SOME TYPES OF $(\in, \in \vee \mathrm{q})$-INTERVAL-VALUED FUZZY IDEALS OF BCI ALGEBRAS

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Abstract. In this paper, we introduce the notions of interval-valued and $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy ($p$-,$q$- and a-) ideals of BCI algebras and investigate some of their properties. We then derive characterization theorems for these generalized interval-valued fuzzy ideals and discuss their relationship.

1. Introduction

In recent years, because of its applicability, the study of t-norm-based logical systems has become an increasingly important topic in the field of logic. As it is well known, BCK and BCI algebras, introduced by Imai and Iséki [12,15] , are two classes of algebras of logic which have been extensively investigated by many researchers [7,8,15-33,35]. Iorgulescu [13,14] showed that pocrims and BCK algebras with condition (S) are categorically isomorphic. Hence, most of the algebras related to an t-norm based logic, for example, MTL algebras [10], BL algebras [11], hoop, MV algebras (i.e., lattice implication algebras) and Boolean algebras, are subclasses of BCK algebras. This shows that BCK/BCI algebras are very general structures.

After the introduction of fuzzy sets by Zadeh [34], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroups, called $(\in, \in \vee \mathrm{q})$-fuzzy subgroups, were introduced in a paper of Bhakat and Das [2] using the combined notions of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, introduced by Pu and Liu [31]. In fact, the $(\in, \in \vee \mathrm{q})$-fuzzy subgroup is an important generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar type of generalizations of existing fuzzy subsystems with other algebraic structures. Jun [21,22] introduced the concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI algebra and investigated related results. Recently, Davvaz [4] applied this theory to near-rings and obtained some useful results. Davvaz and Corsini [5,6] also redefined fuzzy H_v-submodules and fuzzy H_v-ideals. We discuss this topic further in this paper.

In section 2, we recall some basic definitions and results of BCI algebras. In section

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3, we introduce the notions of interval-valued fuzzy (p-, q- and a-) ideals of BCI algebras, which are, respectively, generalizations of fuzzy (p-, q- and a-) ideals, and investigate some of their properties. Section 4 is divided into four subsections. In section 4.1, we briefly review $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy ideals of BCI algebras. In section 4.2, we investigate the properties of $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy p-ideals of BCI algebras. The notions of $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy ($q$- and a-) ideals of BCI algebras and the relationship among these generalized interval-valued fuzzy ideals of BCI algebras are discussed in section 4.3 and 4.4, respectively.

2. **Preliminaries**

By a BCI algebra we mean an algebra $(X, *, 0)$ of type (2,0) satisfying the axioms:
(i) $((x * y) * (x * z)) * (z * y) = 0$;
(ii) $(x * (x * y)) * y = 0$;
(iii) $x * x = 0$;
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define a partial ordering " $\leq$ " by $x \leq y$ if and only if $x * y = 0$. In what follows, X will denote a BCI algebra unless otherwise specified.

Proposition 2.1. [7, 8, 16, 17] *In any BCI algebra X, we have:*
(1) $(x * y) * z = (x * z) * y$,
(2) $(x * z) * (y * z) \leq x * y$,
(3) $(x * y) * (x * z) \leq z * y$,
(4) $x * 0 = x$,
(5) $0 * (x * y) = (0 * x) * (0 * y)$,
(6) $x * (x * (x * y)) = x * y$.

A non-empty subset I of X is called an ideal of X if it satisfies (I1) $0 \in I$; (I2) $x * y \in I$ and $y \in I$ imply $x \in I$. A non-empty subset I of X is called a p-ideal of X if it satisfies (I1) and (I3) $(x * z) * (y * z) \in I$ and $y \in I$ imply $x \in I$. A non-empty subset I of X is called a q-ideal of X if it satisfies (I1) and (I4) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$. A non-empty subset I of X is called an a-ideal of X if it satisfies (I1) and (I5) $(x * z) * (0 * y) \in I$ and $z \in I$ imply $y * x \in I$ (see [25,27]).

Theorem 2.2. [22] *(i) Every p- (resp., q-, a-) ideal of a BCI algebra is an ideal, but the converse is not true.*

(ii) A non-empty subset I of a BCI algebra X is an a-ideal of X if and only if it is both a p-ideal and a q-ideal.

