Counting polyominoes of size 50

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1 BASICS

The OEIS lists the following sequences which enumerate distinct symmetries of polyomino[1], and whose "sum" is consequently equal to A000105, which enumerates all free polyominoes. In the following table, an acronym has been added to each case, for simple reference. The column "Fixed" gives the contribution that each symmetry gives to the count of fixed polyominoes.

Sequence	**Redelmeier[2]**		**Acronym**	**Fixed**	**Polyominoes enumerated**
	Free	Fixed			
A006749	none	N	ASYM	8	No symmetry
A006746	axis	H/V	M90	4	Vertical or horizontal reflective symmetry but no other
A006748	diag	A/D	M45	4	Reflective on just one diagonal and no other symmetry
A006747	rot	R	R180	4	180° rotational symmetry but no other
A056877	axis 2	HVR	BothM90	2	Vertical and horizontal reflective symmetries, 180° rotational symmetry, but no reflective symmetry on a diagonal
A056878	diag 2	ADR	BothM45	2	Reflective on both diagonals, 180° rotational symmetry, but without 90° rotational symmetry
A144553	rot 2	R2	R90	2	With 90° rotational symmetry but no reflective symmetry
A142886	all	HVADR2	ALL	1	With all symmetries

In addition, A001168 enumerates the fixed polyominoes, and A000988 the one-sided.

Therefore 2 simple formulae[3] may be stated.

Formula 1:
Free(n) = ASYM(n) + M90(n) + M45(n) + R180(n) + BothM90(n) + BothM45(n) + R90(n) + ALL(n)

Formula 2:
Fixed(n) = 8*ASYM(n) + 4*M90(n) + 4*M45(n) + 4*R180(n) + 2*BothM90(n) + 2*BothM45(n) + 2*R90(n) + ALL(n)

[1] https://en.wikipedia.org/wiki/Polyomino

[2] Counting Polyominoes: yet another attack. D. Hugh Redelmeier 1980

[3] "The correct Latin plural form of formula is 'formulae', although the less pretentious-sounding 'formulas' is used more commonly." Eric W.Weisstein, CRC Concise Encyclopedia of Mathematics.

As a consequence, eliminating ASYM(n), we have
Formula 3:
Free(n) = (4*M90(n) + 4*M45(n) + 4*R180(n) + 6*BothM90(n) + 6*BothM45(n) + 6*R180(n) + 7*ALL(n) + Fixed(n))/8

Alternatively:
A000105(n) = (4*A006746(n) + 4*A006748(n) + 4*A006747(n) + 6*A056877(n) + 6*A056878(n) + 6*A144553(n) + 7*A142886(n) + A001168(n))/8

As of January 2023, these sequences had been published through to the following maximum indexes:

Sequence	Content	n	a(n)	Contributor
A000105	Free	48	135629410647775553284438364	Mason
A001168	Fixed	56	69150714562532896936574425480218	Jensen
A006746	M90	48	13263652614595	Mason
A006748	M45	48	2614239679253	Mason
A006747	R180	48	13302247932738	Mason
A056877	BothM90	81	154671306896	Russell
A056878	BothM45	87	63185177777	Russell
A144553	R90	93	489906241235	Russell
A142886	ALL	161	29256182414	Russell

In order to calculate Free (49) and Free (50) using Formula 3, it was therefore sufficient to extend to n=50 the 3 sequences corresponding to M90, M45 and R180.

Two of these sequences are available in a more granular form according to the following table:

Sequence	Acronym	Polyominoes enumerated			
A006746	M90	Vertical or horizontal reflective symmetry but no other			
		A349328	M90C	VI/HI[4]	The axis of symmetry passes through the centre of a square of the lattice
		A349329	M90V	VX/HX	The axis of symmetry passes through the vertex of a square
A006747	R180	180° rotational symmetry but no other			
		A351615	R180C	RII	The axis of symmetry coincides with the centre of a square
		A234008	R180M	RXI/RIX	The axis of symmetry coincides with the midpoint of an edge of a square
		A351616	R180V	RXX	The axis of symmetry coincides with a vertex of a square

[4] Redelmeier

The following table gives the availability of terms for these granular sequences. Some of them refer to polyominoes of size 2n, and so two columns are given to indicate their completeness with respect to the job in hand.

