CENTRALIZING MAPS IN PRIME RINGS WITH INVOLUTION

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DMS–586–IR September, 1991

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1991 *Mathematics Subject Classification*: 16N60, 16W10.

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[1]The third author was partially supported by NSERC of Canada.

ABSTRACT

Let R be a prime ring with involution, of characteristic $\neq 2$, with center Z, skew elements K, and extended centroid C. <u>THEOREM</u>. Suppose $[K,K] \neq \{0\}$ and $f : K \longrightarrow K$ is an additive map such that $[f(x),x] \in Z$ for all $x \in K$. Then, unless R is an order in a 16–dimensional central simple algebra, there exists $\lambda \in C$ and an additive map $\mu : K \longrightarrow C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in K$.

1. INTRODUCTION

Let R be a ring with center Z, and let A be a subset of R. A map $f : A \longrightarrow R$ is said to be *centralizing* if $[f(x),x] \in Z$ for all $x \in A$. In the special case where $[f(x),x] = 0$ for all $x \in A$, f is called *commuting*. The study of centralizing maps was initiated by a well-known theorem of Posner [13] which states that the existence of a nonzero commuting derivation in a prime ring implies that R is commutative. An analogous result for centralizing automorphisms on prime rings was obtained by Mayne [12]. A number of authors have extended these theorems of Posner and Mayne; they have showed that derivations, automorphisms, and some related maps cannot be centralizing on certain subsets of noncommutative prime (and some other) rings. For these results we refer the reader to ([1], [6], [3]) where further references can be found.

In [4] the description of all centralizing additive maps of a prime ring R of characteristic not 2 was given and subsequently in [0] the characterization for semiprime rings of characteristic not 2 was given. It was shown that every such map f is of the form $f(x) = \lambda x + \mu(x)$ where $\lambda \in C$, the extended centroid of R, and μ is an additive map of R into C (see also [3] where similar results for some other rings are presented). In [5], using similar methods (although the proof is more complicated), commuting traces of biadditive mappings in prime rings satisfying some additional conditions were characterized. We remark that this characterization was the main tool in removing the requirement of orthogonal idempotents in the determination of Lie isomorphisms [8] and Lie derivations [7] in prime rings.

The main purpose of this paper is to describe the structure of additive centralizing maps of the skew elements in a prime ring with involution. In the case of derivations this has been done by Lanski [6]. If R is a prime ring with involution $*$ it is well-known that $*$ extends in a natural way to an involution on the central closure RC. The skew elements of R will be denoted by K; K is closed under the Lie bracket $[x,y] = xy - yx$. A rough statement of our main result (Theorem 6.4) is as follows:

> Let R be a prime ring with involution, of characteristic not 2, with $[K,K] \neq \{0\}$, and let $f : K \longrightarrow K$ be an additive centralizing map. Then, unless R is an order in a certain 16-dimensional central simple algebra, there exists $\lambda \in C$ and $\mu : K \longrightarrow C$ such that $f(x) = \lambda x + \mu(x)$ for all $x \in K$.

In proving the theorem our approach is to reduce the problem, using the theory of generalized polynomial identities (GPI's) to the case of matrix rings where specific calculations can be made. In section 2 we elaborate on the definitions of centralizing and commuting maps, give some background on the structure of K, and recall various results from GPI theory needed in the sequel. The theory of biderivations in section 3 is used in several places; in particular the case of involutions of the second kind reduces to theorem 3.3. For the case where $*$ is of the first kind and $f : K \longrightarrow K$ is commuting, the result is taken care of by Theorems 4.3 and 4.5 of section 4, and more generally for GPI rings by Theorem 5.5 in section 5. In section 6 the main theorem (Theorem 6.4) is obtained, and we also present, in what may be of independent interest, a result (Theorem 6.5) in which under rather general conditions it is shown that f centralizing implies f commuting.

2. PRELIMINARIES

Let R be a ring with center Z, and let A be an additive subgroup of R. An additive map $f : A \longrightarrow A$ is said to be *centralizing* if

(1) $$[f(x),x] \in Z \quad \text{for all} \quad x \in A,$$

where $[x,y]$ denotes the Lie bracket operation $xy - yx$. Linearization of (1) immediately yields

(2) $$[f(x),y] + [f(y),x] \in Z \quad \text{for all} \quad x,y \in A.$$

An additive map $f : A \longrightarrow A$ is said to be *commuting* if

(3) $$[f(x),x] = 0 \quad \text{for all} \quad x \in A.$$

Linearization of (3) then gives

(4) $$[f(x),y] = [x,f(y)] \quad \text{for all} \quad x,y \in A.$$

Throughout this paper we are assuming that the characteristic of R is not 2, *i.e.* for all $x \in R$, $2x = 0$ implies $x = 0$. Thus, under this assumption, it is clear that (1) is equivalent to (2) and (3) is equivalent to (4).

We now proceed to give a brief resumé of various concepts and results concerning the Lie theory of prime rings with involution, together with GPI theory, which we will need in our study of centralizing and commuting maps.

Many of our remarks can be found in [11], and we recommend this source for further details.

Let R be a prime ring with involution $*$, and with center Z. The subset of skew elements $K = \{x \in R \mid x^* = -x\}$ is closed under addition and the Lie bracket, $[x,y] = xy - yx$. The involution $*$ induces an involution on the *extended centroid* C in a natural way and can be extended to an involution of the *central closure* RC. In general a prime ring R is *closed over a field* F if F is both the centroid and the extended centroid of R, or, equivalently, R is an algebra over F where F is the extended centroid. In particular the central closure RC is closed over C.