We now review some fuzzy logic concepts. Recall that the real unit interval [0,1] with the totally ordered relation " $\leq$ " is a complete lattice, with $\wedge = \min$ and $\vee = \max$, 0 and 1 being the least and greatest element respectively.

Definition 2.3. [34] (i) A fuzzy set of X is a function $\mu : X \to [0, 1]$;

(ii) For a fuzzy set μ of X and $t \in (0, 1]$, the crisp set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called the *level subset* of μ.

Definition 2.4. [23] A fuzzy set μ of X is called a fuzzy ideal of X if it satisfies:
(F1) $\mu(0) \geq \mu(x), \forall x \in X$,

(F2) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}, \forall x, y \in X$.

Definition 2.5 (23,25,29)**.** (i) A fuzzy set μ of X is called a fuzzy p-ideal of X if it satisfies (F1) and (F3) $\mu(x) \geq \min\{\mu((x*z)*(y*z)), \mu(y)\}$, for all $x, y, z \in X$.

(ii) A fuzzy set μ of X is called a fuzzy q-ideal of X if it satisfies (F1) and

(F4) $\mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\}$, for all $x, y, z \in X$.

(iii) A fuzzy set μ of X is called a fuzzy a-ideal of X if it satisfies (F1) and

(F5) $\mu(y*x) \geq \min\{\mu((x*z)*(0*y)), \mu(z)\}$, for all $x, y, z \in X$.

Theorem 2.6. [29] *A fuzzy set μ of X is a fuzzy (p-,q−, a−)ideal of X if and only if, for all $t \in (0,1]$, each non-empty level subset μ_t is a (p-,q−, a−)ideal of X,respectively.*

The following theorem, which is a consequence of Theorem 2.2 and Theorem 2.6, shows the connection between these four types of fuzzy ideals of BCI algebras.

Theorem 2.7. [29] *(i) Every fuzzy p-(resp.,q−, a−)ideal of a BCI algebra is a fuzzy ideal, but the converse is not true.*

(ii) A fuzzy set μ of any BCI algebra X is a fuzzy a-ideal of X if and only if it is both a fuzzy p-ideal and a fuzzy q-ideal.

3. Some Types of Interval-valued Fuzzy Ideals

We now review some interval-valued fuzzy logic concepts. Let $\overline{a} = [a^-, a^+]$ be a closed interval of [0,1], where $0 \leq a^- \leq a^+ \leq 1$. We denote by $D[0,1]$ the set of all such closed intervals of [0,1]. The reader can see [30] for more details on the order relation " $\leq$ ".

Definition 3.1. [3,32] (i) An interval-valued fuzzy set of X is $\overline{F} : X \to D[0,1]$, where, for each $x \in X, \overline{F}(x) = [F^-(x), F^+(x)] \in D[0,1]$.

(ii) Let $\overline{F}$ be an interval-valued fuzzy set of X. Then, for every $[0,0] < \overline{t} \leq [1,1]$, the crisp set $\overline{F}_{\overline{t}} = \{x \in X | \overline{F}(x) \geq \overline{t}\}$ is called the level subset of $\overline{F}$.

Note that since every $a \in [0,1]$ is in correspondence with the interval $[a,a] \in D[0,1]$, hence a fuzzy set is a particular case of interval-valued fuzzy sets. Also, we can consider an interval-valued fuzzy set $\overline{F}$ of X to be a pair of fuzzy sets (F^-, F^+) of X such that $F^-(x) \leq F^+(x)$ for all $x \in X$.

We refer the reader to [30] for more details on operations on two interval-valued fuzzy sets of X.

Define rmin by $\text{rmin}\{\overline{a}_i, \overline{b}_i\} = [\min\{a_i^-, b_i^-\}, \min\{a_i^+, b_i^+\}]$, where $\overline{a}_i = [a_i^-, a_i^+], \overline{b}_i = [b_i^-, b_i^+] \in D[0,1], i \in I$.

