ON m-POLAR FUZZY LIE SUBALGEBRAS

Muhammad Akram

Department of Mathematics
University of the Punjab
New Campus, Lahore
Pakistan
e-mail: makrammath@yahoo.com, m.akram@pucit.edu.pk

Adeel Farooq

Department of Mathematics
COMSATS Institute of Information Technology
Lahore
Pakistan
e-mail: adeelfarooq@ciitlahore.edu.pk

K.P. Shum

Institute of Mathematics
Yunnan University
China
e-mail: kpshum@ynu.edu.cn

Abstract. The notion of an m-polar fuzzy set is a generalization of a bipolar fuzzy set. We apply the concept of m-polar fuzzy sets to Lie algebras. We introduce the concept of m-polar fuzzy Lie subalgebras of a Lie algebra and investigate some of their properties. We also present the homomorphisms between the Lie subalgebras of a Lie algebra and their relationship between the domains and the co-domains of the m-polar fuzzy subalgebras under these homomorphisms.

Keywords: m-polar fuzzy Lie subalgebras, m-polar fuzzy Lie ideal, homomorphisms, Cartesian product.
Subject Classification (2000): 04A72, 17B99.

1. Introduction

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain *smooth* subgroups of general linear groups. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. There are many applications of Lie algebras in many branches of mathematics and physics.

In 1965, Zadeh [12] introduced the concept of fuzzy subset of a set. A fuzzy set on a given set X is a mapping $A : X \to [0, 1]$. In 1994, Zhang [13] extended the idea of a fuzzy set and defined the notion of bipolar fuzzy set on a given set X as a mapping $A : X \to [-1, 1]$, where the membership degree 0 of an element x means that the element x is irrelevant to the corresponding property, the membership degree in $(0, 1]$ of an element x indicates that the element satisfies the property, and the membership degree in $[-1, 0)$ of an element x indicates that the element somewhat satisfies the implicit counter-property. In 2014, Chen et al. [6] introduced the notion of m-polar fuzzy sets as a generalization of bipolar fuzzy set and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain concisely one from the corresponding one in [6]. The idea behind this is that "multipolar information" (not just bipolar information which correspond to two-valued logic) exists because data for a real world problems are sometimes from n agents ($n \geq 2$). For example, the exact degree of telecommunication safety of mankind is a point in $[0, 1]^n (n \approx 7 \times 10^9)$ because different person has been monitored different times. There are many examples such as truth degrees of a logic formula which are based on n logic implication operators ($n \geq 2$), similarity degrees of two logic formula which are based on n logic implication operators ($n \geq 2$), ordering results of a magazine, ordering results of a university and inclusion degrees (accuracy measures, rough measures, approximation qualities, fuzziness measures, and decision preformation evaluations) of a rough set.

The notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were considered first in [10] by Yehia. Since then, the concepts and results of Lie algebras have been broadened to the fuzzy setting frames [1]–[5], [9]–[11]. In this paper, we introduce the concept of m-polar fuzzy Lie subalgebras of a Lie algebra and investigate some of their properties. The Cartesian product of m-polar fuzzy Lie subalgebras will be discussed. In particular, the homomorphisms between the Lie subalgebras of a Lie algebra and their relationship between the domains and the co-domains of the m-polar fuzzy subalgebras under these homomorphisms will be investigated.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper. A *Lie algebra* is a vector space $\mathscr{L}$ over a field $\mathbb{F}$ (equal to $\mathbb{R}$ or $\mathbb{C}$) on which $\mathscr{L} \times \mathscr{L} \to \mathscr{L}$ denoted by $(x, y) \to [x, y]$ is defined satisfying the following axioms:

(L1) $[x, y]$ is bilinear,

(L2) $[x, x] = 0$ for all $x \in \mathscr{L}$,

(L3) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathscr{L}$ (Jacobi identity).