We let C_* denote the subfield of symmetric elements of C and $C_{\#}$ the set of skew elements of C and note that $C = C_* \oplus C_{\#}$. We say that the involution $*$ on R is *of the first kind* if $C = C_*$, and of the *second kind* if $C_{\#} \neq \{0\}$. If $*$ is of the first kind then the skew elements of RC are just CK. If $*$ is of the second kind it is easy to see that $CK = RC$.

For simplifying the structure of R and also for distinguishing between different types of involutions it is useful to have on hand the notion of the *super $*$–closure* of R (see [11], p. 27). If F is the algebraic closure of the field C_*, then $\tilde{R} = RC_* \otimes_{C_*} F$ is called the *super $*$–closure* of R. The involution on $\tilde{R}$ is given by $(r\alpha \otimes \beta)^* = r^*\alpha \otimes \beta$, $r \in R$, $\alpha \in C_*$, $\beta \in F$. We will be only interested in $\tilde{R}$ when $*$ is of the first kind, in which case it is known that $\tilde{R}$ is a closed prime algebra over F.

Now let $*$ be an involution of the first kind of the matrix ring $R = M_n(F)$ where F is an algebraically closed field. It is a classical and well–known result that there is a set of matrix units $\{e_{ij}\}$ in R relative to which either $*$ is ordinary transpose or $*$ is symplectic (*i.e.* $n = 2m$,

$*$ = transpose followed by the inner automorphism determined by the element $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, I the $m \times m$ identity matrix).

The following remark is a consequence of ([11], Lemma 5.5).

Remark 2.1. $[K,K] = \{0\}$ if and only if R is commutative or $\tilde{R} = M_2(F)$ under transpose.

As a consequence of ([11], Theorem 3.3 and Lemma 5.2(d)) we have

Theorem 2.2 *If* $[K,K] \neq \{0\}$ *and* $a \in R$ *is such that* $[K,a] = \{0\}$, *then* $a \in Z$.

At this point we make an easily proved but useful observation concerning commuting maps.

Remark 2.3. Let R be a closed prime algebra over F with involution of the first kind, with $[K,K] \neq \{0\}$, and let $f : K \longrightarrow K$ be an additive commuting map. Then f is F–linear.

Proof. K is an F–space, and using (4) we see that $[f(\lambda x),y] = [\lambda x,f(y)] = \lambda[x,f(y)] = \lambda[f(x),y] = [\lambda f(x),y]$ for all $\lambda \in F$, $x,y \in K$. Therefore $f(\lambda x) - \lambda f(x) = 0$ by Theorem 2.2.

For the moment we let R be any prime ring (with or without involution) and form the free product $RC_C\langle T\rangle$ over C of the central closure RC and the free algebra $C\langle T\rangle$ on a set T of indeterminates. An additive subgroup A of RC is said to satisfy a *generalized polynomial identity* over C

(briefly *A is GPI over C*) if there is a non-zero element $\varphi(t_1,t_2,\cdots,t_n)$ of $RC_C\langle T\rangle$ such that $\varphi(a_1,\cdots,a_n) = 0$ for all $a_i \in A$. A key lemma in the theory is

Lemma 2.4 ([9], Theorem 1). If $a,b \in RC$ are such that $axb = bxa$ for all $x \in R$, then a and b are C-dependent.

Closed prime GPI rings are characterized as follows.

Theorem 2.5 ([9], Theorem 3). *Let* R *be a closed prime* GPI *ring over a field* F. *Then* R *is primitive (acting densely on a vector space* V *over a division ring* D), R *contains nonzero transformations of finite rank (i.e.* R *has a non-zero socle,* H) *and* $(D:F) < \infty$. *Furthermore, if* F *is algebraically closed,* $D = F$.

We return now to the situation where R is a prime ring with involution. An important result for our purpose is the following corollary of ([10], Theorem 4.9):

Theorem 2.6. *If* K *is* GPI *over* C *then* R *is* GPI *over* C.

If K is GPI then Theorems 2.5 and 2.6 together show that the socle H of RC is nonzero, and the following result serves as a useful link between prime rings with nonzero socle and matrix rings.

Theorem 2.7 ([11], Corollary 2.9). *Let* R *be a primitive ring with involution and with nonzero socle* H. *If* $h_1,\cdots,h_m \in H$ *and if* k *is a* *positive integer* $\leq (V:D)$ *there exists a symmetric idempotent* e *in* H *such that* $h_1, \cdots, h_m \in eRe$ *and* $\text{rank } e \geq k$.

3. BIDERIVATIONS OF PRIME RINGS

Let R be a ring. A biadditive map $B : R \times R \longrightarrow R$ is called a *biderivation* if for every $x \in R$ the map $y \longrightarrow B(x,y)$ is a derivation of R, and for every $y \in R$ the map $x \longrightarrow B(x,y)$ is a derivation of R (see [14] where biderivations satisfying some special properties are studied). Typical examples are mappings of the form $(x,y) \longrightarrow c[x,y]$ where c is an element of the center of R. It is our aim to show that in noncommutative prime rings these obvious examples are essentially the only examples.