Definition 3.2 (20)**.** An interval-valued fuzzy set $\overline{F}$ of X is called an interval-valued fuzzy ideal of X if $\overline{F}$ satisfies:

(intF1) $\overline{F}(0) \geq \overline{F}(x)$, for all $x \in X$,

(intF2) $\overline{F}(x) \geq \text{rmin}\{\overline{F}(x*y), \overline{F}(y)\}$, for all $x, y \in X$.

Remark 3.3. Every fuzzy ideal of X is a particular case of an interval-valued fuzzy ideal.

Theorem 3.4. [20] *An interval-valued fuzzy set $\overline{F}$ of X is an interval-valued fuzzy ideal of X if and only if the set $\overline{F}_{\bar{t}}(\neq \emptyset)$ is an ideal of X for all $[0,0] < \bar{t} \leq [1,1]$.*

Definition 3.5. (i) An interval-valued fuzzy ideal $\overline{F}$ of X is called an interval-valued fuzzy p-ideal of X if
(intF3) $\overline{F}(x) \geq \text{rmin}\{\overline{F}((x*z)*(y*z)), \overline{F}(y)\}$, for all $x, y, z \in X$.
(ii) An interval-valued fuzzy ideal $\overline{F}$ of X is called an interval-valued fuzzy q-ideal of X if
(intF4) $\overline{F}(x*z) \geq \text{rmin}\{\overline{F}(x*(y*z)), \overline{F}(y)\}$, for all $x, y, z \in X$.
(iii) An interval-valued fuzzy ideal $\overline{F}$ of X is called an interval-valued fuzzy a-ideal of X if
(intF5) $\overline{F}(y*x) \geq \text{rmin}\{\overline{F}((x*z)*(0*y)), \overline{F}(z)\}$, for all $x, y, z \in X$.

Example 3.6. (i) Let $X = \{0, 1, 2, 3\}$ be a proper BCI algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define an interval-valued fuzzy set $\overline{F}$ of X by $\overline{F}(0) = [0.8, 0.9], \overline{F}(1) = \overline{F}(2) = [0.7, 0.8]$, and $\overline{F}(3) = [0.2, 0.3]$. It is easy to verify that $\overline{F}$ is an interval-valued fuzzy p-ideal of X.

(ii)Let $X = \{0, 1, 2\}$ be a proper BCI algebra with Cayley table as follows:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Define an interval-valued fuzzy set $\overline{F}$ of X by $\overline{F}(0) = [0.8, 0.9]$, and $\overline{F}(1) = \overline{F}(2) = [0.2, 0.3]$. It is easy to verify that $\overline{F}$ is an interval-valued fuzzy q-ideal of X.

(iii) Consider the BCI algebra X as in Example 3.6 (i). Define an interval-valued fuzzy set $\overline{F}$ in X by $\overline{F}(0) = [0.8, 0.9], \overline{F}(1) = [0.7, 0.8]$, and $\overline{F}(2) = \overline{F}(3) = [0.2, 0.3]$. It is easy to verify that $\overline{F}$ is an interval-valued fuzzy a-ideal of X.

Remark 3.7. Every fuzzy (p-,$q-$, $a-$)ideal of X is a particular case of interval-valued fuzzy (p-,$q-$, $a-$) ideal, respectively.

Now we characterize the interval-valued fuzzy (p -,$q-$, $a-$) ideals by their level subsets.

Theorem 3.8. *An interval-valued fuzzy set $\overline{F}$ of X is an interval-valued fuzzy (p-,q−, a−) ideal of X if and only if the set $\overline{F}_{\bar{t}}(\neq \emptyset)$ is respectively a (p -,q−, a−) ideal of X for all $[0,0] < \bar{t} \leq [1,1]$.*

Proof. Similar to the proof of Theorem 2.6. ∎

The following theorem, a consequence of Theorem 2.6 and Theorem 3.8, shows the connections between these four types of interval-valued fuzzy ideals of BCI algebras.

Theorem 3.9. *(i) Every interval-valued fuzzy p- (resp., q−, a-)ideal of X is an interval-valued fuzzy ideal, but the converse is not true;*

(ii) An interval-valued fuzzy set $\overline{F}$ of X is an interval-valued fuzzy a-ideal if and only if it is both an interval-valued fuzzy p-ideal and an interval- valued fuzzy q-ideal.