Throughout this paper, $\mathscr{L}$ is a Lie algebra and $\mathbb{F}$ is a field. We note that the multiplication in a Lie algebra is not associative, i.e., it is not true in general that $[[x, y], z] = [x, [y, z]]$. But it is *anti commutative*, i.e., $[x, y] = -[y, x]$. A subspace H of $\mathscr{L}$ closed under $[\cdot, \cdot]$ will be called a *Lie subalgebra.* A subspace I of $\mathscr{L}$ with the property $[I, \mathcal{L}] \subseteq I$ is called a *Lie ideal* of $\mathscr{L}$. Obviously, any Lie ideal is a subalgebra.

Let μ be a *fuzzy set* on $\mathscr{L}$, i.e., a map $\mu : \mathscr{L} \to [0, 1]$. Then, we call a *fuzzy set* $\mu : \mathscr{L} \to [0, 1]$ a *fuzzy Lie subalgebra* [10] of $\mathscr{L}$ if the following conditions are satisfied:

(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(\alpha x) \geq \mu(x)$,

(c) $\mu([x, y]) \geq \min\{\mu(x), \mu(y)\}$,

for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

Definition 2.1. [6] An *m-polar fuzzy set* (or a $[0, 1]^m$-set) on X is a mapping $A : \mathscr{L} \to [0, 1]^m$. The set of all m-polar fuzzy sets on $\mathscr{L}$ is denoted by $m(\mathscr{L})$.

Note that $[0, 1]^m$(m-power of [0, 1]) is considered a poset with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n \mid n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$ for each $i \in m$ ($x, y \in [0, 1]^m$), and $p_i : [0, 1]^m \to [0, 1]$ is the ith projection mapping ($i \in m$). $\mathbf{0} = (0, 0, ..., 0)$ is the smallest element in $[0, 1]^m$ and $\mathbf{1} = (1, 1, ..., 1)$ is the largest element in $[0, 1]^m$.

3. m-polar fuzzy Lie subalgebras

Definition 3.1. An m-polar fuzzy set C on $\mathscr{L}$ is called an *m-polar fuzzy Lie subalgebra* if the following conditions are satisfied:

(1) $C(x + y) \geq C(x) \wedge C(y))$,

(2) $C(\alpha x) \geq C(x)$,

(3) $C([x, y]) \geq C(x) \wedge C(y))$ for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$.

That is,

(1) $p_i \circ C(x + y) \geq \inf(p_i \circ C(x), p_i \circ C(y))$,

(2) $p_i \circ C(\alpha x) \geq p_i \circ C(x)$,

(3) $p_i \circ C([x, y]) \geq \inf(p_i \circ C(x), p_i \circ C(y))$

for all $x, y \in \mathscr{L}$ and $\alpha \in \mathbb{F}$, $i = 1, 2, 3, ..., m$.

Definition 3.2. An m-polar fuzzy set C on $\mathscr{L}$ is called an *m-polar fuzzy Lie ideal* if it satisfies the conditions (1),(2) and the following additional condition:

(4) $C([x, y]) \geq C(x)$

for all $x, y \in \mathscr{L}$.

From (2) it follows that:

(5) $C(0) \geq C(x)$,

(6) $C(-x) \geq C(x)$.

Example 3.3. Let $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors. Then $\mathbb{R}^3$ with the bracket $[\cdot, \cdot]$ defined as the usual cross product, i.e., $[x, y] = x \times y$, forms a real Lie algebra. We also define an m-polar fuzzy set $C : \mathbb{R}^3 \rightarrow [0, 1]^m$ by

$$C(x, y, z) = \begin{cases} (0.6, 0.6, ..., 0.6) & \text{if } x = y = z = 0, \\ (0.2, 0.2, ..., 0.2) & \text{otherwise.} \end{cases}$$

By routine computations, we can verify that the above m-polar fuzzy set C is an m-polar fuzzy Lie subalgebra and Lie ideal of the Lie algebra $\mathbb{R}^3$.

Proposition 3.4. *Every m-polar fuzzy Lie ideal is an m-polar fuzzy Lie subalgebra.*

We note here that the converse of Proposition 3.4 does not hold in general as can be seen in the following example.