The notion of biderivation arise naturally in the study of additive commuting maps. Namely the linearization

$$(4) \quad [f(x),y] = [x,f(y)]$$

of an additive commuting map f implies that the map $B : R \times R \longrightarrow R$ given by $B(x,y) = [f(x),y]$ is a biderivation.

A special case of the following lemma, where B was a biderivation obtained from an additive commuting map f, was proved in [3].

LEMMA 3.1. *Let* R *be a ring and* $B : R \times R \longrightarrow R$ *a biderivation. Then*

$$B(x,y)z[u,v] = [x,y]zB(u,v) \quad \textit{for all} \quad x,y,z,u,v \in R.$$

Proof. We compute $B(xu,yv)$ in two different ways. Using the fact that B is a derivation in the first argument, we get

$$B(xu,yv) = B(x,yv)u + xB(u,yv). \tag{5}$$

Since B is a derivation in the second argument, it follows from (5) that

$$B(xu,yv) = B(x,y)vu + yB(x,v)u + xB(u,y)v + xyB(u,v).$$

Analogously, we obtain $B(xu,yv) = B(xu,y)v + yB(xu,v) = B(x,y)uv + xB(u,y)v + yB(x,v)u + yxB(u,v)$. Comparing the relations so obtained for $B(xu,yv)$ we arrive at

$$B(x,y)[u,v] = [x,y]B(u,v) \quad \text{for} \quad x,y,u,v \in R.$$

Replacing u by zu and using the relations

$$[zu,v] = [z,v]u + z[u,v], \qquad B(zu,v) = B(z,v)u + zB(u,v)$$

we obtain the assertion of the lemma.

Our next result is a slight generalization of [2], Lemma. Fortunately, the same proof works, but we include it for the sake of completeness.

Lemma 3.2. *Let* S *be any set and* R *be a prime ring. If functions* $F : S \to R$, $G : S \to R$ *satisfy* $F(s)xG(t) = G(s)xF(t)$ *for all* $s,t \in S$, $x \in R$, *and* $F \not\equiv 0$, *then there exists* $\lambda \in C$, *the extended centroid of* R, *such that* $G(s) = \lambda F(s)$ *for all* $s \in S$.

Proof. Given $s \in S$ we have $F(s)xG(s) = G(s)xF(s)$ for all $x \in R$. If $F(s) \neq 0$, Lemma 2.4 implies that $G(s) = \lambda(s)F(s)$ for some $\lambda(s) \in C$. Thus if $s,t \in S$ are such that $F(s) \neq 0$ and $F(t) \neq 0$, the relation $F(s)xG(t) = G(s)xF(t)$ can be written in the form $(\lambda(t)-\lambda(s))F(s)xF(t) = 0$. The primeness of R yields $\lambda(s) = \lambda(t)$. Therefore there exists $\lambda \in C$ such that $G(s) = \lambda F(s)$ for all $s \in S$ such that $F(s) \neq 0$. However, if $F(s) = 0$, then we see from the relation $F(s)xG(t) = G(s)xF(t)$ that $G(s) = 0$ since R is prime and $F \neq 0$. Thus $G(s) = \lambda F(s)$ for all $s \in S$.

We are now in a position to prove

Theorem 3.3. *Let* R *be a noncommutative prime ring and let* $B : R \times R \longrightarrow R$ *be a biderivation. Then there exists* $\lambda \in C$ *such that* $B(x,y) = \lambda[x,y]$ *for all* $x,y \in R$.

Proof. Let $S = R \times R$ and define $A : S \longrightarrow R$ by $A(x,y) = [x,y]$; $A \neq 0$ since R is noncommutative. According to Lemma 3.1 the functions $A,B : S \longrightarrow R$ satisfy all the requirements of Lemma 3.2. Hence the result follows.

As a consequence of Theorem 3.3 we get the principal result of [4].

Corollary 3.4. *Let* R *be a prime ring. If* $f : R \longrightarrow R$ *is an additive commuting map, then there exists* $\lambda \in C$ *and an additive map* $\mu : R \longrightarrow C$ *such that* $f(x) = \lambda x + \mu(x)$ *for all* $x \in R$.

Proof. Linearizing $[f(x),x] = 0$ we see that the map $(x,y) \longrightarrow [f(x),y]$ is a biderivation. Clearly we may assume R is noncommutative. Therefore by Theorem 3.3 there exists $\lambda \in C$ such that $[f(x),y] = [\lambda x,y]$ for all $x,y \in R$. Hence we see that for any $x \in R$ the element $\mu(x) = f(x) - \lambda x \in C$, and the proof is complete.

4. THE MATRIX RING CASE

In this section we determine all additive commuting maps $f : K \longrightarrow K$ in the case where $R = M_n(F)$, F a field under ordinary transpose or symplectic involution. Since these involutions are of the first kind we know from Remark 2.3 that f is necessarily F–linear unless $n = 2$ and $*$ is transpose.

We first consider $M_n(F)$ under transpose. If $\{e_{ij}\}$ are the usual matrix units for $M_n(F)$ we set $E_{ij} = e_{ij} - e_{ji}$, $i \neq j$ and note that $\{E_{ij} \mid i < j\}$ is a basis for K. For convenience if $x = \sum_{i<j} \alpha_{ij} E_{ij} \in K$, the *support* of $x = \{(i,j) \mid \alpha_{ij} \neq 0\}$. The E_{ij} satisfy $E_{ij} = -E_{ji}$, $[E_{ij},E_{jk}] = E_{ik}$, $i \neq k$, and the consequences thereof.