4. **Some Types of** $(\in, \in\vee \mathrm{q})$**-interval-valued Fuzzy Ideals**

For any $\overline{F}(x) = [F^-(x), F^+(x)]$ and $\overline{t} = [t^-, t^+]$, we define $\overline{F}(x) + \overline{t} = [F^-(x) + t^-, F^+(x) + t^+]$, for all $x \in X$. In particular, if $t^- + F^-(x) > 1$, we write $\overline{F}(x) + \overline{t} > [1, 1]$.

Let $x \in X$ and $\overline{t} \in D[0, 1]$. An interval-valued fuzzy set $\overline{G}$ of a BCI algebra X is said to be an interval-valued fuzzy point $x_{\overline{t}}$, with support x and interval value $\overline{t}$, if

$$\overline{G}(y) = \begin{cases} \overline{t}(\neq[0,0]) & \text{if} \quad y = x, \\ [0,0] & \text{if} \quad y \neq x, \end{cases}$$

for all $y \in X$. We say $x_{\overline{t}}$ belongs to (resp., is quasi-coincident with) an interval-valued fuzzy set $\overline{F}$, written by $x_{\overline{t}} \in \overline{F}$ (resp., $x_{\overline{t}}\, q\overline{F}$) if $\overline{F}(x) \geq \overline{t}$ (resp. $\overline{F}(x) + \overline{t} > [1, 1]$). If $x_{\overline{t}} \in \overline{F}$ or $x_{\overline{t}}\, q\overline{F}$, then we write $x_{\overline{t}} \in\vee \mathrm{q}\ \overline{F}$; if $\overline{F}(x) < \overline{t}$ (resp., $\overline{F}(x) + \overline{t} \leq [1, 1]$), then we say that $x_{\overline{t}}\, \overline{\in}\overline{F}$ (resp., $x_{\overline{t}}\, \overline{q}\overline{F}$). The symbol $\overline{\in\vee \mathrm{q}}$ means that $\in\vee \mathrm{q}$ does not hold.

An interval-valued fuzzy set $\overline{F}(x) = [F^-(x), F^+(x)]$ of X is said to satisfy the condition (E) if the following holds:

(E) $\overline{F}(x) \leq [0.5, 0.5]$ or $[0.5, 0.5] < \overline{F}(x)$, for all $x \in X$.

In what follows, unless otherwise specified, we all interval-valued fuzzy sets of X must satisfy the comparability conditions and condition (E) .

4.1. $(\in, \in\vee \mathrm{q})$**-interval-valued Fuzzy Ideals.** In this subsection, we review $(\in, \in\vee \mathrm{q})$-interval-valued fuzzy ideals of X and state some of their properties [30].

Definition 4.1.1. [30] An interval-valued fuzzy set $\overline{F}$ of X is said to be an $(\in, \in\vee \mathrm{q})$- interval-valued fuzzy ideal of X if, for all $[0, 0] < \overline{t} \leq [1, 1], [0, 0] < \overline{r} \leq [1, 1]$ and for all $x, y \in X$,

(F1') $x_{\overline{t}} \in \overline{F}$ implies $0_{\overline{t}} \in\vee \mathrm{q}\overline{F}$,

(F2') $(x * y)_{\overline{t}} \in \overline{F}$ and $y_{\overline{r}} \in \overline{F}$ imply $x_{\mathrm{rmin}\{\overline{t},\overline{r}\}} \in\vee \mathrm{q}\overline{F}$.

Definition 4.1.2. [30] An interval-valued fuzzy set $\overline{F}$ of X is said to be an $(\overline{\in}, \overline{\in}\vee\overline{q})$-interval-valued fuzzy ideal of X if, for all $[0, 0] < \overline{t} \leq [1, 1], [0, 0] < \overline{r} \leq [1, 1]$ and all $x, y \in X$,

(F3') $0_{\overline{t}}\ \overline{\in}\ \overline{F}$ implies $x_{\overline{t}}\ \overline{\in} \vee \overline{q}\ \overline{F}$,

(F4') $x_{\mathrm{rmin}\{\overline{t},\overline{r}\}}\ \overline{\in}\ \overline{F}$ implies $(x * y)_{\overline{t}}\ \overline{\in} \vee \overline{q}\ \overline{F}$ or $y_{\overline{r}}\ \overline{\in} \vee \overline{q}\ \overline{F}$.

