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A generalization of truncated M-fractional derivative and applications to fractional differential equations

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Abstract

In this paper, our aim is to generalize the truncated M-fractional derivative which was recently introduced [Sousa and de Oliveira, A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties, Inter. of Jour. Analy. and Appl., 16 (1), 83–96, 2018]. To do that, we used generalized M-series, which has a more general form than Mittag-Leffler and hypergeometric functions. We called this generalization as truncated $\mathcal{M}$-series fractional derivative. This new derivative generalizes several fractional derivatives and satisfies important properties of the integer-order derivatives. Finally, we obtain the analytical solutions of some $\mathcal{M}$-series fractional differential equations.

Keywords: Truncated M-fractional derivative, alternative fractional derivative, conformable fractional derivative, M-series
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1 Introduction

Fractional analysis is a field that is frequently studied by scientists because of its many applications used to model real-world problems. In some recent studies, it is seen that mathematical models obtained by using various fractional derivatives have better overlapping with experimental data rather than the models with integer order derivatives. However, unlike integer order derivatives, different fractional derivative definitions may be used for different types of problems. This situation led scientists to identify more general fractional operators.

Especially in the last five years, several generalizations of some well-known fractional derivative operators have been addressed by many authors (see, for example [2, 3, 5, 6, 11, 18, 19, 33]). In addition to these studies, different fractional derivative operators having many features provided by the integer order derivative operator were also studied (see [16, 17, 27–31] and the references therein).

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In 2014, Khalil et al. [17] introduced a new type of fractional derivative for $f:[0,\infty)\to\mathbb{R}$, $t>0$ and $\alpha\in(0,1)$ as

$$T_\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}. \tag{1}$$

They called it conformable fractional derivative.

In the same year, Katugampola [16] introduced the alternative and truncated alternative fractional derivatives for $f:[0,\infty)\to\mathbb{R}$ as

$$D^\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(te^{\varepsilon t^{-\alpha}})-f(t)}{\varepsilon},\quad t>0,\ \alpha\in(0,1) \tag{2}$$

and

$$D_i^\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(te_i^{\varepsilon t^{-\alpha}})-f(t)}{\varepsilon},\quad t>0,\ \alpha\in(0,1) \tag{3}$$

respectively. Here $e_i^x=\sum_{k=0}^{i}\frac{x^k}{k!}$ is the truncated exponential function.

Recently, Sousa and de Oliveira [27, 29] introduced the M-fractional and truncated M-fractional derivatives for $f:[0,\infty)\to\mathbb{R}$ as

$$D_M^{\alpha;\beta}f(t)=\lim_{\varepsilon\to 0}\frac{f(tE_\beta(\varepsilon t^{-\alpha}))-f(t)}{\varepsilon},\quad \beta,t>0,\ \alpha\in(0,1) \tag{4}$$

and

$$_iD_M^{\alpha;\beta}f(t)=\lim_{\varepsilon\to 0}\frac{f(t_iE_\beta(\varepsilon t^{-\alpha}))-f(t)}{\varepsilon},\quad \beta,t>0,\ \alpha\in(0,1) \tag{5}$$

respectively, by means of one parameter Mittag-Leffler function [12]

$$E_\beta(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)},\quad \Re(\beta)>0,\ z\in\mathbb{C},$$

and its truncated version.

All the derivatives given above satisfies some properties of classical calculus, e.g. linearity, product rule, quotient rule, function composition rule and chain rule. Besides, for all the operators given above the α-order derivative of a function is a multiple of $t^{1-\alpha}\frac{df}{dt}$.

In 2009, generalized M-series defined by Sharma and Jain [25, 26]

$${}_pM_q^{\beta,\gamma}(z):={}_pM_q^{\beta,\gamma}\left[\begin{matrix}a_1\cdots a_p\\ c_1\cdots c_q\end{matrix};z\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{z^k}{\Gamma(\beta k+\gamma)}$$

where $\beta,\gamma,z\in\mathbb{C}$, $p,q\in\mathbb{N}$, $\Re(\beta)>0$, $c_i\neq 0,-1,-2,\ldots(i=1,2,\ldots,q)$. Here, $(\alpha)_k$ is the Pochhammer symbol [1] which given by

$$(\alpha)_\nu=\frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)},\qquad \alpha,\nu\in\mathbb{C}$$

with the assume $(\alpha)_0=1$. Note that if a_j $(j=1,2,\ldots,p)$ equals to zero or a negative integer, then the series reduces to a polynomial.

Generalized M-series is convergent for all z if $p\le q$; it is convergent for $|z|<\delta=\alpha^\alpha$ if $p=q+1$; and divergent if $p>q+1$. When $p=q+1$ and $|z|=\delta$, the series can converge on conditions depending on the parameters. For more information about M-series we refer [25, 26] and the references therein.

Most of the famous special functions can be described as the special cases of the generalized M-series:

$$
\begin{aligned}
{}_1\overset{1,1}{M}_1\,(1;1;z) &= \sum_{k=0}^{\infty}\frac{z^k}{k!} = e^z,\\
{}_1\overset{\beta,1}{M}_1\,(1;1;z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)} = E_\beta(z),\\
{}_1\overset{\beta,\gamma}{M}_1\,(1;1;z) &= \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\gamma)} = E_{\beta,\gamma}(z),\\
{}_1\overset{\beta,\gamma}{M}_1\,(\sigma;1;z) &= \sum_{k=0}^{\infty}\frac{(\sigma)_k\, z^k}{\Gamma(\beta k+\gamma)} = E^{\sigma}_{\beta,\gamma}(z),\\
{}_1\overset{1,1}{M}_1\,(a;c;z) &= \sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!} = \Phi(a;c;z),\\
{}_2\overset{1,1}{M}_1\,(a,b;c;z) &= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} = {}_2F_1(a,b;c;z),\\
{}_p\overset{1,1}{M}_q\,(z) &= \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{z^k}{k!} = {}_pF_q\left[\begin{matrix}a_1\cdots a_p\\ c_1\cdots c_q\end{matrix};z\right].
\end{aligned}
$$

Here, E_β, $E_{\beta,\gamma}$, $E^{\sigma}_{\beta,\gamma}$ are the one [23], two [32] and three parameters [24] Mittag-Leffler functions; and also Φ, ${}_2F_1$, ${}_pF_q$ are the confluent, Gauss and generalized hypergeometric functions [1], respectively.

Motivated by the above studies and the frequent use of M-series in fractional operator theory (see [8–10, 14, 21]), with the help of M-series, we first define a more general fractional derivative (truncated $\mathcal{M}$-series fractional derivative) and investigate its properties like linearity, product rule, the chain rule, etc. Then we extend some of the classical results in calculus like Rolle's theorem, mean value theorem etc. We also introduce the $\mathcal{M}$-series fractional integral and finally, we obtain the analytical solutions of ordinary and partial $\mathcal{M}$-series fractional linear differential equations.

2 Truncated $\mathcal{M}$-series Fractional Derivative

We first present the definitions of the truncated M-series and truncated $\mathcal{M}$-series fractional derivative operator.