Now let $f : K \longrightarrow K$ be an additive commuting map, K the skew elements of $M_n(F)$ under transpose and $n \neq 2$ or 4. (The cases $n = 2$ and $n = 4$ are special and will be discussed later.)

Lemma 4.1. $f(E_{ij}) = \alpha E_{ij} + \sum_{k<\ell} \alpha_{k\ell}E_{k\ell}$, $\alpha,\alpha_{k\ell} \in F$, where $\{k,\ell\} \cap \{i,j\} = \emptyset$.

Proof. We may assume $(i,j) = (1,2)$. Writing $f(E_{12}) = \sum_{k<\ell} \alpha_{k\ell}E_{k\ell}$ we may deduce from

$$0 = \left[\Sigma\alpha_{k\ell} \cdot E_{k\ell},\ E_{12}\right] = \Sigma\alpha_{k\ell}\left[E_{k\ell},E_{12}\right]$$

that $\alpha_{k\ell} = 0$ when $\{k,\ell\} \cap \{1,2\} = \{1\}$ or $\{2\}$.

Lemma 4.2. $f(E_{ij}) = \alpha_{ij}E_{ij},\ \ i < j.$

Proof. By Lemma 4.1 the result holds for $n = 3$ and so we may assume that $n \geq 5$. We may assume $(i,j) = (1,2)$ and in view of Lemma 4.1 we may write

$$f(E_{12}) = \alpha E_{12} + \sum_{i<j} \alpha_{ij}E_{ij}, \qquad \{i,j\} \cap \{1,2\} = \emptyset.$$

Suppose, for example, that $\alpha_{34} \neq 0$. In (4) we set $x = E_{35}$, $y = E_{12}$ and, using Lemma 4.1 again, we write

$$f(E_{35}) = \beta E_{35} + \sum_{k<\ell} \beta_{k\ell}E_{k\ell}, \qquad \{k,\ell\} \cap \{3,5\} = \emptyset.$$

Consider the equation

$$a = \left[\alpha E_{12} + \Sigma\alpha_{ij}E_{ij},\ E_{35}\right] = \left[E_{12},\ \beta E_{35} + \Sigma\beta_{k\ell}E_{k\ell}\right] = b.$$

The term $\alpha_{34}[E_{34},E_{35}] = \alpha_{34}E_{54} \neq 0$ appears in a and every member of the support of a has 3 or 5 as an index. However no member of the support of b has either 3 or 5 since $\{k,\ell\} \cap \{3,5\} = \emptyset$ which contradicts $a = b$.

Theorem 4.3. *Let* $R = M_n(F)$ *under transpose involution,* $n \neq 2$, $n \neq 4$ *and let* $f : K \longrightarrow K$ *be an additive commuting map. Then there exists* $\lambda \in F$ *such that* $f(x) = \lambda x$ *for all* $x \in K$.

Proof. By Lemma 4.2 $f(E_{ij}) = \alpha_{ij}E_{ij}$, $i < j$ and we need only show that all the α_{ij} are equal (and thus we let λ be this common value). In fact, without loss of generality, it is enough to prove $\alpha_{12} = \alpha_{23}$. To this end, using (4), we see immediately from $[\alpha_{12}E_{12},E_{13}] = [E_{12},\alpha_{23}E_{23}]$ that $\alpha_{12} = \alpha_{23}$.

For $n = 2$, K is 1-dimensional, generated by E_{12}. Let φ be any additive endomorphism of F. Then the map $f : K \longrightarrow K$ given by $f(\alpha E_{12}) = \varphi(\alpha)E_{12}$ is additive commuting but in general there is no $\lambda \in F$ such that $f(x) = \lambda x$. Thus the theorem fails in this case.

For $n = 4$ it is well-known that $K = U \oplus V$ with U and V 3-dimensional simple Lie algebras. If we define $f(u+v) = u - v$, $u \in U$, $v \in V$ it is clear that f is additive commuting but that the conclusion of the theorem fails.

Next consider $M_n(F)$ under symplectic involution. Here $n = 2m$ and it is well-known that K consists of all matrices of the form

$$\begin{bmatrix} A & S \\ T & -A^t \end{bmatrix}$$

where A is an arbitrary $m \times m$ matrix and S and T are symmetric $m \times m$ matrices. We now let $f : K \longrightarrow K$ be an additive commuting map.

We first study the "diagonal" case, writing

(6) $$f\left(\begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}\right) = \begin{bmatrix} \varphi(A) & \chi_1(A) \\ \chi_2(A) & -\varphi(A)^t \end{bmatrix}$$

where φ, $\chi_i : M_m(F) \longrightarrow M_m(F)$ are linear maps with $\chi_i(A)$ symmetric, $i = 1,2$.

Lemma 4.4.

(a) $\chi_i = 0$ *for* $i = 1,2$.

(b) *There exists* $\lambda \in F$ *such that* $\varphi(A) = \lambda A + \mu(A)I$, $\mu(A) \in F$. (*In the case* $m = 1$ *we may take* $\mu(A) = 0$.)

