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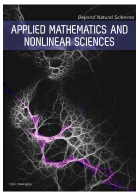

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Some Improvements on Relativistic Positioning Systems

Màrius Josep Fullana i Alfonso[1]†, Diego Pascual Sáez Milán[2,3], Josep Vicent Arnau i Córdoba[4], Neus Puchades Colmenero[2]

1 Institut de Matemàtica Multidisplinar, Universitat Politècnica de València, E-46022, València

2 Departament d'Astronomia i Astrofísica, Universitat de València, E-46100, Burjassot, València

3 Observatori Astronòmic, Universitat de València, E-46980, Paterna, València

4 Departament de Matemàtica Aplicada, Universitat de València, E-46100, Burjassot, València
Spain

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Abstract

We make some considerations about Relativistic Positioning Systems (RPS). Four satellites are needed to position a user. First of all we define the main concepts. Errors should be taken into account. Errors depend on the Jacobian transformation matrix. Its Jacobian is proportional to the tetrahedron volume whose vertexes are the four tips of the receiver-satellite unit vectors. If the four satellites are seen by the user on a circumference in the sky, then, the Jacobian and the tetrahedron volume vanish. The users we consider are spacecraft. Spacecraft to be positioned cannot be close to a null Jacobian satellites-user configuration. These regions have to be avoided choosing an appropriate set of four satellites which are not seen too close to the same circumference in the sky. Errors also increase as the user spacecraft separates from the emission satellite region, since the tetrahedron volume decreases. We propose a method to autonomously potion a user-spacecraft which can test our method. This positioning should be compared with those obtained by current methods. Finally, a proposal to position a user-spacecraft moving far from Earth, with suitable devices (autonomous), is presented.

Keywords: Astronomy and Astrophysics, Mathematical Physics, General Relativity, Relativistic Positioning Systems, Numerical Methods
AMS 2010 codes:35Q85, 37M99, 83C05.

1 Introduction

This paper gives some considerations about Relativistic Positioning Systems (RPS). First of all we define the main concepts. In Minkowski Space-Time (M-ST), the analytical transformation from emission to inertial

†Corresponding author.
Email address: mfullana@mat.upv.es

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coordinates was found by [1]. A numerical code (Newton-Raphson) to find the inverse transformation has also been developed by our team. A quasi-inertial system should be taken to describe satellites and user coordinates.

Errors depend on the Jacobian transformation matrix, J. Its Jacobian, $|J|$, is proportional to the tetrahedron volume whose vertexes are the four tips of the receiver-satellite unit vectors. If the four satellites are seen by the user on a circumference in the sky, then, the Jacobian and the tetrahedron volume vanish. Two kind of errors may be defined: U-errors associated to uncertainties in the satellite world lines (see [2] for details) and S-errors due to an approximate description of the photon world lines: the metric used in this description which takes into account some of the astronomical bodies but not all of them.

Both S and Uerrors have infinite limit as $|J|$ tends to zero; hence, there are great errors in a certain region near null Jacobian points. The users we consider are spacecraft. Spacecraft to be positioned cannot be close to a null Jacobian satellites-user configuration. These regions have to be avoided choosing an appropriate set of four satellites which are not seen too close to the same circumference in the sky. Errors also increase as the user spacecraft separates from the emission satellite region, since the tetrahedron volume decreases.

We present the basic ideas to understand the following prospects: 1) Improve on the description of the Solar System gravitational field in the positioning region. 2) Look for criteria to select the best sets of four satellites. It is not enough to calculate the Jacobian because its maximum value does not correspond to the minimum U and S errors. We only know that the errors tend to infinite when J vanishes. 3) If a spacecraft carries devices to get the unit vectors at emission times, the tetrahedron volume may be estimated for any set of four visible satellites and, consequently, sets leading to excessively small Jacobian values should not be used for positioning. 4) Only configurations of four satellites leading to big enough Jacobian values will be considered for positioning to get small U and S-errors. Each of these configurations gives a position, and the distribution of positions will be used to get the most probable position and the probability of any other location. 5) We propose a method to autonomously potion a user-spacecraft which can test our method. This positioning should be compared with those obtained by current methods. 6) Finally, a proposal to position a user-spacecraft moving far from Earth, with suitable devices (autonomous), should be presented.

2 The method

In the 0-order RPS, positioning coordinates are inertial coordinates in the M-ST asymptotic to Schwarszchild-ST and, consequently, they will be called *inertial asymptotic coordinates or inertial coordinates* x^α, as done in previous papers [3], [4] and [5].