Definition 1. The *truncated M-Series* is defined for $\beta > 0$ as

$$
{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(t) = {}_i\mathcal{M}^{\beta,\gamma}_{p,q}\left[\begin{matrix}a_1\cdots a_p\\ c_1\cdots c_q\end{matrix};t\right] := \sum_{k=0}^{i}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{t^k}{\Gamma(\beta k+\gamma)} \tag{6}
$$

where $\beta,\gamma,t\in\mathbb{R}$, $p,q\in\mathbb{N}$, $a_n,c_m\in\mathbb{R}$, $c_m\neq 0,-1,-2,\ldots(n=1,2,\ldots,p;\ m=1,2,\ldots,q)$.

Definition 2. Let $f:[0,\infty)\to\mathbb{R}$. For $\beta>0$, $t>0$ and $\alpha\in(0,1)$, the *truncated $\mathcal{M}$-series fractional derivative* of order α of a function f is

$$
\begin{aligned}
{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t) &= {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left[\begin{matrix}a_1\cdots a_p\\ c_1\cdots c_q\end{matrix};\beta,\gamma\right]f(t)\\
&:= \lim_{\varepsilon\to 0}\frac{f\big(\Gamma(\gamma)t\ {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\big)-f(t)}{\varepsilon},
\end{aligned} \tag{7}
$$

where $\alpha,\beta,\gamma\in\mathbb{R}$, $p,q\in\mathbb{N}$, $a_n,c_m\in\mathbb{R}$, $c_m\neq 0,-1,-2,\ldots(n=1,2,\ldots,p;\ m=1,2,\ldots,q)$ and ${}_i\mathcal{M}^{\beta;\gamma}_{p,q}$ is the truncated M-series given with (6). If a truncated $\mathcal{M}$-series fractional derivative of a function f exists then we called the function f is $\mathcal{M}$-*differentiable*.

Note that, if f is $\mathcal{M}$-differentiable in some interval $(0,a)$, $a>0$ and

$$\lim_{t\to 0^+} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t)$$

exists, then we define

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(0) = \lim_{t\to 0^+} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t).$$

Because Sousa and de Oliveira showed in [29] that, truncated M-fractional derivative (5) is the generalization of the fractional derivative operators (1)-(4), it is enough to choose $\gamma=p=q=1$ and $a_1=c_1$ in (7) for proving that the all the fractional derivative operators (1)-(5) given above are the special cases of our definition.

For the sake of shortness, throughout the paper we assume that $\alpha,\beta,\gamma\in\mathbb{R},\ p,q\in\mathbb{N},\ \beta>0,\ p>0,\ q>0,$ $a_n,c_m\in\mathbb{R}$ and $c_m\neq 0,-1,-2,\ldots (n=1,2,\ldots,p;\, m=1,2,\ldots,q)$. Also, we use the notation $\mathcal{K}$ instead of the constant $\frac{a_1\cdots a_p}{c_1\cdots c_q}\frac{\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$.

Now we begin our investigation with an important theorem.

Theorem 1. *If a function* $f:[0,\infty)\to\mathbb{R}$ *is* $\mathcal{M}$*-differentiable at* $t_0>0$ *for* $\alpha\in(0,1]$*, then* f *is continuous at* t_0.

Proof. Consider the identity

$$f\left(\Gamma(\gamma)t_0\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) - f(t_0) = \frac{f\left(\Gamma(\gamma)t_0\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) - f(t_0)}{\varepsilon}\varepsilon.$$

Applying the limit for $\varepsilon\to 0$ on both sides, we get

$$\begin{aligned}\lim_{\varepsilon\to 0} & f\left(\Gamma(\gamma)t_0\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) - f(t_0)\\ &= \lim_{\varepsilon\to 0}\left(\frac{f\left(\Gamma(\gamma)t_0\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) - f(t_0)}{\varepsilon}\right)\lim_{\varepsilon\to 0}\varepsilon\\ &= {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t)\lim_{\varepsilon\to 0}\varepsilon\\ &= 0.\end{aligned}$$

Then, f is continuous at t_0.

Besides, using the definition of the truncated M-series, we can write

$$f\left(\Gamma(\gamma)t\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) = f\left(\Gamma(\gamma)t\sum_{n=0}^{i}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{(\varepsilon t^{-\alpha})^k}{\Gamma(\beta k+\gamma)}\right).$$

If we apply the limit for $\varepsilon\to 0$ on both sides and since f is continuous, we get

$$\lim_{\varepsilon\to 0} f\left(\Gamma(\gamma)t\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) = f\left(\Gamma(\gamma)t\lim_{\varepsilon\to 0}\sum_{k=0}^{i}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{(\varepsilon t^{-\alpha})^k}{\Gamma(\beta k+\gamma)}\right).$$

Because

$$\lim_{\varepsilon\to 0}\sum_{k=0}^{i}\frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k}\frac{(\varepsilon t^{-\alpha})^k}{\Gamma(\beta k+\gamma)} = \frac{1}{\Gamma(\gamma)},$$

we can write

$$\lim_{\varepsilon\to 0} f\left(\Gamma(\gamma)t\, {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) = f(t).$$

The following theorem is about the basic properties of $\mathcal{M}$-series fractional derivative:

Theorem 2. *Let $\alpha \in (0,1]$, $a,b \in \mathbb{R}$ and f,g $\mathcal{M}$-differentiable functions at a point $t>0$. Then*

(a) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(af+bg)(t) = a\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t) + b\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t)$,

(b) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f\cdot g)(t) = f(t){}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t) + g(t){}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t)$,

(c) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{f}{g}\right)(t) = \dfrac{g(t){}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t) - f(t){}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t)}{[g(t)]^2}$,

(d) If f is differentiable, then

$$ {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f) = \mathcal{K}t^{1-\alpha}\frac{df(t)}{dt}, \tag{8} $$

(e) If $f'(g(t))$ exists, then

$$ {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f\circ g)(t) = f'(g(t)){}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t). $$

Proof. The proof of the first three cases are quite simple and easily obtainable by following the same way with the corresponding proofs of classical calculus. For (d): from the definition of truncated M-series we can write

$$ \begin{aligned} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t) &= \lim_{\varepsilon\to 0}\frac{f\big(\Gamma(\gamma)t\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\big) - f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f\big(\Gamma(\gamma)t(\frac{1}{\Gamma(\gamma)} + \frac{a_1\cdots a_p}{c_1\cdots c_q}\frac{\varepsilon t^{-\alpha}}{\Gamma(\beta+\gamma)} + \mathcal{O}(\varepsilon^2)\big) - f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f\big(t+\varepsilon t^{1-\alpha}(\mathcal{K}+\mathcal{O}(\varepsilon))\big) - f(t)}{\varepsilon} \end{aligned} $$

Choosing $h = \varepsilon t^{1-\alpha}\big(\mathcal{K}+\mathcal{O}(\varepsilon)\big)$ we get the result

$$ \begin{aligned} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t) =& t^{1-\alpha}\lim_{\varepsilon\to 0}\frac{f\big(t+h\big)-f(t)}{\frac{h}{\mathcal{K}+\mathcal{O}(\varepsilon)}} \\ =& \mathcal{K}t^{1-\alpha}\frac{df(t)}{dt}. \end{aligned} $$