Proof. Commuting (6) with $\begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}$ results in

(7) $$[A,\varphi(A)] = 0,$$

(8) $$A\chi_1(A) + \chi_1(A)A^t = 0, \quad \text{and}$$

(9) $$\chi_2(A)A + A^t\chi_2(A) = 0.$$

Setting $A = I$ in (8) results in $2\chi_1(I) = 0$ whence $\chi_1(I) = 0$. Using this and replacing A by $A + I$ in (8) we obtain $\chi_1(A) = 0$. Similarly one shows that $\chi_2(A) = 0$ and so (a) has been established. From (7) we see that φ is a commuting map on $M_m(F)$. Therefore by Corollary 3.4 there exists $\lambda \in F$ such that $\varphi(A) = \lambda A + \mu(A)I$ for all $A \in M_m(F)$, with $\mu : M_m(F) \longrightarrow F$ an additive map, which proves (b).

We next write $f\left(\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} g(S) & h_1(S) \\ h_2(S) & -g(S)^t \end{bmatrix}$ and set $x = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$, $y = \begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}$ in the equation $[x,f(y)] = [f(x),y]$. In view of Lemma 4.4 this reads

$$(10) \quad \left[\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda A+\mu(A)I & 0 \\ 0 & -\lambda A^t-\mu(A)I \end{bmatrix}\right] = \left[\begin{bmatrix} g(S) & h_1(S) \\ h_2(S) & -g(S)^t \end{bmatrix}, \begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}\right].$$

Expansion of (10) yields

$$(11) \qquad [g(S),A] = 0$$

$$(12) \qquad h_2(S)A + A^t h_2(S) = 0$$

$$(13) \qquad S(\lambda A^t+\mu(A)I) + (\lambda A+\mu(A)I)S = h_1(S)A^t + Ah_1(S).$$

From (11) we conclude that $g(S) = \gamma(S)I$, $\gamma(S) \in F$, and setting $A = I$ in (12) we see that $h_2(S) = 0$. Setting $A = I$ in (13) results in

$$(14) \qquad h_1(S) = \beta S, \qquad \beta = \lambda + \mu(I).$$

Setting $S = I$ in (13), together with (14), yields

$$(15) \qquad 2\mu(A)I = (\beta-\lambda)(A^t+A) \quad \text{for all} \quad A \in M_m(F).$$

In case $m = 1$ we already have $\mu(A) = 0$ and $\beta = \lambda$ so we may assume

$m > 1$. If $\beta \neq \lambda$, (15) says that every symmetric matrix is central, which is false. Therefore $\beta = \lambda$ and so (15) implies $\mu(A) = 0$ for all $A \in M_m(F)$.

Thus far we have shown that

$$f\left(\begin{bmatrix} A & 0 \\ 0 & -A^t \end{bmatrix}\right) = \begin{bmatrix} \lambda A & 0 \\ 0 & -\lambda A^t \end{bmatrix}, \quad \text{and} \quad f\left(\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \gamma(S)I & \lambda S \\ 0 & -\gamma(S)I \end{bmatrix}.$$

Finally, from $\left[\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \gamma(S)I & \lambda S \\ 0 & -\gamma(S)I \end{bmatrix}\right] = 0$ we see that $2\gamma(S)S = 0$ whence $\gamma(S) = 0$. A similar argument shows that $f\left(\begin{bmatrix} 0 & 0 \\ T & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ \lambda T & 0 \end{bmatrix}$ and so, by the linearity of f we have proved

Theorem 4.5. *Let* $R = M_n(F)$ *under symplectic involution and let* $f : K \longrightarrow K$ *be an additive commuting map. Then there exists* $\lambda \in F$ *such that* $f(x) = \lambda x$ *for all* $x \in K$.

5. THE GPI CASE

Throughout this section we assume that R is a prime ring with involution $*$ of the first kind and with $[K,K] \neq \{0\}$. We let $f : K \longrightarrow K$ be an additive commuting map.

Lemma 5.1. f *can be extended to a map* $g : CK \longrightarrow CK$ *which is* C-*linear, commuting and satisfies* $g(\lambda x) = \lambda f(x)$, $\lambda \in C$, $x \in K$.

Proof. We define

$$g\left[\Sigma\, \lambda_i x_i\right] = \Sigma\, \lambda_i\, f(x_i), \quad \lambda_i \in C, \quad x_i \in K.$$

To show g is well-defined suppose $\Sigma\, \lambda_i x_i = 0$. Then for all $y \in K$ we have

$$\left[\Sigma\, \lambda_i\, f(x_i), y\right] = \Sigma\, \lambda_i [f(x_i), y] = \Sigma\, \lambda_i [x_i, f(y)] = \left[\Sigma\, \lambda_i x_i, f(y)\right] = 0,$$

showing that $[\Sigma\, \lambda_i\, f(x_i),\, CK] = \{0\}$. By Theorem 2.2 we conclude that $\Sigma\, \lambda_i\, f(x_i)$ is central, and hence 0 since $*$ is of the first kind. That g is commuting follows easily from the fact that f is commuting. Since CK is a C-space we see that g must be C-linear by Remark 2.3.

Lemma 5.2. *If* R *is a closed prime algebra over* C *and* F *is any extension field of* C *then* f *can be extended to a map* $g : K \otimes_C F \longrightarrow K \otimes_C F$ *which is* F*-linear, commuting, and satisfies* $g(x \otimes \lambda) = f(x) \otimes \lambda$, $x \in K$, $\lambda \in F$.