Example 3.5. Consider $\mathbb{F} = \mathbb{R}$. Let $\mathscr{L} = \mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ be the set of all 3-dimensional real vectors which forms a Lie algebra and define

$$\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$$
$$[x, y] \rightarrow x \times y,$$

where $\times$ is the usual cross product. We define an m-polar fuzzy set $C : \mathbb{R}^3 \rightarrow [0, 1]^m$ by

$$C(x, y, z) = \begin{cases} (1, 1, ..., 1) & \text{if } x = y = z = 0, \\ (0.5, 0.5, ..., 0.5) & \text{if } x \neq 0, y = z = 0, \\ (0, 0, ..., 0) & \text{otherwise.} \end{cases}$$

Then C is an m-polar fuzzy Lie subalgebra of $\mathscr{L}$ but C is not an m-polar fuzzy Lie ideal of $\mathscr{L}$ since

$$\begin{aligned} C([(1, 0, 0)\ (1, 1, 1)]) &= C(0, -1, 1) = (0, 0, ..., 0), \\ C(1, 0, 0) &= (0.5, 0.5, ..., 0.5) \end{aligned}$$

That is,

$$C([(1, 0, 0)\ (1, 1, 1)]) \ngeq C(1, 0, 0).$$

Theorem 3.6. *Let C be an m-polar fuzzy Lie subalgebra in a Lie algebra $\mathscr{L}$. Then C is an m-polar fuzzy Lie subalgebra of $\mathscr{L}$ if and only if the non-empty upper s-level cut $C_{[s]} = U(C; s) = \{x \in \mathscr{L} \mid C(x) \geq s\}$ is a Lie subalgebra of L, for all $s \in [0, 1]^m$.*

Example 3.7. Consider the group algebra $\mathbb{C}[S_3]$, where S_3 is the Symmetric group. Then $\mathbb{C}[S_3]$ assumes the structure of a Lie algebra via the bracket (commutator) operation. Clearly, the linear span of the elements $\hat{g} = g - g^{-1}$ for $g \in S_3$ is the subalgebra of $\mathbb{C}[S_3]$, which is also known as Plesken Lie algebra and denoted by $\mathscr{L}(S_3)_{\mathbb{C}}$. It is easy to see that $\mathscr{L}(S_3)_{\mathbb{C}} = Span_{\mathbb{C}}\{\widehat{(1,2,3)}\}$ and $\widehat{(1,2,3)} = (1,2,3) - (1,3,2)$.

We define an m-polar fuzzy set $C : \mathscr{L}(S_3)_{\mathbb{C}} \to [0,1]^m$ by

$$C(g) = \begin{cases} (t_1, t_2, ..., t_m), & g = \gamma(1,2,3) - \gamma(1,3,2), \text{ where } \gamma \in \mathbb{C}, g \in \mathbb{C}[S_3] \\ (s_1, s_2, ..., s_m), & \text{otherwise, where } s_i < t_i \end{cases}$$

By routine calculations, we have $\{g \in \mathbb{C}[S_3] : C(g) > (s_1, s_2, ..., s_m)\} = \mathscr{L}(S_3)_{\mathbb{C}}$. Then we see that $\mathscr{L}(S_3)_{\mathbb{C}}$ can be realized $C_{[s]}$ as an upper s_i-level cut and C is an m-polar fuzzy Lie ideal of $\mathscr{L}(S_3)_{\mathbb{C}}$.

Definition 3.8. Let C and D be two m-polar fuzzy sets of $\mathscr{L}$. We define the *sup-inf* product $[CD]$ of C and D as follows: for all x, y, $z \in \mathscr{L}$

$$[CD](x) = \begin{cases} \sup\limits_{x=[yz]} \{\inf(C(y), D(z))\} \\ \mathbf{0}, \quad \text{if x} \neq \text{[yz]}. \end{cases}$$

Let C and D be m-polar fuzzy Lie subalgebras of the Lie algebra $\mathscr{L}$. Then $[CD]$ may not be an m-polar fuzzy Lie subalgebra of $\mathscr{L}$ as this can be seen in the following counterexample:

Example 3.9. Let $\{e_1, e_2, ..., e_8\}$ be a basis of a vector space over a field $\mathbb{F}$. Then,it is not difficult to see that, by putting:

$$\begin{array}{llll} [e_1, e_2] = e_5, & [e_1, e_3] = e_6, & [e_1, e_4] = e_7, & [e_1, e_5] = -e_8, \\ [e_2, e_3] = e_8, & [e_2, e_4] = e_6, & [e_2, e_6] = -e_7, & [e_3, e_4] = -e_5, \\ [e_3, e_5] = -e_7, & [e_4, e_6] = -e_8, & [e_i, e_j] = -[e_j, e_i] & \end{array}$$

and $[e_i, e_j] = 0$ for all $i \leq j$, we can obtain a Lie algebra over a field $\mathbb{F}$. The following fuzzy sets

$$C(x) := \begin{cases} (1, 1, ..., 1) & \text{if } x \in \{0, e_1, e_5, e_6, e_7, e_8\}, \\ (0, 0, ..., 0) & \text{otherwise,} \end{cases}$$

$$D(x) := \begin{cases} (1, 1, ..., 1) & \text{if } x = 0, \\ (0.5, 0.5, ..., 0.5) & \text{if } x \in \{e_2, e_5, e_6, e_7, e_8\}, \\ (0, 0, ..., 0) & \text{otherwise,} \end{cases}$$

are clearly fuzzy Lie subalgebras of a Lie algebra $\mathscr{L}$. Thus C and D are m-polar fuzzy Lie subalgebras of $\mathscr{L}$ because the level Lie subalgebras

$$\begin{aligned} U(C;(1,1,...,1)) &= \langle e_1,e_5,e_6,e_7,e_8\rangle, \\ U(D;(0.5,0.5,...,0.5)) &= \langle e_2,e_5,e_6,e_7,e_8\rangle \end{aligned}$$

are Lie subalgebras of $\mathscr{L}$. But $[CD]$ is not an m-polar fuzzy Lie subalgebra because the following condition does not hold:

$$[CD](e_7+e_8)\geq \inf\{[CD](e_7),[CD](e_8)\}.$$

$$(1)\ [CD](e_7)=\sup\begin{cases} \inf\{C(e_1),D(e_4)\}=(0,0,...,0), & e_7=[e_1,e_4], \\ \inf\{C(e_2),D(e_6)\}=(0,0,...,0), & e_7=-[e_2,e_6], \\ \inf\{C(e_3),D(e_5)\}=(0,0,...,0), & e_7=-[e_3,e_5], \\ \inf\{C(e_4),D(e_1)\}=(0,0,...,0), & e_7=-[e_4,e_1], \\ \inf\{C(e_6),D(e_2)\}=(0.5,0.5,...,0.5), & e_7=[e_6,e_2], \\ \inf\{C(e_5),D(e_3)\}=(0,0,...,0), & e_7=[e_5,e_3]. \end{cases}$$

Thus $[CD](e_7)=(0.5,0.5,...,0.5)$.

(2) By using similar arguments, we can show that $[CD](e_8)=(0.5,0.5,\cdots,0.5)$.

(3) $[CD](e_7+e_8)=\sup\{(i)-(vi)\}$

(i) if $e_7+e_8=[e_1(e_4-e_5)]$, then $\inf\{C(e_1),D(e_4-e_5)\}=\inf\{C(e_1),D(e_4),D(e_5)\}=(0,0,...,0)$, since $D(e_4)=(0,0,...,0)$, and if $e_7+e_8=[(e_5-e_4)e_1]$, then, $\inf\{C(e_5-e_4),D(e_1)\}=\inf\{C(e_5),D(e_4),D(e_1)\}=(0,0,...,0)$, since $C(e_4)=(0,0,...,0)$.

By using similar method, we can also obtain the following numerical results:

(ii) If $e_7+e_8=[e_2(e_3-e_6)]$, then $\inf(C(e_2),D(e_3-e_6))=(0,0,...,0)$.

(iii) If $e_7+e_8=[e_3(-e_2-e_5)]$, then $\inf(C(e_3),D(e_2-e_5))=(0,0,...,0)$.