For (e): If g is a constant function in a neighborhood of a. Then clearly ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(g(a)) = 0$. Now, assume that g is not a constant function, that is, we can find an $\varepsilon > 0$ for any $t_1,t_2 \in (a-\varepsilon, a+\varepsilon)$ such that $g(t_1) \neq g(t_2)$. Since g is continuous at a and for small enough ε, we have

$$ \begin{aligned} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(f\circ g)(a) &= \lim_{\varepsilon\to 0}\frac{f\big(g(\Gamma(\gamma)a\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon a^{-\alpha}))\big) - f\big(g(a)\big)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f\big(g(\Gamma(\gamma)a\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon a^{-\alpha}))\big) - f\big(g(a)\big)}{g(\Gamma(\gamma)a\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon a^{-\alpha})) - g(a)}\frac{g(\Gamma(\gamma)a\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon a^{-\alpha})) - g(a)}{\varepsilon} \\ &= \lim_{\varepsilon_1\to 0}\frac{f\big(g(\Gamma(\gamma)a\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon a^{-\alpha})\big) - f\big(g(a)\big)}{\varepsilon_1}.\lim_{\varepsilon\to 0}\frac{g(\Gamma(\gamma)a\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon a^{-\alpha})) - g(a)}{\varepsilon} \\ &= f'(g(a)){}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(a), \end{aligned} $$

with $a>0$.

Example 3. *Now we give the truncated $\mathcal{M}$-series fractional derivatives of some well-known functions by using the result* (8)*. Let $n \in \mathbb{R}$ and $\alpha \in (0,1]$. Then we have the following results*

(a) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(const.) = 0$,

(b) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(e^{nt}) = \mathcal{K}nt^{1-\alpha}e^{nt}$,

(c) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(\sin nt) = \mathcal{K}nt^{1-\alpha}\cos nt$,

(d) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(\cos nt) = -\mathcal{K}nt^{1-\alpha}\sin nt$,

(e) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(t^{n}) = \mathcal{K}nt^{n-\alpha}$,

(f) ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(\frac{t^{\alpha}}{\alpha}\right) = \mathcal{K}$.

Theorem 4 (Rolle's theorem)**.** *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function such that:*

(a) *f is continuous on $[a,b]$,*

(b) *f is $\mathcal{M}$-differentiable on (a,b) for some $\alpha \in (0,1)$,*

(c) $f(a) = f(b)$.

Then, there exists $c \in (a,b)$, such that ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(c) = 0$.

Proof. Let f is a continuous function on $[a,b]$ and $f(a) = f(b)$, then there exists a point $c \in (a,b)$ at which the function f has a local extreme. Then,

$$\begin{aligned} {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(c) &= \lim_{\varepsilon\to 0^-} \frac{f\left(\Gamma(\gamma)c\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) - f(c)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0^+} \frac{f\left(\Gamma(\gamma)c\,{}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha})\right) - f(c)}{\varepsilon}, \end{aligned}$$

Since

$$\lim_{\varepsilon\to 0^{\pm}} {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{-\alpha}) = \frac{1}{\Gamma(\gamma)},$$

the two limits have opposite sings. So ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(c) = 0$.

Theorem 5 (Mean value theorem)**.** *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function such that:*

(a) *f is continuous on $[a,b]$;*

(b) *f is $\mathcal{M}$-differentiable on (a,b) for some $\alpha \in (0,1)$.*

Then, there exists $c \in (a,b)$, such that

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(c) = \mathcal{K}\frac{f(b) - f(a)}{\frac{b^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}}.$$

Proof. Consider the following function:

$$g(t) = f(t) - f(a) - \left(\frac{f(b) - f(a)}{\frac{b^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}}\right)\left(\frac{t^{\alpha}}{\alpha} - \frac{a^{\alpha}}{\alpha}\right). \tag{9}$$

The function g provides the conditions of the Rolle's theorem. Then, there exists a point $c \in (a,b)$, such that ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(c) = 0$. Applying the new truncated $\mathcal{M}$-series fractional derivative on both sides of the equality (9) and using the properties (a) and (f) of Example 1, we have the result.

Theorem 6 (Extended mean value theorem)**.** *Let $f,g:[a,b]\to\mathbb{R}$, $a>0$ be two functions such that:*

(a) f,g are continuous on $[a,b]$;

(b) f,g are $\mathcal{M}$-differentiable on (a,b) for some $\alpha\in(0,1)$.

Then, there exists $c\in(a,b)$, such that:

$$\frac{{}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(c)}{{}_i\mathcal{D}_{\mathcal{M}}^{\alpha}g(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof. Consider the following function:

$$F(x)=f(t)-f(a)-\left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(t)-g(a)). \tag{10}$$

The function F provides the conditions of the Rolle's theorem. Then, there exists a point $c\in(a,b)$, such that ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}F(c)=0$. Applying the truncated $\mathcal{M}$-series fractional derivative on both sides of the equality (10) and using the property (a) of Example 1, we have the result.

Theorem 7. *Let $a>0$ and $f:[a,b]\to\mathbb{R}$ be a function such that:*

(a) f is continuous on $[a,b]$;

(b) f is $\mathcal{M}$-differentiable on (a,b) for some $\alpha\in(0,1)$.

If for all $t\in(a,b)$ ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(t)=0$, then f is a constant function on $[a,b]$.

Proof. Assume that, for all $t\in(a,b)$, ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(t)=0$, and let, $t_1,t_2\in[a,b]$, with $t_1<t_2$. Since f is also continuous in $[t_1,t_2]$ and $\mathcal{M}$-differentiable in (t_1,t_2), from Rolle's theorem, there exist a point $c\in(t_1,t_2)$ with

$${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(c)=\mathcal{K}\frac{f(t_2)-f(t_1)}{\frac{t_2^{\alpha}}{\alpha}-\frac{t_1^{\alpha}}{\alpha}}=0.$$

So, $f(t_1)=f(t_2)$. Since $t_1<t_2$ are arbitrary chosen from $[a,b]$, f has to be a constant function.

Corollary 8. *Let $a>0$ and $f,g:[a,b]\to\mathbb{R}$ be functions such that for all $\alpha\in(0,1)$ and $t\in(a,b)$,*

$${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(t)={}_i\mathcal{D}_{\mathcal{M}}^{\alpha}g(t).$$

Then, there exists a constant c such that $f(t)=g(t)+c$

Proof. Apply Theorem 7 with choosing $h(t)=f(t)-g(t)$.