Proof. Since K is a C-space we know f is C-linear by Remark 2.3. Therefore by a well-known property of the tensor product the map $g : K \otimes F \longrightarrow K \otimes F$ given by $x \otimes \lambda \longrightarrow f(x) \otimes \lambda$ is a well-defined F-linear map. It is also clear that g is commuting since f is.

We recall from Theorem 2.5 that if R is a closed prime GPI ring over a field F then R has nonzero socle H and acts densely on a vector space V over a division ring D where $(D : F) < \infty$. Furthermore, if F is algebraically closed, then $D = F$.

Lemma 5.3. *If* R *is closed prime* GPI *with socle* H *then* $H \cap K$ *is invariant under* f.

Proof. We may assume $(V : D) = \infty$ since otherwise $R = H$ and we are done. Let $h \in H \cap K$. By Theorem 2.7 there exists a symmetric idempotent e in H such that $h \in eRe$. Setting $e_1 = e$ and (formally) $e_2 = 1 - e$ we may express R in its Pierce decomposition relative to e_1, e_2:

$$R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22} \quad \text{where} \quad R_{ij} = e_i R e_j .$$

We use the suggestive terminology x_{ij} for an element of R_{ij} and, given $x_{ij} \in R_{ij}$, we note that $x_{ij}^* \in R_{ji}$ and we denote x_{ij}^* by x_{ji}. The skew elements of R are of the form $x_{11} + (x_{12}-x_{21}) + x_{22}$, $x_{ii} \in K \cap R_{ii}$, $x_{12} \in R_{12}$. With this machinery in hand we write $f(h) = y_{11} + y_{12} - y_{21} + y_{22}$ and for $x_{22} \in K \cap R_{22}$ we consider the relation

$$(16) \qquad \left[f(x_{22}),\, h\right] = \left[x_{22},\, f(h)\right] = \left[x_{22},\, y_{11} + y_{12} - y_{21} + y_{22}\right].$$

Expansion of (16) together with the fact that $h \in R_{11}$ shows that $[x_{22}, y_{22}] = 0$ for all $x_{22} \in K \cap R_{22}$. By Theorem 2.2 applied to the ring R_{22} we see that y_{22} lies in the center of R_{22}, and therefore $y_{22} = 0$ since $*$ is of the first kind. It follows that $f(h) \in H$ since e lies in H and the proof is complete.

Lemma 5.4. *If* R *is closed prime* GPI *over an algebraically closed field* F *with* $[K,K] \neq \{0\}$ *and* H *is the socle of* R, *then there exists* $\lambda \in F$

such that $f(h) = \lambda h$ *for all* $h \in H \cap K$, *or* $R \cong M_4(F)$ *under transpose involution.*

Proof. If $R = M_n(F)$ then $K = H$ and we are finished by Theorems 4.3 and 4.5. Hence we may assume $(R:F) = \infty$. It suffices to show that given h_1, h_2 distinct nonzero elements of $H \cap K$, there exists $\lambda \in F$ such that $f(h_1) = \lambda h_1$ and $f(h_2) = \lambda h_2$. Choosing such $h_1, h_2 \in H \cap K$ we know from Lemma 5.3 that $f(h_1)$ and $f(h_2)$ lie in $H \cap K$. By Theorem 2.7 there is a symmetric idempotent e in H of rank > 4 such that h_1, h_2, $f(h_1)$, $f(h_2)$ all lie in eRe. We now define a map $g : eKe \longrightarrow eKe$ by

$$g(exe) = ef(exe)e, \quad x \in K,$$

which is additive and commuting since f is. But $eRe \cong M_n(F)$ and, since F is algebraically closed, we have already seen in section 2 that the involution is either transpose or symplectic. By Theorems 4.3 and 4.5 applied to eRe there exists $\lambda \in F$ such that $g(exe) = \lambda\, exe$ for all $x \in K$. In particular, for $i = 1,2$, $f(h_i) = ef(h_i)e = g(h_i) = \lambda h_i$ and the lemma is proved.

We now have all the pieces with which to prove the main result of this section. We recall from section 2 the notation $\tilde{R}$ for the super $*$-closure $RC \otimes_C F$ of R where F is the algebraic closure of C.

THEOREM 5.5. *Let* R *be a prime* GPI *ring with involution* $*$ *of the first kind,* $[K,K] \neq \{0\}$, *and* $\tilde{R} \neq M_4(F)$ *under transpose. If* $f : K \longrightarrow K$ *is an additive commuting map there exists* $\lambda \in C$ *such that* $f(x) = \lambda x$ *for* *all* $x \in K$.

Proof. We claim that it suffices to prove the theorem in the special case that R is closed prime GPI over an algebraically closed field F. Indeed, by Lemmas 5.1 and 5.2 f can be extended to an F-linear commuting map $g : CK \otimes F \longrightarrow CK \otimes F$ given by $\alpha x \otimes \beta \longrightarrow \alpha f(x) \otimes \beta$, $\alpha \in C$, $\beta \in F$, $x \in K$, where $\tilde{R} = CR \otimes_C F$ is the super $*$-closure of R. By assumption there exists $\lambda \in F$ such that $g(y) = \lambda y$ for all $y \in CK \otimes F$. In particular $f(x) \otimes 1 = x \otimes \lambda$ for all $x \in K$. It follows by tensor product considerations that $\lambda \in C$ and so we see that $f(x) = \lambda x$ for all $x \in K$.