The inertial coordinates (user position) may be found by using the satellite world lines and the emission proper times, excepting some cases in which the emission coordinates are compatible with two user positions (bifurcation); in these cases, a criterion –based on additional data– is necessary to choose the true position [6], [7], [8], [9], [10], [11] and [3]. Bifurcation does not play a role in this paper.

User -simultaneously- receives proper times (*emission coordinates* τ^A) from 4 satellites. User space-time position is given by four *Quasi-Inertial coordinates* x^α, associated to a quasi-inertial reference system. From emission (*satellite*) to reception (*user spacecraft*), photons travel through *null geodesics*: If the satellite world lines are known, *user quasi-inertial coordinates* may be computed from *emission ones*.

In M-ST, the analytical transformation from emission to inertial coordinates was found by [1]:

$$\tau^A(past) \longrightarrow x^\alpha \tag{1}$$

This analytical transformation is valid for any known satellite world line $x^\alpha_A = x^\alpha_A(\tau^\alpha)$ in $M-ST$. Only past emission proper times lead to positioning (past solutions).

$x^\alpha_A = x^\alpha_A(\tau^\alpha)$ are the world lines of satellites (known) in the quasi-inertial system parameterized by satellite proper times. These are the photon null geodesics:

$$\eta_{\alpha\beta}\,[x^\alpha_A - x^\alpha_A(\tau^\alpha)]\left[x^\beta_A = x^\beta_A(\tau^\alpha)\right] = 0 \tag{2}$$

An analytical solution was found for these four equations to give x^α as a function of τ^A, see [1]. We also built a numerical code (Newton-Raphson) to find the inverse transformation giving τ^A in terms of x^α. The Jacobian of the transformation, $|J|$, of $\tau^A = \tau^A(x^\alpha)$ is:

$$|J_{A\alpha}| = \frac{\partial \tau^A}{\partial x^\alpha} \cong \frac{\zeta[x_A^\alpha(\tau^A) - x^\alpha]}{D_A} \tag{3}$$

where $\zeta = 1$ for $\alpha = 1,2,3$ and $\zeta = -1$ for $\alpha = 4$. Quantities D_A are given by:

$$\eta_{\mu\nu} U_A^\mu(\tau^A)[x^\nu - x_A^\nu] \cong x_A^4(\tau^A) - x^4 = -D_A \tag{4}$$

And the explicit form of $|J|$ is:

$$|J| = \begin{vmatrix} \frac{x_1^1 - x^1}{D_1} & \frac{x_1^2 - x^2}{D_1} & \frac{x_1^3 - x^3}{D_1} & 1 \\ \frac{x_2^1 - x^1}{D_2} & \frac{x_2^2 - x^2}{D_2} & \frac{x_2^3 - x^3}{D_2} & 1 \\ \frac{x_3^1 - x^1}{D_3} & \frac{x_3^2 - x^2}{D_3} & \frac{x_3^3 - x^3}{D_3} & 1 \\ \frac{x_4^1 - x^1}{D_4} & \frac{x_4^2 - x^2}{D_4} & \frac{x_4^3 - x^3}{D_4} & 1 \end{vmatrix} . \tag{5}$$

In Fig. 1 we can see the Jacobian geometrical interpretation. V_T is the tetrahedron volume whose vertex are the four tips of the receiver-satellite unit vectors n_A^i (red vectors in Fig. 1). This is the relation between the $|J|$ and the tetrahedron volume.

$$|J| \cong 6V_T \tag{6}$$

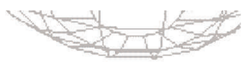

Fig. 1 Name of the figure.

If the four satellites are seen by the user on a circumference in the sky, then $|\alpha_1 - \alpha_4|$, $|J|$ and V_T vanish. For more details see [12]. This can be seen in Fig. 2.

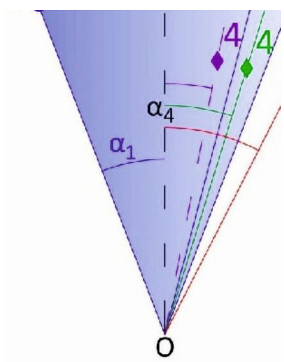

Fig. 2 Name of the figure.

3 Positioning Errors in RPS

Two kind of errors may be defined:

U-errors are associated to uncertainties in the satellite world lines (see [2] for details) S-errors due to an approximate description of the photon world lines. The metric used in this description which takes into account some of the astronomical bodies but not all of them. Both S and U errors have infinite limit as $|J|$ tends to zero; hence, there are great errors in a certain region near $|J| = 0$ points. Spacecraft to be positioned cannot be close to a $|J| = 0$ satellites-user configuration.

We should avoid these regions choosing an appropriate set of four satellites which are not seen too close to the same circumference in the sky (unit vectors n^i_A are not in the same cone). Errors increase as the user spacecraft separates from the emission satellite region, since the tetrahedron volume decreases (small solid angle).