Theorem 9. *Let $\mathcal{K}>0$ and $f:[a,b]\to\mathbb{R}$ be a function which continuous on $[a,b]$ and $\mathcal{M}$-differentiable on (a,b) for some $\alpha\in(0,1)$. Then, for all $t\in(a,b)$*

- *if ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(t)>0$, then f is increasing on $[a,b]$,*
- *if ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(t)<0$, then f is decreasing on $[a,b]$.*

Proof. From Theorem 7 we know that for $t_1,t_2\in[a,b]$ there exist a $c\in(t_1,t_2)$ such as

$${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(c)=\mathcal{K}\frac{f(t_2)-f(t_1)}{\frac{t_2^{\alpha}}{\alpha}-\frac{t_1^{\alpha}}{\alpha}}.$$

If ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(c)>0$ then $f(t_2)>f(t_1)$ while $t_2>t_1$, so f is increasing since t_1 and t_2 chosen arbitrary. But if ${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}f(c)<0$ then $f(t_2)>f(t_1)$ while $t_2<t_1$ (or $f(t_2)<f(t_1)$ while $t_2>t_1$), so f is decreasing.

Theorem 10. *Let $\mathcal{K} > 0$ and $f, g : [a,b] \to \mathbb{R}$ be functions which continuous on $[a,b]$, $\mathcal{M}$-differentiable on (a,b) for some $\alpha \in (0,1)$ and for all $t \in [a,b]$, ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}} f(t) \leq {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} g(t)$. Then,*

- *if $f(a) = g(a)$, then $f(t) \leq g(t)$ for all $t \in [a,b]$,*
- *if $f(b) = g(b)$, then $f(t) \geq g(t)$ for all $t \in [a,b]$.*

Proof. The proof is trivial when you consider the function $h(t) = g(t) - f(t)$.

Theorem 11. *Let $f : [0,\infty) \to \mathbb{R}$ be a two times differentiable function with $t > 0$ and $\alpha_1, \alpha_2 \in (0,1)$. Then*

$$ {}_i\mathcal{D}^{\alpha_1+\alpha_2}_{\mathcal{M}} f(t) \neq {}_i\mathcal{D}^{\alpha_1}_{\mathcal{M}} \left({}_i\mathcal{D}^{\alpha_2}_{\mathcal{M}} f \right)(t). $$

Proof. From the equality (8) we have

$$ {}_i\mathcal{D}^{\alpha_1+\alpha_2}_{\mathcal{M}} f(t) = \mathcal{K} t^{1-\alpha_1-\alpha_2} f'(t), \tag{11} $$

but for the other side we have

$$ \begin{aligned} {}_i\mathcal{D}^{\alpha_1}_{\mathcal{M}} \left({}_i\mathcal{D}^{\alpha_2}_{\mathcal{M}} f \right)(t) &= {}_i\mathcal{D}^{\alpha_1}_{\mathcal{M}} \left(\mathcal{K} t^{1-\alpha_2} f'(t) \right) \\ &= \mathcal{K}^2 t^{1-\alpha_1} \left(t^{1-\alpha_2} f'(t) \right)' \\ &= \mathcal{K}^2 t^{1-\alpha_1-\alpha_2} \left((1-\alpha_2) f'(t) + t f''(t) \right). \end{aligned} \tag{12} $$

The proof is clear from (11) and (12).

The following result is the direct consequences of the previous theorem.

Corollary 12. *Let $f : [0,\infty) \to \mathbb{R}$ be a two times differentiable function with $t > 0$ and $\alpha_1, \alpha_2 \in (0,1)$. Then*

$$ {}_i\mathcal{D}^{\alpha_1}_{\mathcal{M}} \left({}_i\mathcal{D}^{\alpha_2}_{\mathcal{M}} f \right)(t) \neq {}_i\mathcal{D}^{\alpha_2}_{\mathcal{M}} \left({}_i\mathcal{D}^{\alpha_1}_{\mathcal{M}} f \right)(t). $$

The following definition is about the $\mathcal{M}$-series fractional derivative operator for $\alpha \in (n, n+1]$, $n \in \mathbb{N}$.

Definition 3. Let $\alpha \in (n, n+1]$, $n \in \mathbb{N}$ and for $t > 0$, f be a n times differentiable function. The *truncated $\mathcal{M}$-series fractional derivative* of order α of f is given as

$$ {}_i\mathcal{D}^{\alpha;n}_{\mathcal{M}} f(t) := \lim_{\varepsilon \to 0} \frac{f^{(n)}\left(\Gamma(\gamma) t \; {}_i\mathcal{M}^{\beta,\gamma}_{p,q}(\varepsilon t^{n-\alpha}) \right) - f^{(n)}(t)}{\varepsilon}, \tag{13} $$

if and only if the limit exists.

Remark 1. For $t > 0$, $\alpha \in (n, n+1]$ and for $(n+1)$ times differentiable function f, it is easy to show that

$$ {}_i\mathcal{D}^{\alpha;n}_{\mathcal{M}} f(t) = \mathcal{K} t^{n+1-\alpha} f^{(n+1)}(t). $$

by using (13), (8) and induction on n.

3 $\mathcal{M}$-series Fractional Integral

In this section, we defined the corresponding $\mathcal{M}$-series fractional integral operator $\mathcal{I}^{\alpha}_{\mathcal{M}} f(t)$. We want that our integral operator satisfies ${}_i\mathcal{D}^{\alpha}_{\mathcal{M}} \left(\mathcal{I}^{\alpha}_{\mathcal{M}} f(t) \right) = f(t)$. Let $F(t) = \mathcal{I}^{\alpha}_{\mathcal{M}} f(t)$ be a differentiable function, then from (8) we have the following differential equation

$$ f(t) = {}_i\mathcal{D}^{\alpha}_{\mathcal{M}} (F(t)) = \mathcal{K} t^{1-\alpha} \frac{dF(t)}{dt}, $$

which have a solution of the form for $a_n \neq 0$, $(n = 1, 2, \ldots, p)$

$$ F(t) = \mathcal{K}^{-1} \int \frac{f(t)}{t^{1-\alpha}} dt. $$

This yields the following definition.

Definition 4. Let $a \geq 0$ and $t \geq a$, and f is defined in $(a,t]$. If the following improper Riemann integral exists, then for $\alpha \in (0,1)$, the α order $\mathcal{M}$-series fractional integral of a function f is defined by

$$\mathcal{I}_{\mathcal{M}}^{\alpha} f(t) := \mathcal{I}_{\mathcal{M}}^{\alpha}\left[\begin{matrix} a_1 \cdots a_p \\ c_1 \cdots c_q \end{matrix}; \beta, \gamma\right] f(t) = \mathcal{K}^{-1}\int_a^t \frac{f(t)}{t^{1-\alpha}} dt, \tag{14}$$

where the conditions are same as (7) with $a_n \neq 0,\ n = 1,2,\dots,p$.

Remark 2. It can easily seen from the definition of $\mathcal{M}$-series fractional integral that, the integral operator is linear and $\mathcal{I}_{\mathcal{M}}^{\alpha} f(a) = 0$.

For the rest of the paper we assume that $a_n \neq 0,\ n = 1,2,\dots,p$.