(iv) If $e_7+e_8=[e_4(-e_1-e_6)]$, then $\inf(C(e_4),D(-e_3-e_1))=(0,0,...,0)$.

(v) If $e_7+e_8=[e_5(-e_3-e_1)]$, then $\inf(C(e_5),D(-e_3-e_1))=(0,0,...,0)$.

(vi) If $e_7+e_8=[e_6(-e_2-e_4)]$, then $\inf(C(e_6),D(-e_2-e_4))=(0,0,...,0)$.

Thus, $[CD](e_7+e_8)=\sup\{(0,0,...,0),(0,0,...,0),(0,0,...,0),(0,0,...,0),(0,0,...,0),(0,0,...,0)\}=(0,0,...,0)$. Hence, we have proved that

$$[CD](e_7+e_8)\ngeq \inf\{[CD](e_7),[CD](e_8)\}.$$

We now refine the product of two m-polar fuzzy Lie subalgebras C and D of $\mathscr{L}$ to an extended form.

Definition 3.10. Let C and D be two m-polar fuzzy sets of $\mathscr{L}$. Then,we define the *sup-inf* product $\ll CD \gg$ of C and D as follows, for all x, y, $z \in \mathscr{L}$

$$\ll CD \gg (x) = \begin{cases} \sup\limits_{x=\sum\limits_{i=1}^{n}[x_i y_i]} \left\{ \inf\limits_{i \in \mathbb{N}} \{\inf(C(x_i), D(y_i))\} \right\} \\ \mathbf{0}, \ \ \text{if } \mathrm{x} \neq \sum\limits_{\mathrm{i}=1}^{\mathrm{n}} [\mathrm{x_i y_i}]. \end{cases}$$

From the definitions of $[CD]$ and $\ll CD \gg$, we can easily see that $[CD] \subseteq \ll CD \gg$ and $[CD] \neq \ll CD \gg$ hold generally even if C and D are both m-polar fuzzy Lie subalgebras of $\mathscr{L}$, and in this case, $\ll CD \gg$ is also an m-polar fuzzy Lie subalgebra of $\mathscr{L}$.

Theorem 3.11. *Let C be an m-polar fuzzy Lie subalgebra of Lie algebra $\mathscr{L}$. Define a binary relation $\sim$ on $\mathscr{L}$ by $x \sim y$ if and only if $C(x-y) = C(0)$ for all x, $y \in \mathscr{L}$. Then $\sim$ is a congruence relation on $\mathscr{L}$.*

Proof. We first prove that "$\sim$" is an equivalent relation.We only need to show the transitivity of "$\sim$" because the reflectivity and symmetry of "$\sim$" hold trivially. Let x, y, $z \in \mathscr{L}$, If $x \sim y$ and $y \sim z$, then $C(x-y) = C(0)$, $C(y-z) = C(0)$. Hence it follows that

$$C(x-z) \ = \ C(x-y+y-z) \geq \inf(C(x-y), C(y-z)) = C(0),$$

Consequently $x \sim z$. We now verify that "$\sim$" is a congruence relation on $\mathscr{L}$. For this purpose,we let $x \sim y$ and $y \sim z$. Then $C(x-y) = C(0)$, $C(y-z) = C(0)$. Now, for x_1, x_2, y_1, $y_2 \in \mathscr{L}$, we have

$$\begin{aligned} C((x_1+x_2)-(y_1+y_2)) &= C((x_1-y_1)+(x_2-y_2)) \\ &\geq \inf(C(x_1-y_1), C(x_2-y_2)) = C(0), \end{aligned}$$

$$\begin{aligned} C((\alpha x_1 - \alpha y_1) &= C(\alpha(x_1-y_1)) \geq C(x_1-y_1) = C(0), \\ C([x_1,x_2]-[y_1,y_2]) &= C([x_1-y_1],[x_2-y_2]) \\ &\geq \inf\{C(x_1-y_1), C(x_2-y_2)\} = C(0). \end{aligned}$$

That is, $x_1+x_2 \sim y_1+y_2$, $\alpha x_1 \sim \alpha y_1$ and $[x_1,x_2] \sim [y_1,y_2]$. Thus, "$\sim$" is indeed a congruence relation on $\mathscr{L}$. ∎

Definition 3.12. Let C be an m -polar fuzzy set on a set $\mathscr{L}$. An *m-polar fuzzy relation* on C is an m-polar fuzzy set D of $\mathscr{L} \times \mathscr{L}$ such that $D(xy) \leq \inf(C(x), C(y))$ $\forall x, y \in \mathscr{L}$.