We may therefore assume that R is closed prime GPI over an algebraically closed field F. The socle H is nonzero and by Lemma 5.4 there exists $\lambda \in C$ such that $f(h) = \lambda h$ for all $h \in H \cap K$. Letting $x \in K$ we see from $[h, f(x)] = [f(h), x] = [\lambda h, x] = [h, \lambda x]$ that $[f(x)-\lambda x, h] = 0$ for all $h \in H \cap K$ and hence for all $h \in H$ by Theorem 2.2 applied to H. Thus $f(x) - \lambda x$ is central. But $*$ is of the first kind which results in $f(x) = \lambda x$ and the proof is complete.

6. THE GENERAL CASE

We begin this section by showing that centralizing maps of K are in fact commuting.

Lemma 6.1. *If* R *is prime with involution and* $f : K \longrightarrow K$ *is an additive centralizing map then* f *is commuting.*

Proof. Let $a \in K$ and set $z = [f(a),a] \in Z \cap K$, noting that $za^2 \in K$ and $[f(a), a^2] = 2za$. Setting $v = [f(za^2), a]$ from (2) we see that $[f(za^2), a] + [f(a), za^2] = v + 2z^2a \in Z$. In particular $[v, a] = 0$. We also see from this that $0 = [f(a), v+2z^2a] = [f(a), v] + 2z^3$, that is

$$(17) \qquad [f(a), v] = -2z^3.$$

Now $[f(za^2), za^2] \in Z$, so $[f(za^2), za^2] = z\left\{[f(za^2), a]a + a[f(za^2),a]\right\} = z(va+av) = 2zva \in Z$, since $[v, a] = 0$. Therefore $0 = [f(a), 2zva] = 2z\left\{[f(a), v]a + v[f(a), a]\right\} = 2z\{-2z^3a + zv\}$, *i.e.* $2z^2v = 4z^4a$. If $z \neq 0$ then $v = 2z^2a$, whence $[f(a), v] = 2z^3$ a contradiction to (17). Therefore $z = 0$ and the lemma is proved.

At this point we remark that modulo characteristic restrictions Lemma 6.1 holds true under far weaker conditions and at the end of this section we indicate this in the form of Theorem 6.5, which the reader may find of independent interest.

In view of Lemma 6.1 we may hereon focus our attention on commuting maps. We first consider the situation where the involution is of the second kind.

Theorem 6.2. *If* R *is prime with* $*$ *of the second kind and* $f : K \longrightarrow K$ *is an additive commuting map then there exists* $\lambda \in C_*$ *and* $\mu : K \longrightarrow C_\#$ *such that* $f(x) = \lambda x + \mu(x)$ *for all* $x \in K$.

Proof. We let $B : K \times K \longrightarrow K$ be the biderivation given by $B(x,y) = [f(x), y] = [x, f(y)]$, $x,y \in K$. We note that $CK = RC$ (since $*$ is of the second kind) and we define $B' : CK \times CK \longrightarrow CK$ according to

$$B'\left[\sum_i \lambda_i x_i, \sum_j \mu_j y_j\right] = \sum_{i,j} \lambda_i \mu_j B(x_i, y_i)$$

where $\lambda_i, \mu_j \in C$, $x_i, y_i \in K$. By symmetry it is readily seen from the equations

$$\sum_{i,j} \lambda_i \mu_j B(x_i, y_j) = \Sigma \lambda_i \mu_j [x_i, f(y_j)] = \left[\Sigma \lambda_i x_i, \Sigma \mu_j f(y_i)\right]$$

that B' is well-defined and is also a biderivation. By Theorem 3.3 there exists $\gamma \in C$ such that $B'(x,y) = \gamma[x,y]$ for all $x,y \in RC$. In particular for $x \in K$, $y \in R$ we have $[f(x), y] = B'(x,y) = \gamma[x,y]$, whence $f(x) - \gamma x = \delta(x) \in C$. Letting λ be the symmetric component of γ and $\mu(x)$ the skew component of $\delta(x)$ we finally see that $f(x) = \lambda x + \mu(x)$, $\lambda \in C_*$, $\mu(x) \in C_\#$ for $x \in K$.

Again let $f : K \to K$ be an additive commuting map in a prime ring with involution. The following lemma will be useful in reducing the proof of the main theorem to the GPI case.

Lemma 6.3. *For* $a,x \in K$, $[[a,x], [f(a),x]] = 0$.

Proof. The conclusion will follow from expanding $[f([x,a]),[x,a]] = 0$ while making use of (4).

$$\begin{aligned}
0 &= [f([x,a]),[x,a]] \\
&= [[f([x,a]),x],a] + [x,[f([x,a]),a]] \\
&= [[[x,a],f(x)],a] + [x,[[x,a],f(a)]] \\
&= [[x,[a,f(x)]],a] + [x,[[x,a],f(a)]] \\
&= [[x,[f(a),x]],a] + [x,[[x,a],f(a)]] \\
&= [[x,a],[f(a),x]] + [x,[[f(a),x],a]] + [x,[[x,a],f(a)]] \\
&= [[x,a],[f(a),x]]
\end{aligned}$$

We are now in a position to prove the main result of this paper.