Sequence	Content	n	Polyomino size	a(n)
A349328	M90C	48	48	10643659275967
A349329	M90V	48	96	542517642591030589186344612
A351615	R180C	32	32	8058790
A234008	R180M	18	36	2741031257
A351616	R180V	16	32	35665587

M90V is easily calculated according to the formula:
M90V(n) = 2*(R180(2n) + M45(2n)) + R90(2n) + BothM45(2n) + M90(2n) + 4*ASYM(2n)

The following are therefore the sequences that need to be extended in order to calculate Free(n) through to n=50:

Sequence	Content
A349328	M90C
A351615	R180C
A234008	R180M
A351616	R180V
A006748	M45

The enumeration in these cases was programmed according to the basic principles outlined in Redelmeier's paper to which reference should be made for a more complete explanation:

1. From the starting cell, choose an adjacent square. Then explore recursively the two possibile forks: (i) that square is occupied, and (ii) that square is not occupied. This method will generally count fixed polyominoes.
2. For R180 polyominoes, start with the set of R180 rings and for each one, grow inwards and/or outwards until the required size of polyomino has been built. Note that many rings, like the degenerate 2x1 ring (aka "domino"), permit only outer growth. The implementation of this algorithm has as a prerequisite the generation of all R180 rings through to the target size. The R180C rings have sizes 1, 8, 12, …, 48. For R180M, sizes 2, 10, 14, 18, …, 50. For R180V, sizes 4, 8, 12, …, 48.

Principle 1 may be optimised in the calculation of M90C polyominoes. Consider as the starting point the lowest square on the (assume) vertical axis of symmetry. In this case:

a. There is no need to count the cells on the left; any cell to the right of the starting cell corresponds automatically to another square on the left.
b. There is no need to extrapolate squares directly below the starting point, which is, by definition, the lowest square of that column.
c. As we are counting polyominoes that are exclusively M90, we must test for and exclude any polyominoes that have greater symmetry.
d. The end result must be divided by 2 to obtain the count of free M90C polyominoes.

A similar approach applies to M45 polyomino counting: consider as the starting point the lowest square on the (assume) south-west to north-east diagonal.

Possible improvement

The approach, as outlined above, aims to count all polyominoes that have exclusively some specific symmetry S (e.g., M45). In order to obtain that, during propagation, each polyomino must be tested for symmetry S+ (e.g., BothM45) so that those having S+ can be eliminated from the count. Testing for S+ has a cost, estimated in one case to be around 25% of total runtime.

A different approach that was not implemented[5], is to not test for S+. The resulting count of S is therefore polluted by the count of S+. This can be corrected by counting separately polyominoes satisfying S+ which are far fewer and therefore contribute a negligible extra runtime.

In other words, if S(n) is the number of polyominoes of size n satisfying precisely S, and similarly for S+(n), then S+(n) / S(n) is so small that the extra cost of counting S+(n) is negligible with respect to the time saving of not testing S(n) polyominoes for symmetry S+.

The choice to be made therefore is between extra implementation and longer runtimes.

In some cases, no extra implementation is needed. After the activity was already complete, I verified that if M90C polyominoes were counted as described above, without testing for symmetry, it would be sufficient to subtract 2*A351190(n) + A351127(n) + A346799(n/2) and obtain the correct result. This would have saved 30 days of CPU, if the above 25% estimate is to be believed.

[5] Although in another context it had been suggested to me by Walter Trump.