Theorem 4.1.3. [30] *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\in, \in\vee \mathrm{q})$ *-interval-valued fuzzy ideal (resp.,* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy ideal) of* X *if and only if* $\overline{F}_{\overline{t}}(\neq \emptyset)$ *is an ideal of* X *for all* $[0,0] < \overline{t} \leq [0.5, 0.5]$ *(resp.,* $[0.5, 0.5] < \overline{t} \leq [1,1]$*).*

4.2. $(\in, \in\vee \mathrm{q})$**-interval-valued Fuzzy** p**-ideals.** In this subsection, we introduce the concept of $(\in, \in\vee \mathrm{q})$-interval-valued fuzzy p-ideals of X and derive their properties.

Definition 4.2.1. An $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy ideal $\overline{F}$ of X is called an $(\in, \in\vee \mathrm{q})$-interval-valued fuzzy p- ideal of X if

(F5') $\overline{F}(x) \geq \mathrm{rmin}\{\overline{F}((x*z)*(y*z)), \overline{F}(y), [0.5, 0.5]\}$, for all $x, y, z \in X$.

Example 4.2.2. Consider the BCI algebra X of Example 3.6(i) and define an interval-valued fuzzy set $\overline{F}$ in X by $\overline{F}(0) = [0.7, 0.8], \overline{F}(1) = \overline{F}(2) = [0.8, 0.9]$, and $\overline{F}(3) = [0.2, 0.3]$. Then $\overline{F}$ is an $(\in, \in\vee \mathrm{q})$- interval-valued fuzzy p-ideal of X, but it is not an interval-valued fuzzy p-ideal.

Proposition 4.2.3. *Every interval-valued fuzzy p-ideal of* X *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy p-ideal, but the converse is not true in general.*

Proof. Similar to the proof of Theorem 5.2.3 in [30]. □

Proposition 4.2.4. *Let* $\overline{F}$ *be an* $(\in, \in\vee \mathrm{q})$*-interval-valued fuzzy ideal of* X*. Then the following are equivalent:*

(i) $\overline{F}$ *is an* $(\in, \in\vee \mathrm{q})$*- interval-valued fuzzy p-ideal.*

(ii) $\forall x \in X, \quad \overline{F}(x) \geq \mathrm{rmin}\{\overline{F}(0*(0*x)), [0.5, 0.5]\}$.

Proof. The proposition follows from Proposition 2.3 and Theorem 2.10 in [23]. □

Next, we characterize the $(\in, \in\vee \mathrm{q})$*-interval-valued fuzzy p-ideals by their level subsets.*

Theorem 4.2.5. *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy p-ideal of* X *if and only if* $\overline{F}_{\overline{t}}(\neq \emptyset)$ *is a p-ideal of* X *for all* $[0,0] < \overline{t} \leq [0.5, 0.5]$.

Proof. Similar to the proof of Theorem 4.1.3. □

We have a corresponding result when $\overline{F}_{\overline{t}}$ *is a p-ideal of* X*, for all* $[0.5, 0.5] < \overline{t} \leq [1,1]$.

Definition 4.2.6. *An* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy ideal* $\overline{F}$ *of* X *is called an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*- interval-valued fuzzy p-ideal of* X *if*

(F6') $\mathrm{rmax}\{\overline{F}(x), [0.5, 0.5]\} \geq \mathrm{rmin}\{\overline{F}((x*z)*(y*z)), \overline{F}(y)\}$*, for all* $x, y, z \in X$.