Theorem 13. *Let $a \geq 0,\ \alpha \in (0,1)$ and f is a continuous function such that $\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)$ exists. Then for $t \geq a$,*

$${}_i\mathcal{D}_{\mathcal{M}}^{\alpha}(\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)) = f(t).$$

Proof. Since f is continuous, $\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)$ is differentiable. Then from (8) we have

$$\begin{aligned} {}_i\mathcal{D}_{\mathcal{M}}^{\alpha}(\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)) &= \mathcal{K} t^{1-\alpha}\frac{d}{dt}\mathcal{I}_{\mathcal{M}}^{\alpha} f(t) \\ &= t^{1-\alpha}\frac{d}{dt}\left(\int_a^t \frac{f(t)}{t^{1-\alpha}} dt\right) \\ &= f(t), \end{aligned}$$

which completes the proof.

Theorem 14. *Let $f : (a,b) \to \mathbb{R}$ be a differentiable function and $\alpha \in (0,1]$. Then, for all $t > a$, we have*

$$\mathcal{I}_{\mathcal{M}}^{\alpha}({}_i\mathcal{D}_{\mathcal{M}}^{\alpha} f(t)) = f(t) - f(a).$$

Proof. Since the function f is differentiable, by using the fundamental theorem of calculus for the integer-order derivatives and (8), we get

$$\begin{aligned} \mathcal{I}_{\mathcal{M}}^{\alpha}({}_i\mathcal{D}_{\mathcal{M}}^{\alpha} f(t)) &= \mathcal{K}^{-1}\int_a^t \frac{{}_i\mathcal{D}_{\mathcal{M}}^{\alpha} f(t)}{t^{1-\alpha}} dx \\ &= \int_a^t \frac{df(t)}{dt} dx \\ &= f(t) - f(a), \end{aligned}$$

which gives the result.

Remark 3. If $f(a) = 0$ then $\mathcal{I}_{\mathcal{M}}^{\alpha}\left({}_i\mathcal{D}_{\mathcal{M}}^{\alpha} f(t)\right) = {}_i\mathcal{D}_{\mathcal{M}}^{\alpha}\left(\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)\right) = f(t)$.

Theorem 15. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function with $0 < a < b$ and $\alpha \in (0,1)$. Then for $\mathcal{K} > 0$ we have*

$$|\mathcal{I}_{\mathcal{M}}^{\alpha} f|(t) \leq \mathcal{I}_{\mathcal{M}}^{\alpha}|f|(t).$$

Proof. From the definition of $\mathcal{M}$-series fractional integral we have

$$\begin{aligned} |\mathcal{I}_{\mathcal{M}}^{\alpha} f(t)| &= \left|\mathcal{K}^{-1}\int_a^t \frac{f(x)}{x^{1-\alpha}} dx\right| \\ &\leq |\mathcal{K}^{-1}|\left|\int_a^t \frac{f(x)}{x^{1-\alpha}} dx\right| \\ &\leq \mathcal{K}^{-1}\int_a^t \left|\frac{f(x)}{x^{1-\alpha}}\right| dx \\ &= \mathcal{K}^{-1}\int_a^t \frac{|f(x)|}{x^{1-\alpha}} dx, \end{aligned}$$

which completes the proof.

Corollary 16. *Let $f:[a,b]\to\mathbb{R}$ be a continuous function such that*

$$N=\sup_{t\in[a,b]}|f(t)|.$$

Then, for all $t\in[a,b]$ with $0<a<b$, $\alpha\in(0,1)$ and $\mathcal{K}>0$ we have

$$|\mathcal{I}^{\alpha}_{\mathcal{M}}f(t)|\le\mathcal{K}^{-1}N\left(\frac{t^{\alpha}}{\alpha}-\frac{a^{\alpha}}{\alpha}\right).$$

Proof. From the previous theorem we have

$$\begin{aligned}|\mathcal{I}^{\alpha}_{\mathcal{M}}f|(t)&\le\mathcal{I}^{\alpha}_{\mathcal{M}}|f|(t)\\&=\mathcal{K}^{-1}\int_a^t\frac{|f(x)|}{x^{1-\alpha}}dx\\&=\mathcal{K}^{-1}N\int_a^t x^{\alpha-1}dx,\end{aligned}$$

which gives the result.

Theorem 17. *Let $f,g:[a,b]\to\mathbb{R}$ be two differentiable functions and $\alpha\in(0,1)$. Then*

$$\int_a^b f(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t)d_{\alpha}t=f(t)g(t)\Big|_a^b-\int_a^b g(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t)d_{\alpha}t,$$

where $d_{\alpha}t=\mathcal{K}^{-1}t^{\alpha-1}dt$.

Proof. Using the definition of $\mathcal{M}$-series fractional integral (14), (8) and applying fundamental theorem of calculus for integer-order derivatives, we get

$$\begin{aligned}\int_a^b f(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t)d_{\alpha}t&=\mathcal{K}^{-1}\int_a^b\frac{f(t)}{t^{1-\alpha}}\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}g(t)dt\\&=\int_a^b f(t)\frac{dg(t)}{dt}dt\\&=f(t)g(t)\Big|_a^b-\int_a^b g(t)\frac{df(t)}{dt}dt\\&=f(t)g(t)\Big|_a^b-\int_a^b g(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f(t)d_{\alpha}t,\end{aligned}$$

which completes the proof.

Now we define the $\mathcal{M}$-series fractional integral for $\alpha\in(n,n+1]$ as follows.

Definition 5. Let $a\ge0$ and $t\ge a$, and f is defined in $(a,t]$. If the following improper Riemann integral exists, then for $\alpha\in(n,n+1)$, the α order $\mathcal{M}$-series fractional integral of a function f is defined by

$$\mathcal{I}^{\alpha;n}_{\mathcal{M}}f(t):=\mathcal{I}^{\alpha;n}_{\mathcal{M}}\left[\begin{matrix}a_1\cdots a_p\\c_1\cdots c_q\end{matrix};\beta,\gamma\right]f(t)=\mathcal{K}^{-1}\underbrace{\int_a^t dt\int_a^t dt\cdots\int_a^t}_{n+1\text{ times}}\frac{f(t)}{t^{n+1-\alpha}}dt, \tag{15}$$

where the conditions are same as (7) with $a_n\neq0,\ n=1,2,\dots,p$.

The following theorem is a generalization of Theorem 14.

Theorem 18. *Let $\alpha \in (n, n+1]$ and $f : [a, \infty) \to \mathbb{R}$ be $(n+1)$ times differentiable function for $t > a$. Then we have*

$$\mathcal{I}_{\mathcal{M}}^{\alpha;n}\left({}_i\mathcal{D}_{\mathcal{M}}^{\alpha;n} f\right)(t) = f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)(t-a)^k}{k!}.$$

Proof. From (7) and (15) we have

$$\mathcal{I}_{\mathcal{M}}^{\alpha;n}\left({}_i\mathcal{D}_{\mathcal{M}}^{\alpha;n} f\right)(t) = \mathcal{K}^{-1} \underbrace{\int_a^t dt \int_a^t dt \cdots \int_a^t}_{n+1 \text{ times}} \frac{{}_i\mathcal{D}_{\mathcal{M}}^{\alpha;n} f(t)}{t^{n+1-\alpha}} dt$$
$$= \underbrace{\int_a^t dt \int_a^t dt \cdots \int_a^t}_{n+1 \text{ times}} f^{(n+1)}(t) dt,$$

which gives the result.