We now show a representation of $|J|$, U-errors and $|\alpha_1 - \alpha_4|$ for a particular direction from Earth up to a distance of 10^5 km in Fig. 3. $|J|$ does not vanish, but it decreases in terms of the distance Consistently, U-errors increase, whereas $|\alpha_1 - \alpha_4|$ decreases.

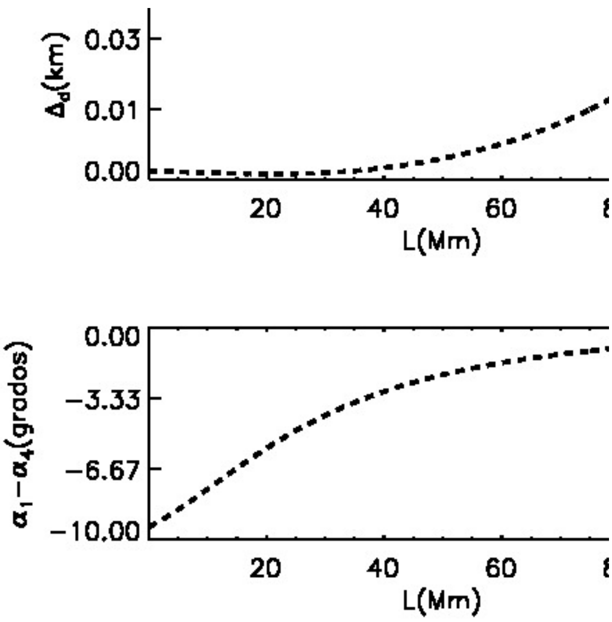

Fig. 3 Name of the figure.

We have choosen another particular direction with $|J| = 0$, $|\alpha_1 - \alpha_4|$ and infinite errors. See fig. 4. Those are the kind of tracks that should be avoided.

4 Conclusions and Perspectives

We have explained the basic ideas to understand the following prospects, which are listed below: 1) We should improve on the description of the Solar System gravitational field in the positioning region. 2) Look for the criteria to select the best sets of four satellites. It is not enough to calculate the Jacobian $|J|$ because the maximum value of $|J|$ does not correspond to the minimum U and S errors. We only know that the errors tend to infinite when $|J|$ vanishes. 3) If a spacecraft carries devices to get the unit vectors n^1_A at emission times τ^A, the tetrahedron volume ($|J|$) may be estimated for any set of four visible satellites and, consequently, sets

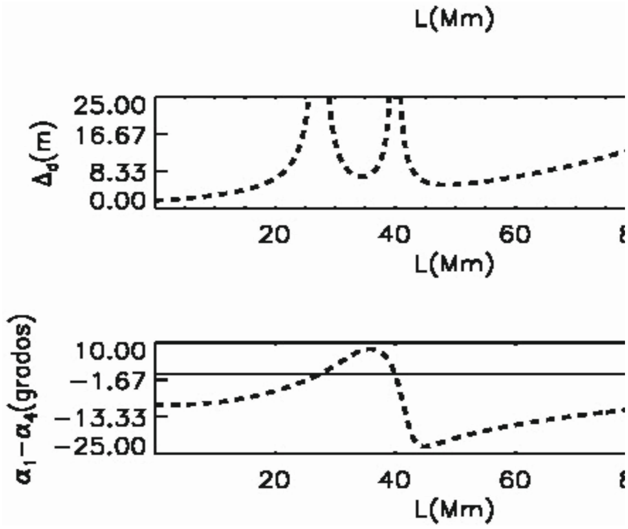

Fig. 4 Name of the figure.

leading to excessively small $|J|$ values should not be used for positioning. Only configurations of four satellites leading to big enough $|J|$ values will be considered for positioning to get small U and S-errors. Each of these configurations gives a position, and the distribution of positions will be used to get the most probable position and the probability of any other location. 4) The International Space Station (ISS) would be considered as a user-spacecraft to be autonomously positioned. To do that, this station would be provided with appropriate devices: detectors for the signals giving the emission coordinates τ_A (to get the position), and instruments to measure angles and vectors n^i_A (to select suitable sets of four satellites). This would be a good way to test the method that we have described in this paper, our resulting positions could be compared with those obtained by current methods. 5) Since spacecrafts moving far from Earth cannot be positioned with GNSS, it should be interesting to design a distribution of emissors in the Solar System placed in adequate planets or spacecrafts. This distribution would be designed in such a way that a user-spacecraft moving far from Earth, with suitable devices (autonomous), can see sets of four emissors having large enough tetrahedron volumes (small enough errors). This work is in progress.

Acknowledgements

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