Example 4.2.7. *Consider the BCI algebra* X *of Example 3.6(i) and define an interval-valued fuzzy set* $\overline{F}$ *of* X *by* $\overline{F}(0) = [0.8, 0.9], \overline{F}(1) = [0.6, 0.7], \overline{F}(2) = [0.2, 0.3]$ *and* $\overline{F}(3) = [0.3, 0.4]$*. Then* $\overline{F}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy p-ideal of* X*, but it is not an interval-valued fuzzy p-ideal of* X*.*

Theorem 4.2.8. *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*- interval-valued fuzzy p-ideal of* X *if and only if* $\overline{F}_{\bar{t}}(\neq \emptyset)$ *is a p-ideal of* X *for all* $[0.5, 0.5] < \bar{t} \leq [1, 1]$.

Proof. Similar to the proof of Theorem 4.1.3. □

Remark 4.2.9. *Let* $\overline{F}$ *be an interval-valued fuzzy set of* X *and* $J = \{\bar{t} \mid \bar{a} < \bar{t} \leq \bar{b}$ *and* $\overline{F}_{\bar{t}}$ *is an empty set or a p-ideal of* $X\}$*. In particular, if* $\bar{a} = [0, 0]$ *and* $\bar{b} = [1, 1]$*, then* $\overline{F}$ *is an ordinary interval-valued fuzzy p-ideal of* X *(Theorem 3.8); if* $\bar{a} = [0, 0]$ *and* $\bar{b} = [0.5, 0.5]$*, then* $\overline{F}$ *is an* $(\in, \in \vee q)$*-interval-valued fuzzy p-ideal of* X *(Theorem 4.2.5); if* $\bar{a} = [0.5, 0.5]$ *and* $\bar{b} = [1, 1]$*, then* $\overline{F}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy p-ideal of* X *(Theorem 4.2.8).*

4.3. $(\in, \in\vee q)$**-interval-valued Fuzzy** q**-ideals.** *In this subsection, we introduce the concept of* $(\in, \in\vee q)$*-interval-valued fuzzy q-ideals of* X *and obtain some related properties.*

Definition 4.3.1. *An* $(\in, \in \vee q)$*-interval-valued fuzzy ideal* $\overline{F}$ *of* X *is called an* $(\in, \in\vee q)$*-interval-valued fuzzy q- ideal of* X *if*

(F7') $\overline{F}(x * z) \geq \text{rmin}\{\overline{F}(x * (y * z)), \overline{F}(y), [0.5, 0.5]\}$*, for all* $x, y, z \in X$.

Example 4.3.2. *Consider the BCI algebra* X *of Example 3.6(ii) and define an interval-valued fuzzy set* $\overline{F}$ *of* X *by* $\overline{F}(0) = [0.7, 0.8], \overline{F}(1) = [0.8, 0.9]$ *and* $\overline{F}(2) = [0.2, 0.3]$*. Then* $\overline{F}$ *is an* $(\in, \in\vee q)$*- interval-valued fuzzy q-ideal of* X*, but it is not an interval-valued fuzzy q-ideal.*

Proposition 4.3.3. *Every interval-valued fuzzy q-ideal of* X *is an* $(\in, \in\vee q)$*-interval-valued fuzzy q-ideal, but the converse is not true in general.*

Proof. Similar to the proof of Proposition 4.2.3. □

Proposition 4.3.4. *Let* $\overline{F}$ *be an* $(\in, \in\vee q)$*-interval-valued fuzzy ideal of* X*. Then the following are equivalent:*

(i) $\overline{F}$ *is an* $(\in, \in\vee q)$*- interval-valued fuzzy q-ideal.*
(ii) $\overline{F}(x * y) \geq \text{rmin}\{\overline{F}(x * (0 * y)), [0.5, 0.5]\}, \forall x, y \in X$.

Proof. the proof follows from Proposition 3.16 in [23]. □

Next, we characterize the $(\in, \in\vee q)$*-interval-valued fuzzy q-ideals by their level subsets.*

Theorem 4.3.5. *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\in, \in \vee q)$*-interval-valued fuzzy q-ideal of* X *if and only if* $\overline{F}_{\bar{t}}(\neq \emptyset)$ *is a q-ideal of* X *for all* $[0, 0] < \bar{t} \leq [0.5, 0.5]$.

Proof. Similar to the proof of Theorem 4.1.3. □

We have a corresponding result when $\overline{F}_{\bar{t}}$ *is a q-ideal of* X, *for all* $[0.5, 0.5] < \bar{t} \leq [1, 1]$.