4 Applications to $\mathcal{M}$-series Fractional Differential Equations

In this section, we obtained the general solutions of linear fractional differential equations including the $\mathcal{M}$-series fractional derivative operator.

Example 19. *Let $u = u(t)$ is a $\mathcal{M}$-differentiable function and assume that for $\alpha \in (0, 1]$ the linear $\mathcal{M}$-series fractional differential equation*

$${}_i\mathcal{D}_{\mathcal{M}}^{\alpha} u(t) + p(t)u(t) = q(t) \tag{16}$$

is given. If u is also a differentiable function then by using (8), we get a linear ordinary differential equation

$$\frac{du(t)}{dt} + \mathcal{K}^{-1} t^{\alpha-1} p(t)u(t) = \mathcal{K}^{-1} t^{\alpha-1} q(t).$$

The integrating factor of the equation can be found as $\mu(t) = e^{\mathcal{K}\int t^{\alpha-1} p(t) dt}$, which yields the solution as

$$u(t) = e^{-\mathcal{K}^{-1}\int \frac{p(t)}{t^{1-\alpha}} dt} \left[\mathcal{K}^{-1} \int \frac{q(t)}{t^{1-\alpha}} e^{\mathcal{K}^{-1}\int \frac{p(t)}{t^{1-\alpha}} dt} dt + C\right],$$

where C is a constant. By definition of the $\mathcal{M}$-series integral operator we can write the last equality as

$$u(t) = e^{-\mathcal{I}_{\mathcal{M}}^{\alpha} p(t)} \left[\mathcal{I}_{\mathcal{M}}^{\alpha}\left(q(t) e^{\mathcal{I}_{\mathcal{M}}^{\alpha} p(t)}\right) + C\right]. \tag{17}$$

If we choose $p(t) = -\lambda,\ q(t) = 0$, then the linear $\mathcal{M}$-series fractional differential equation (16) turns to

$${}_i\mathcal{D}_{\mathcal{M}}^{\alpha} u(t) = \lambda u(t),$$

and the general solution can be found from (17) as

$$u(t) = Ce^{-\mathcal{K}^{-1}\frac{\lambda}{\alpha} t^{\alpha}}.$$

Since $e^t = {}_{\infty}\mathcal{M}_{1,1}^{1,1}(t)$, we can write the solution by means of truncated $\mathcal{M}$-series as

$$u(t) = C_{\infty}\mathcal{M}_{1,1}^{1,1}\left(-\mathcal{K}^{-1}\frac{\lambda}{\alpha} t^{\alpha}\right). \tag{18}$$

For the fixed values $a_n = 1,\ c_m = 1,\ (n = 1, 2, \ldots, p;\ m = 1, 2, \ldots, q)$, this result coincides with the results given in [27] when $\lambda = 1$ and coincides with the corresponding integer-order result when $\alpha = \beta = \lambda = 1$.

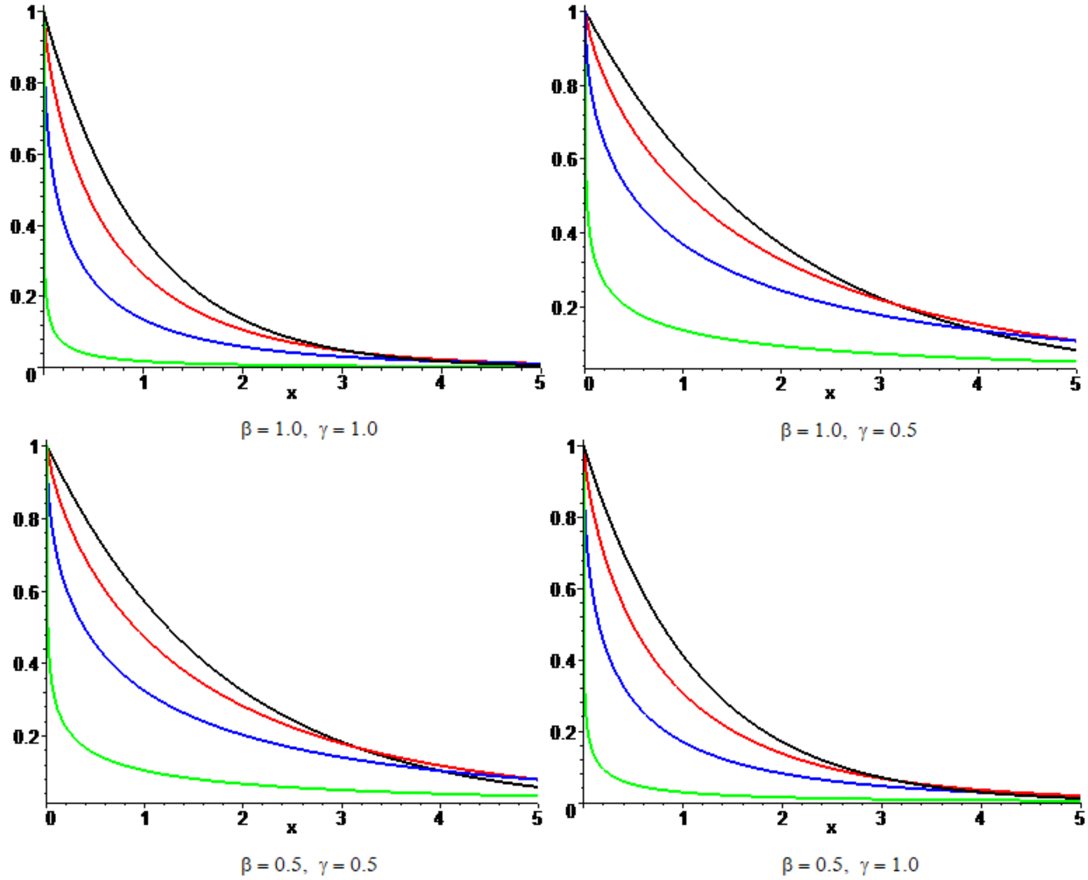

Fig. 1 The graphs of (18) from $\alpha = 0.25$ (green) to $\alpha = 1.00$ (black) by step size 0.25.

In the following, the reader can find the graphs of the solution function (18) for different α, β and γ values with the fixed values $C = \lambda = 1$ and $a_n = 1$, $c_m = 1$, $(n = 1, 2, \ldots, p;\ m = 1, 2, \ldots, q)$.