Definition 3.13. Let C and D be m-polar fuzzy sets on a set $\mathscr{L}$. If C is an m-polar fuzzy relation on a set $\mathscr{L}$, then C is said to be an *m-polar fuzzy relation* on D if $C(x,y) \leq \inf(D(x), D(y))$ for all x, $y \in \mathscr{L}$.

Theorem 3.14. *Let C and D be two m-polar fuzzy Lie subalgebras of a Lie algebras $\mathscr{L}$. Then $C \times D$ is an m-polar fuzzy Lie subalgebra of $\mathscr{L} \times \mathscr{L}$.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in \mathscr{L} \times \mathscr{L}$. Then

$$\begin{aligned}
(C \times D)(x+y) &= (C \times D)((x_1, x_2) + (y_1, y_2)) \\
&= (C \times D)((x_1 + y_1, x_2 + y_2)) \\
&= \inf(C(x_1 + y_1), D(x_2 + y_2)) \\
&\geq \inf(\inf(C(x_1), C(y_1)), T(D(x_2), D(y_2))) \\
&= \inf(\inf(C(x_1), D(x_2)), \inf(C(y_1), D(y_2))) \\
&= \inf((C \times D)(x_1, x_2)), (C \times D)(y_1, y_2)) \\
&= \inf((C \times D)(x), (C \times D)(y)),
\end{aligned}$$

$$\begin{aligned}
(C \times D)(\alpha x) &= (C \times D)(\alpha(x_1, x_2)) = (C \times D)((\alpha x_1, \alpha x_2)) \\
&= \inf(C(\alpha x_1), D(\alpha x_2)) \geq \inf(C(x_1), D(x_2)) \\
&= (C \times D)(x_1, x_2) = (C \times D)(x),
\end{aligned}$$

$$\begin{aligned}
(C \times D)([x, y]) &= (C \times D)([(x_1, x_2), (y_1, y_2)]) \\
&\geq \inf(\inf(C(x_1), D(x_2)), \inf(C(y_1), D(y_2))) \\
&= \inf((C \times D)(x_1, x_2)), (C \times D)(y_1, y_2)) \\
&= \inf((C \times D)(x), (C \times D)(y)),
\end{aligned}$$

This shows that $C \times D$ is an m-polar fuzzy Lie subalgebra of $\mathscr{L} \times \mathscr{L}$. ■

Definition 3.15. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two Lie algebras over a field $\mathbb{F}$. Then a linear transformation $f : \mathscr{L}_1 \to \mathscr{L}_2$ is called a *Lie homomorphism* if $f([x, y]) = [f(x), f(y)]$ holds for all x, $y \in \mathscr{L}_1$.

For the Lie algebras $\mathscr{L}_1$ and $\mathscr{L}_2$, it can be easily observed that if $f : \mathscr{L}_1 \to \mathscr{L}_2$ is a Lie homomorphism and C is an m-polar fuzzy Lie subalgebra of $\mathscr{L}_2$, then the m-polar fuzzy set $f^{-1}(CA)$ of $\mathscr{L}_1$ is also an m-polar fuzzy Lie subalgebra.

Definition 3.16. Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two Lie algebras. Then, a Lie homomorphism $f : \mathscr{L}_1 \to \mathscr{L}_2$ is said to have a natural extension $f : I^{\mathscr{L}_1} \to I^{\mathscr{L}_2}$ defined by for all $C \in I^{\mathscr{L}_1}$, $y \in \mathscr{L}_2$:

$$f(C)(y) = \sup\{C(x) : x \in f^{-1}(y)\}.$$

We now call these sets the homomorphic images of the m-polar fuzzy set C.