Definition 4.3.6. *An* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy ideal* $\overline{F}$ *of* X *is called an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*- interval-valued fuzzy q-ideal of* X *if*

(F8') $\mathrm{rmax}\{\overline{F}(x * z), [0.5, 0.5]\} \geq \mathrm{rmin}\{\overline{F}(x * (y * z)), \overline{F}(y)\}$*, for all* $x, y, z \in X$*.*

Example 4.3.7. *Consider the BCI algebra* X *of Example 3.6(ii) and define an interval-valued fuzzy set* $\overline{F}$ *of* X *by* $\overline{F}(0) = [0.8, 0.9], \overline{F}(1) = [0.2, 0.3]$ *and* $\overline{F}(2) = [0.5, 0.5]$*. Then* $\overline{F}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy q-ideal of* X*, but it is not an interval-valued fuzzy q-ideal of* X*.*

Theorem 4.3.8. *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*- interval-valued fuzzy q-ideal of* X *if and only if* $\overline{F}_{\overline{t}}(\neq \emptyset)$ *is a q-ideal of* X *for all* $[0.5, 0.5] < \overline{t} \leq [1, 1]$*.*

Proof. Similar to the proof of Theorem 4.1.3. □

Remark 4.3.9. *Let* $\overline{F}$ *be an interval-valued fuzzy set of* X *and* $J = \{\overline{t} \mid \overline{a} < \overline{t} \leq \overline{b}$ *and* $\overline{F}_{\overline{t}}$ *is an empty set or a q-ideal of* $X\}$*. In particular, if* $\overline{a} = [0, 0]$ *and* $\overline{b} = [1, 1]$*, then* $\overline{F}$ *is an ordinary interval-valued fuzzy q-ideal of* X *(Theorem 3.8). If* $\overline{a} = [0, 0]$ *and* $\overline{b} = [0.5, 0.5]$*, then* $\overline{F}$ *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy q-ideal of* X *(Theorem 4.3.5). If* $\overline{a} = [0.5, 0.5]$ *and* $\overline{b} = [1, 1]$*, then* $\overline{F}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy q-ideal of* X *(Theorem 4.3.8).*

4.4. $(\in, \in \vee \mathrm{q})$**-interval-valued Fuzzy** a**-ideals.** *In this subsection, we introduce the concept of* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideals of* X *and obtain some related properties.*

Definition 4.4.1. *An* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy ideal* $\overline{F}$ *of* X *is called an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a- ideal of* X *if*

(F9') $\overline{F}(y * x) \geq \mathrm{rmin}\{\overline{F}((x * z) * (0 * y)), \overline{F}(z), [0.5, 0.5]\}$*, for all* $x, y, z \in X$*.*

Example 4.4.2. *Let* $X = \{0, 1, 2\}$ *be a proper BCI algebra with Cayley table as follows:*

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Now define an interval-valued fuzzy set $\overline{F}$ *of* X *by* $\overline{F}(0) = [0.7, 0.8]$ *and* $\overline{F}(1) = \overline{F}(2) = [0.8, 0.9]$*. Then* $\overline{F}$ *is an* $(\in, \in \vee \mathrm{q})$*- interval-valued fuzzy a-ideal of* X*, but it is not an interval-valued fuzzy a- ideal.*

Proposition 4.4.3. *Every interval-valued fuzzy a-ideal of* X *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideal, but the converse is not true in general.*

Proof. Similar to the proof of Proposition 4.2.3. □

Proposition 4.4.4. *Let* $\overline{F}$ *be an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy ideal of* X*. Then the following are equivalent:*

(i) $\overline{F}$ *is an* $(\in, \in \vee \mathrm{q})$*- interval-valued fuzzy a-ideal.*
(ii) $\overline{F}(y * (x * z)) \geq \mathrm{rmin}\{\overline{F}((x * z) * (0 * y)), [0.5, 0.5]\}, \forall x, y, z \in X.$

(iii) $\overline{F}(y * x) \geq \mathrm{rmin}\{\overline{F}(x * (0 * y)), [0.5, 0.5]\}, \forall x, y \in X.$

Proof. The theorem follows from Theorem 3.5 in [29]. □

Next, we characterize the $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideals by their level subsets.*

Theorem 4.4.5. *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideal of* X *if and only if* $\overline{F}_{\bar{t}}(\neq \emptyset)$ *is an a-ideal of* X *for all* $[0, 0] < \bar{t} \leq [0.5, 0.5]$.