Example 20. *Consider the heat equation in one dimension*

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < L,\ t > 0, \tag{19}$$

with the initial and boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0,\ u(x,0) = f(x), \quad t \geq 0,\ 0 \leq x \leq L.$$

Here $\frac{\partial^\alpha}{\partial t^\alpha} = {}_i\mathcal{D}^\alpha_{\mathcal{M}}$, $u = u(x,t)$ *is a* $\mathcal{M}$*-differentiable function,* $\alpha \in (0,1]$ *and* k *is a positive constant. Suppose that* $u(x,t) = P(x)Q(t)$*. Using separation of variables method we get a system of differential equations*

$$\frac{d^\alpha}{dt^\alpha} Q(t) - k\xi Q(t) = 0,$$
$$\frac{d^2}{dx^2} P(x) - \xi P(x) = 0.$$

From the above example and the ordinary differential equations theory, we know that these equations have solutions of the form

$$Q_n(t) = e^{-\mathcal{K}^{-1}\left(\frac{n\pi}{L}\right)^2 \frac{k}{\alpha} t^\alpha}, \quad n = 1, 2, 3, \ldots$$
$$P_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \ldots$$

So, the formal solution of the heat equation (19) *is*

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\mathcal{K}^{-1}\left(\frac{n\pi}{L}\right)^2 \frac{k}{\alpha} t^{\alpha}}, \tag{20}$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

Let us fixed the values $a_n = 1,\ c_m = 1,\ (n = 1,2,\dots,p;\ m = 1,2,\dots,q)$ *in* (20). *Choosing* $\gamma = 1$ *yields us the same result in [29];* $\gamma = \beta = 1$ *yields us the same result in [7], and* $\alpha = \beta = \gamma = 1$ *yields us the same result with the integer-order heat equation.*

If we choose $f(x) = \sin(x),\ L = \pi,\ k = 1$ *in* (19) *we have*

$$u(x,t) = \frac{2}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi}\sin(x)\sin(nx)dx\sin(nx)e^{-\mathcal{K}^{-1}\frac{n^2}{\alpha}t^{\alpha}},$$

which differ from 0 only for $n = 1$. *So, the solution of the problem is*

$$u(x,t) = \sin(x)e^{-\mathcal{K}^{-1}\frac{t^{\alpha}}{\alpha}}. \tag{21}$$

This result is the same as the corresponding integer-order problem when $a_n = 1,\ c_m = 1,\ (n = 1,2,\dots,p;\ m = 1,2,\dots,q)$ *and* $\alpha = \beta = \gamma = 1$.

In the following, the reader can find the graphs which obtained by (21), for different values of α, β and γ with the fixed values $a_n = 1$, $c_m = 1$, $(n = 1,2,\dots,p;\ m = 1,2,\dots,q)$.

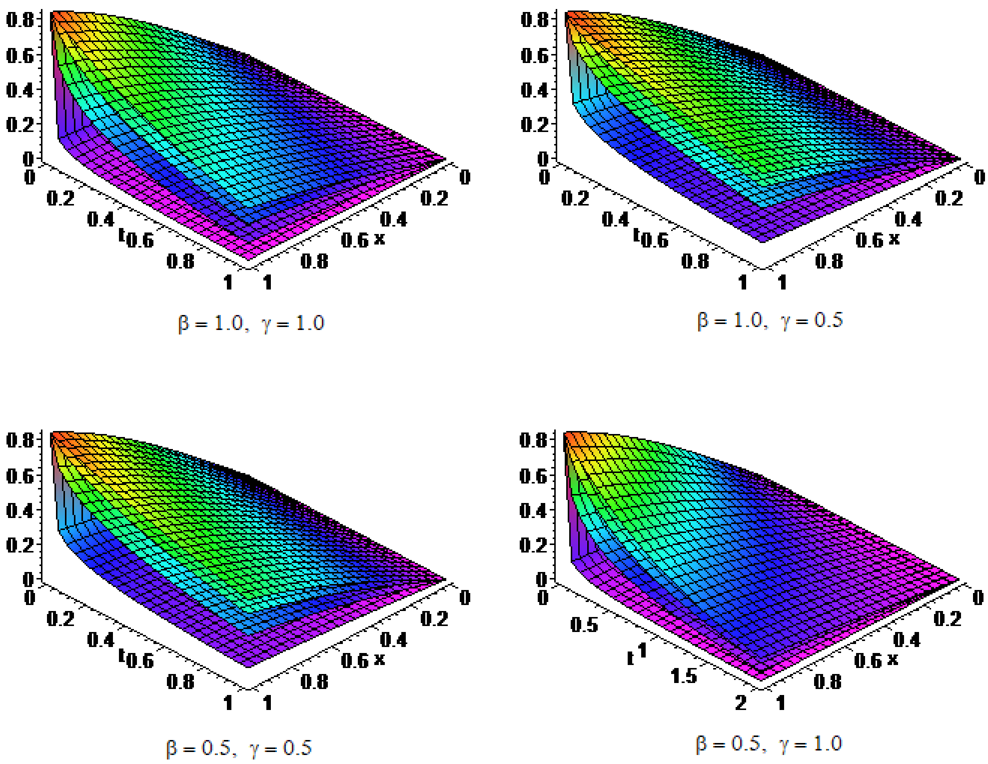

Fig. 2 The graphs of (21) from $\alpha = 0.25$ (bottom) to $\alpha = 1.00$ (top) by step size 0.25.

Example 21. *Let $f:[0,\infty)\to\mathbb{R}$, $t>a>0$. Consider the following $\mathcal{M}$-series fractional differential equation*

$$ {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f\right)+p(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f+q(t)f=0 \tag{22}$$

where p and q are $\mathcal{M}$-differentiable functions of t. Assume that (22) has a solution, say f_1. To find the second linearly independent solutions of (22), we start by assuming that $f_2(t)=v(t)f_1(t)$ where v is an $\mathcal{M}$-differentiable function. So, from the chain rule, we have

$$\begin{aligned}
{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_2(t) &= {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}(vf_1)(t)=v(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)+f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t),\\
{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_2\right)(t) &= {}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left(v(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)+f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t)\right)\\
&= v(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1\right)(t)+{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t)+f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v\right)(t)+{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t).
\end{aligned}$$

Substituting these in (22) and remembering that f_1 is a solution of it, we get

$$f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}\left({}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v\right)(t)+2\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t)+p(t)f_1(t)\,{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t)=0.$$

Now, if we let $w(t)={}_i\mathcal{D}^{\alpha}_{\mathcal{M}}v(t)$, then it becomes

$${}_i\mathcal{D}^{\alpha}_{\mathcal{M}}w(t)+\left(p(t)+2\frac{{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)}{f_1(t)}\right)w(t)=0.$$

From Example 19, the solution of this equation can be found as

$$w(t)=Ce^{-\mathcal{I}^{\alpha}_{\mathcal{M}}\left(p(t)+2\frac{{}_i\mathcal{D}^{\alpha}_{\mathcal{M}}f_1(t)}{f_1(t)}\right)}=C\frac{e^{-\mathcal{I}^{\alpha}_{\mathcal{M}}p(t)}}{f_1^2(t)},\quad (C\in\mathbb{R}),$$

which yields

$$v(t)=C\mathcal{I}^{\alpha}_{\mathcal{M}}\left(\frac{e^{-\mathcal{I}^{\alpha}_{\mathcal{M}}p}}{f_1^2(t)}\right).$$

Then we find the second solution as

$$f_2(t)=Cf_1(t)\mathcal{I}^{\alpha}_{\mathcal{M}}\left(\frac{e^{-\mathcal{I}^{\alpha}_{\mathcal{M}}p}}{f_1^2(t)}\right). \tag{23}$$