Theorem 3.17. *The homomorphic image of an m-polar fuzzy Lie subalgebra is still an m-polar fuzzy Lie subalgebra of its co-domain.*

Proof. Let y_1, $y_2 \in \mathscr{L}_2$. Then

$$\{x \mid x \in f^{-1}(y_1 + y_2)\} \supseteq \{x_1 + x_2 \mid x_1 \in f^{-1}(y_1) \; and, x_2 \in f^{-1}(y_2)\}.$$

Now, we have

$$\begin{aligned} f(C)(y_1+y_2) &= \sup\{C(x) \mid x \in f^{-1}(y_1+y_2)\} \\ &\geq \{C(x_1+x_2), \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\inf\{C(x_1), C(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &= \inf\{\sup\{C(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{C(x_2) \mid x_2 \in f^{-1}(y_2)\}\} \\ &= \inf\{f(C)(y_1), f(C)(y_2)\}. \end{aligned}$$

For $y \in \mathscr{L}_2$ and $\alpha \in \mathbb{F}$, we have

$$\{x \mid x \in f^{-1}(\alpha y)\} \supseteq \{\alpha x \mid x \in f^{-1}(y)\}.$$

$$\begin{aligned} f(C)(\alpha y) &= \sup\{C(\alpha x) \mid x \in f^{-1}(y)\} \\ &\geq \{C(\alpha x) \mid x \in f^{-1}(\alpha y)\} \\ &\geq \sup\{C(x) \mid x \in f^{-1}(y)\} \\ &= f(C)(y), \end{aligned}$$

If y_1, $y_2 \in \mathscr{L}_2$ then

$$\{x \mid x \in f^{-1}([y_1, y_2])\} \supseteq \{[x_1, x_2] \mid x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\}.$$

Now

$$\begin{aligned} f(C)([y_1, y_2]) &= \sup\{C(x) \mid x \in f^{-1}([y_1, y_2])\} \\ &\geq \{C([x_1, x_2]) \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &\geq \sup\{\inf\{C(x_1), C(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\ &= \inf\{\sup\{C(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{C(x_2) \mid x_2 \in f^{-1}(y_2)\}\} \\ &= \inf\{f(C)(y_1), f(C)(y_2)\}. \end{aligned}$$

Thus, $f(C)$ is a fuzzy Lie algebra of $\mathscr{L}_2$. ∎

Theorem 3.18. *Let $f : \mathscr{L}_1 \to \mathscr{L}_2$ be a surjective Lie homomorphism. If C and D are m-polar fuzzy Lie subalgebras of $\mathscr{L}_1$ then $f(\ll CD \gg) = \ll f(C)f(D) \gg$.*

Proof. Assume that $f(\ll CD \gg) < \ll f(C)f(D) \gg$. Now, we choose a number $t \in [0,1]$ such that $f(\ll CD \gg)(x) < t < \ll f(C)f(D) \gg (x)$. Then,there exist y_i, $z_i \in \mathscr{L}_2$ such that $x = \sum_{i=1}^{n}[y_i z_i]$ with $f(C)(y_i) > t$ and $f(D)(z_i) > t$. Since f is surjective, there exists $y \in \mathscr{L}_1$ such that $f(y) = x$ and $y = \sum_{i=1}^{n}[a_i b_i]$ for some $a_i \in f^{-1}(y_i)$, $b_i \in f^{-1}(z_i)$ with $f(a_i) = y_i$, $f(b_i) = z_i$, $C(a_i) > t$ and $D(b_i) > t$. Since $f(\sum_{i=1}^{n}[a_i b_i]) = [\sum_{i=1}^{n} f([a_i b_i]) = [\sum_{i=1}^{n}[f(a_i)f(b_i)] = [\sum_{i=1}^{n}[y_i z_i] = x$, we have $f(\ll CD \gg)(x) > t$. This is a contradiction. Similarly, for the case $f(\ll CD \gg) > \ll f(C)f(D) \gg$, we can also obtain a contradiction. Hence, $f(\ll CD \gg) = \ll f(C)f(D) \gg$. ∎

Definition 3.19. Let C and D be m-polar fuzzy subalgebras of $\mathscr{L}$. Then C is said to be of the same type of D if there exists $f \in Aut(L)$ such that $C = D \circ f$, i.e., $C(x) = D(f(x))$ for all $x \in \mathscr{L}$.