Proof. Similar to the proof of Theorem 4.1.3. □

We have a corresponding result when $\overline{F}_{\bar{t}}$ *is an a-ideal of* X*, for all* $[0.5, 0.5] < \bar{t} \leq [1, 1]$.

Definition 4.4.6. *An* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy ideal* $\overline{F}$ *of* X *is called an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*- interval-valued fuzzy a-ideal of* X *if*

(F10') $\mathrm{rmax}\{\overline{F}(y*x), [0.5, 0.5]\} \geq \mathrm{rmin}\{\overline{F}((x*z)*(0*y)), \overline{F}(z)\}$*, for all* $x, y, z \in X$.

Example 4.4.7. *Consider the BCI algebra* X *of Example 3.6(i) and define an interval-valued fuzzy set* $\overline{F}$ *of* X *by* $\overline{F}(0) = \overline{F}(1) = [0.8, 0.9], \overline{F}(2) = [0.5, 0.5]$ *and* $\overline{F}(3) = [0.2, 0.3]$*. Then* $\overline{F}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy a-ideal of* X*, but it is not an interval-valued fuzzy a-ideal of* X.

Theorem 4.4.8. *An interval-valued fuzzy set* $\overline{F}$ *of* X *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*- interval-valued fuzzy a-ideal of* X *if and only if* $\overline{F}_{\bar{t}}(\neq \emptyset)$ *is an a-ideal of* X *for all* $[0.5, 0.5] < \bar{t} \leq [1, 1]$.

Proof. Similar to the proof of Theorem 4.1.3. □

Remark 4.4.9. *Let* $\overline{F}$ *be an interval-valued fuzzy set of* X *and* $J = \{\bar{t} \mid \bar{a} < \bar{t} \leq \bar{b}$ *and* $\overline{F}_{\bar{t}}$ *is an empty set or an a-ideal of* $X\}$*. In particular, if* $\bar{a} = [0, 0]$ *and* $\bar{b} = [1, 1]$*, then* $\overline{F}$ *is an ordinary interval-valued fuzzy a-ideal of* X *(Theorem 3.8). If* $\bar{a} = [0, 0]$ *and* $\bar{b} = [0.5, 0.5]$*, then* $\overline{F}$ *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideal of* X *(Theorem 4.4.5). If* $\bar{a} = [0.5, 0.5]$ *and* $\bar{b} = [1, 1]$*, then* $\overline{F}$ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-interval-valued fuzzy a-ideal of* X *(Theorem 4.4.8).*

Theorem 4.4.10. *Every* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideal of* X *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy p-ideal, but the converse may not be true.*

Proof. The theorem is an immediate consequence of Theorem 2.2(ii), 4.3.5 and 4.4.5. The last part is shown in Example 3.6 (i), where $\overline{F}$ is an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy p-ideal of X, but not an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy a-ideal of X. □

Theorem 4.4.11. *Every* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy a-ideal of* X *is an* $(\in, \in \vee \mathrm{q})$*-interval-valued fuzzy q-ideal, but the converse may not be true.*

Proof. The theorem is an immediate consequence of Theorem 2.2(ii), 4.2.5 and 5.4.5. The last part is shown in Example 3.6 (ii), where $\overline{F}$ is an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy q-ideal of X, but not an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy a-ideal of X. □

The following theorem shows the relationship between these generalized interval-valued fuzzy ideals of BCI algebras.

Theorem 4.4.12. *An interval-valued fuzzy set $\overline{F}$ of X is an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy a-ideal of X if and only if it is both an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy p-ideal and an $(\in, \in \vee \mathrm{q})$-interval-valued fuzzy q-ideal.*

Proof. Necessity: Theorem 4.4.10 and 4.4.11.
Sufficiency: Proposition 4.3.4, 4.2.4 and 4.4.4. □

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