THEOREM 6.4. *Let* R *be a prime ring with involution* $*$, *of characteristic* $\neq 2$, *with center* Z, *extended centroid* C, *and skew elements* K. *Assume further that* $[K,K] \neq \{0\}$ *and that the super* $*$*–closure* $\tilde{R}$ *is unequal to* $M_4(F)$ *under transpose. If* $f : K \longrightarrow K$ *is an additive centralizing map, i.e.* $[f(x),x] \in Z$ *for all* $x \in K$, *then there exists* $\lambda \in C_*$ *and an additive map* $\mu : K \longrightarrow C_{\#}$ *such that* $f(x) = \lambda x + \mu(x)$ *for all* $x \in K$.

Proof. By Lemma 6.1 we may assume that f is commuting and by Lemma 6.2 that $*$ is of the first kind. If R is GPI we are finished by Theorem 5.5. Therefore we may assume R is not GPI.

We claim that for any $a \in K$, a and $f(a)$ are C-dependent, *i.e.*, $f(a) = \lambda(a)a$, $\lambda(a) \in C$. Let $A_C\langle t\rangle$ denote the free product over C of $A = RC$ and the free algebra $C\langle t\rangle$, and consider the element

$$\varphi(t) = [[a,t],[f(a),t]] \in A_C\langle t\rangle .$$

By Lemma 6.3, $\varphi(t)$ is satisfied by K and if $\varphi(t) \neq 0$ then by Theorem 2.6 R is GPI. Therefore $\varphi(t) = 0$. But expansion of $\varphi(t)$ using the fact that $[a,f(a)] = 0$ and that a, f(a) are C-independent, shows that $\varphi(t)$ cannot be 0, and so our claim that a, f(a) are C-dependent is established.

It remains to show that $\lambda(a)$ is independent of a. For $a,x \in K$ we conclude from $\lambda(a+x)(a+x) = f(a+x) = f(a) + f(x) = \lambda(a)a + \lambda(x)x$ that

$$(18) \qquad \left[\lambda(a+x)-\lambda(a)\right]a + \left[\lambda(a+x)-\lambda(x)\right]x = 0.$$

We fix $a \neq 0 \in K$ and let $\lambda = \lambda(a)$. If $a,x \in K$ are C–independent then from (18) we see $\lambda = \lambda(a) = \lambda(x)$. If $x = \alpha a$, $\alpha \in C$ then $f(x) = \alpha f(a) = \alpha\lambda a = \lambda x$. Therefore there exists $\lambda \in C$ such that $f(x) = \lambda x$ for all $x \in K$ and the proof of the theorem is complete.

As promised at the beginning of this section we now present a result of independent interest which gives rather general conditions under which f centralizing implies f commuting. For convenience we shall say that a ring has characteristic $> m$ if for each $0 < k \leq m$ and each $x \in R$, $kx = 0$ implies $x = 0$.

Theorem 6.5. *Let* R *be a ring having no nonzero central nilpotent elements and let* A *be an additive subgroup of* R *for which there is a fixed integer* $n \geq 2$ *such that* $a^n \in A$ *for all* $a \in A$. *Furthermore assume that* R *has characteristic* $> 2n$. *If* $f : A \longrightarrow A$ *is an additive centralizing map then* f *is commuting.*

Proof. The assumption of characteristic $> 2n$ will be frequently used without specific mention. Let $a \in A$ and set $z = [f(a), a]$,

$u = [f(a^n), a]$. It is easily seen that $[f(a), a^k] = kza^{k-1}$ for all $k \geq 1$ since $z \in Z$ (the center of R). By (2) we see that

$$[f(a^n), a] + [f(a), a^n] = u + nza^{n-1} \in Z,$$

and so $[u,a] = 0$. Moreover

$$0 = [f(a), u + nza^{n-1}] = [f(a), u] + n(n-1)z^2a^{n-2},$$

that is

$$[f(a), u] = -n(n-1)z^2a^{n-2}. \tag{19}$$

Now $[f(a^n), a^n] \in Z$ so $[f(a^n), a^n] = nua^{n-1} \in Z$ since $[u,a] = 0$, so that $ua^{n-1} \in Z$. Therefore

$$\begin{aligned} 0 = [f(a), ua^{n-1}] &= [f(a), u]a^{n-1} + u[f(a), a^{n-1}] \\ &= -n(n-1)z^2a^{2n-3} + (n-1)uza^{n-2} \quad \text{using (19).} \end{aligned}$$

From this we conclude that $uza^{n-2} = nz^2a^{2n-3}$ from which it follows that $zua^{n-1} = nz^2a^{2n-2}$. But $ua^{n-1} \in Z$ and so we obtain $0 = [f(a), nz^2a^{2n-2}] = n(2n-2)z^3a^{2n-3}$, or $0 = z^3a^{2n-3}$. An easy induction then shows that for $3 \leq k \leq 2n$ we have $z^ka^{2n-k} = 0$. Indeed, assuming $z^ka^{2n-k} = 0$ we see that $[f(a), z^ka^{2n-k}] = (2n-k)z^{k+1}a^{2n-k-1} = 0$, whence $z^{k+1}a^{2n-(k+1)} = 0$. In particular for $k = 2n$ we have $z^{2n} = 0$ from which we conclude $z = 0$ and the proof is complete.

We remark that if R is prime with involution and $A = K$ then $a^3 \in A$ for all $a \in A$. A close examination of the proof of Theorem 6.5 reveals that for $n = 3$ Theorem 6.5 holds if char $R > 3$. Hence, with the extra restriction of char $R \neq 3$, Lemma 6.1 is a corollary to Theorem 6.5.

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