Theorem 3.20. *Let C and D be two m-polar fuzzy subalgebras of $\mathscr{L}$. Then C is an m-polar fuzzy subalgebra having the same type of D if and only if C is isomorphic to D.*

Proof. We only need to prove the necessity part because the sufficiency part is trivial. Let C be an m-polar fuzzy subalgebra having the same type of D. Then there exists $\phi \in Aut(L)$ such that $C(x) = D(\phi(x))\ \ \forall x \in \mathscr{L}$.

Let $f : C(L) \to D(L)$ be a mapping defined by $f(C(x)) = D(\phi(x))$ for all $x \in \mathscr{L}$, that is, $f(C(x)) = D(\phi(x)) \quad \forall x \in \mathscr{L}$. Then, it is clear that f is surjective. Also, f is injective because if $f(C(x)) = f(C(y))$ for all $x, y \in \mathscr{L}$, then $D(\phi(x)) = D(\phi(y))$ and hence $C(x) = D(y)$. Finally, f is a homomorphism because for $x, y \in \mathscr{L}$,

$$\begin{aligned} f(C(x+y)) &= D(\phi(x+y)) = D(\phi(x)+\phi(y)), \\ f(C(\alpha x)) &= D(\phi(\alpha x)) = \alpha D(\phi(x)), \\ f(C([x,y])) &= D(\phi([x,y])) = D([\phi(x),\phi(y)]). \end{aligned}$$

Hence C is isomorphic to D. This completes the proof. ∎

References

[1] Akram, M., *Anti fuzzy Lie ideals of Lie algebras*, Quasigroups and Related Systems **14** (2006) 123-132.

[2] Akram, M., *Intuitionistic fuzzy Lie ideals of Lie algebras*, Journal of Fuzzy Mathematics **16**(4)(2008) 991-1008.

[3] Akram, M., *Generalized fuzzy Lie subalgebras*, Journal of Generalized Lie Theory and Applications **2**(4) (2008)261-268.

[4] Akram, M., *Fuzzy Lie ideals of Lie algebras with interval-valued membership function*, Quasigroups and Related Systems **16**(1) (2008) 1-12.

[5] Akram, M., Shum, K.P., *Intuitionistic fuzzy Lie algebras*, Southeast Asian Bulletin of Mathematics **31**(2007) 843-855.

[6] Chen, J., Li, S.-G., Ma, S., Wang, X., *m-polar fuzzy sets: An extention of m-polar fuzzy sets*, The Scientific World Journal, Vol. 2014, pp. 8, 2014.

[7] Coelho, P., Nunes, U., *Lie algebra application to mobile robot control: a tutorial*, Robotica **21**(2003) 483-493.

[8] Humphreys, J.E., *Introduction to Lie Algebras and Representation Theory*, Springer, New York 1972.

[9] Kim, C.G., Lee, D.S., *Fuzzy Lie ideals and fuzzy Lie subalgebras*, Fuzzy Sets and Systems **94** (1998) 101-107.

[10] Yehia, S.E., *Fuzzy ideals and fuzzy subalgebras of Lie algebras*, Fuzzy Sets and Systems **80**(1996) 237-244.

[11] Yehia, S.E., *The adjoint representation of fuzzy Lie algebras*, Fuzzy Sets and Systems, **119**(2001)409-417.

[12] Zadeh, L.A., *Fuzzy sets*, Information and Control **8**(1965) 338-353.

[13] Zhang, W.R., *Bipolar fuzzy sets and relations: a computational framework forcognitive modeling and multiagent decision analysis*, Proc. of IEEE conf., (1994), pp. 305-309.

[14] Zhang, W.R., *Bipolar fuzzy sets*, Proc. of FUZZ-IEEE, (1998), pp. 835-840.

Accepted: 11.01.2016