Example 22. *Consider the following differential equation for $f:[0,\infty)\to\mathbb{R}$, $t>a>0$:*

$${}_i\mathcal{D}^{\frac{2}{3}}_{\mathcal{M}}\,{}_i\mathcal{D}^{\frac{2}{3}}_{\mathcal{M}}f-t^{\frac{1}{3}}\;{}_i\mathcal{D}^{\frac{2}{3}}_{\mathcal{M}}f=0.$$

Clearly, $f_1(t)=1$ is a solution of this equation and $p(t)=-t^{\frac{1}{3}}$. Using formula (23) we obtain the second solution as

$$f_2(t)=C\mathcal{I}^{\frac{2}{3}}_{\mathcal{M}}\left(e^{\mathcal{I}^{\frac{2}{3}}_{\mathcal{M}}(t^{\frac{1}{3}})}\right).$$

The $\mathcal{M}$-series fractional integral

$$\mathcal{I}^{\frac{2}{3}}_{\mathcal{M}}(t^{\frac{1}{3}})=\mathcal{K}^{-1}(t-a),$$

can be found by using the definition of $\mathcal{I}^{\frac{2}{3}}_{\mathcal{M}}$. From here we get,

$$f_2(t)=C\mathcal{I}^{\frac{2}{3}}_{\mathcal{M}}\left(e^{\mathcal{K}^{-1}(t-a)}\right)=C\mathcal{K}^{-1}e^{-\mathcal{K}^{-1}a}\int_a^t x^{-\frac{1}{3}}e^{\mathcal{K}^{-1}x}dx.$$

With transformation $u = -\mathcal{K}^{-1}x$ *we have,*

$$f_2(t) = C\mathcal{K}^{-\frac{1}{3}}e^{-\mathcal{K}^{-1}a}\left[\int_{-\mathcal{K}^{-1}a}^{-\mathcal{K}^{-1}t} u^{-\frac{1}{3}}e^{-u}du\right]$$
$$= C\mathcal{K}^{-\frac{1}{3}}e^{-\mathcal{K}^{-1}a}\left[\int_{-\mathcal{K}^{-1}a}^{\infty} u^{-\frac{1}{3}}e^{-u}du - \int_{-\mathcal{K}^{-1}t}^{\infty} u^{-\frac{1}{3}}e^{-u}du\right], \ (t > a > 0)$$
$$= C\mathcal{K}^{-\frac{1}{3}}e^{-\mathcal{K}^{-1}a}\left[\Gamma(2/3, -\mathcal{K}^{-1}a) - \Gamma(2/3, -\mathcal{K}^{-1}t)\right],$$

where γ *is the incomplete gamma function which defined as*

$$\Gamma(\delta, \nu) = \int_{\nu}^{\infty} t^{\delta-1}e^{-t}dt,$$

for $\delta > 0$. *From here we get the solution as*

$$f(t) = 1 + C\mathcal{K}^{-\frac{1}{3}}e^{-\mathcal{K}^{-1}a}\left[\Gamma(2/3, -\mathcal{K}^{-1}a) - \Gamma(2/3, -\mathcal{K}^{-1}t)\right]. \tag{24}$$

For the fixed values $a_n = 1$, $c_m = 1$, $(n = 1, 2, \ldots, p;\ m = 1, 2, \ldots, q)$, *this result coincides with the results given in [15] when* $c = \beta = \gamma = 1$.

In the following, we plotted the graphs of solution function which obtained by (24), for different values of β and γ with the fixed values $a = 0$, $c = 1$, $\alpha = \frac{2}{3}$ and $\frac{c_1 \cdots c_q}{a_1 \cdots a_p} = -1$.

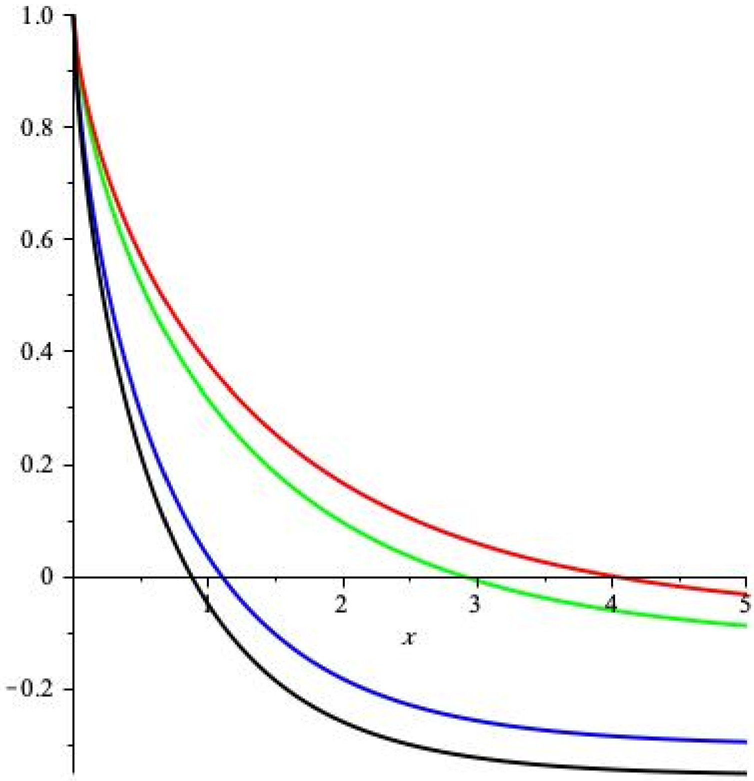

Fig. 3 The graphs of (24) for the values $\beta = \gamma = 1$ (black); $\beta = 0.5, \gamma = 1$ (blue); $\beta = 1, \gamma = 0.5$ (red) and $\beta = \gamma = 0.5$ (green).

5 Concluding Remarks and Observations

In this paper, we first presented a fractional derivative operator, which is also a generalization of truncated M-fractional derivative, by using generalized M-series. Then we gave a definition of corresponding integral operator. Unlike fractional operators with different kernels, we showed that there are many common properties provided by both these and the corresponding integer-order operators. We also used these operators in differential equation problems as application and we plotted the graphs of the solutions for various values of α, β and γ. These problems are hard to solve by means of the classical definitions of fractional derivatives.

Besides, from equality (e) of Example 1, we observed that, for polynomials, truncated $\mathcal{M}$-series fractional derivative coincides with the Riemann-Liouville and Caputo fractional derivatives [20] up to a constant multiple. In this case, we can say that the truncated $\mathcal{M}$-series fractional derivative operator can be used instead of Riemann-Liouville or Caputo type derivatives (and also their generalizations) to solve some difficult problems.

Our definition is also a generalization of the $\mathcal{V}$-fractional derivative for $p = q = 1$ which defined in [28]. It is also possible to define new fractional derivatives by using other special functions instead of M-series. Since M-series is a general class of special functions, all future definitions have chance to be the special cases of our definition. Further properties and applications of $\mathcal{M}$-series fractional operators will be discussed in forthcoming papers.

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