polynomial hierarchy. We want to understand what happens in this case to $DL\text{-}Lite_{Horn}$ and $DL\text{-}Lite_{Bool}$. Interestingly, for basic patterns in $DL\text{-}Lite^{\mathcal{H}}$, pointwise circumscription seems to make things more challenging, and may even cause higher complexity.

In this work, we disallowed *role disjointness axioms*, usually allowed in classical DL-Lite: the latter affects the complexity under circumscription with priorities [10]. It is open whether this happens also in our settings. We plan to extend circumscription patterns by allowing priorities over minimized predicates, a key ingredient for defeasible DLs. Under the latter assumptions, there are no results for pointwise circumscribed DLs, while for global circumscription some very recent results can be found in [10]. The data complexity of circumscribed DL-Lite family is also a future research direction worth investigating. Lastly, another interesting direction is to look at $\mathcal{EL}$ and extensions, especially under pointwise circumscription.

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