2 SUBDIVISION OF ACTIVITIES

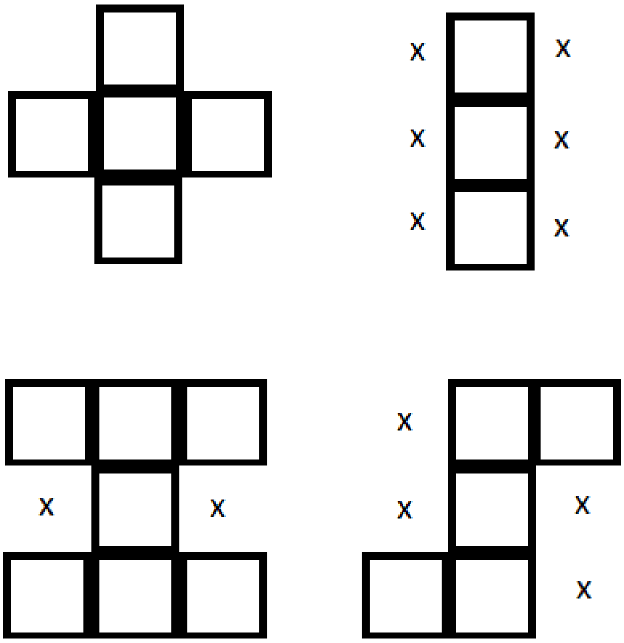
In order to obtain results for Free(50) in “reasonable” times, it was chosen to subdivide the activities over multiple (5) servers, each with 2 CPU’s, according to the following table, which shows running time, the number of the term reached, and the maximum value reached. The explanation of the phases follows the table.

Sequence	Job	Days	#term	Max term (billion)
A351615	R180 polyominoes about 1x1 ring phase A	24	49	2464
A351615	R180 polyominoes about 1x1 ring phase B	9	49	1968
A351615	R180 polyominoes about 1x1 ring phase C	12	49	5811
A351615	R180 polyominoes about 1x1 ring phase D	14	49	2924
A351615	R180 polyominoes about 3x3 ring	11	50	1138
A351615	R180 polyominoes about 5x3 ring	2	50	337
A351615	R180 polyominoes about rings, with axis of rotation at the centre of a cell of the square lattice, of sizes 16 through 48	2	50	647
	Total	75		
A234008	R180 polyominoes about 2x1 ring phase A	46	50	22820
A234008	R180 polyominoes about 2x1 ring phase B	25	50	5876
A234008	R180 polyominoes about 2x1 ring phase C	51	50	11245
A234008	R180 polyominoes about 4x3 ring	4	50	878
A234008	R180 polyominoes about rings, with axis of rotation at the middle of an edge of a square of the square lattice, of sizes 14 through 50	5	50	1365
	Total	131		
A351616	R180 polyominoes about 2x2 ring phase A	6	50	2919
A351616	R180 polyominoes about 2x2 ring phase B	5	50	1055
A351616	R180 polyominoes about 2x2 ring phase C	4	50	890
A351616	R180 polyominoes about 2x2 ring phase D	6	50	1558
A351616	R180 polyominoes about 2x2 ring phase E	2	50	1046
A351616	R180 polyominoes about 2x2 ring phase F	1	50	88
A351616	R180 polyominoes about rings, with axis of rotation at a vertex of a square of the square lattice, of sizes 12 through 48	6	50	1058
	Total	30		
A349328	M90C polyominoes phase A1	17	50	9479
A349328	M90C polyominoes phase A2	12	50	7270
A349328	M90C polyominoes phase A3	20	50	12214
A349328	M90C polyominoes phase B	32	50	20340
A349328	M90C polyominoes phase E[6]	16	50	9777
A349328	M90C polyominoes phase F	10	50	9301
A349328	M90C polyominoes phase G	25	50	15873
	Total	131		
A006748	M45 polyominoes	42	50	10251
Total		409		

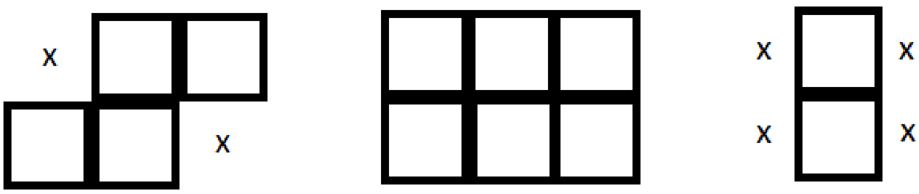
[6] Phases ABEFG instead of ABCDE? Because.

- R180 1x1 ring

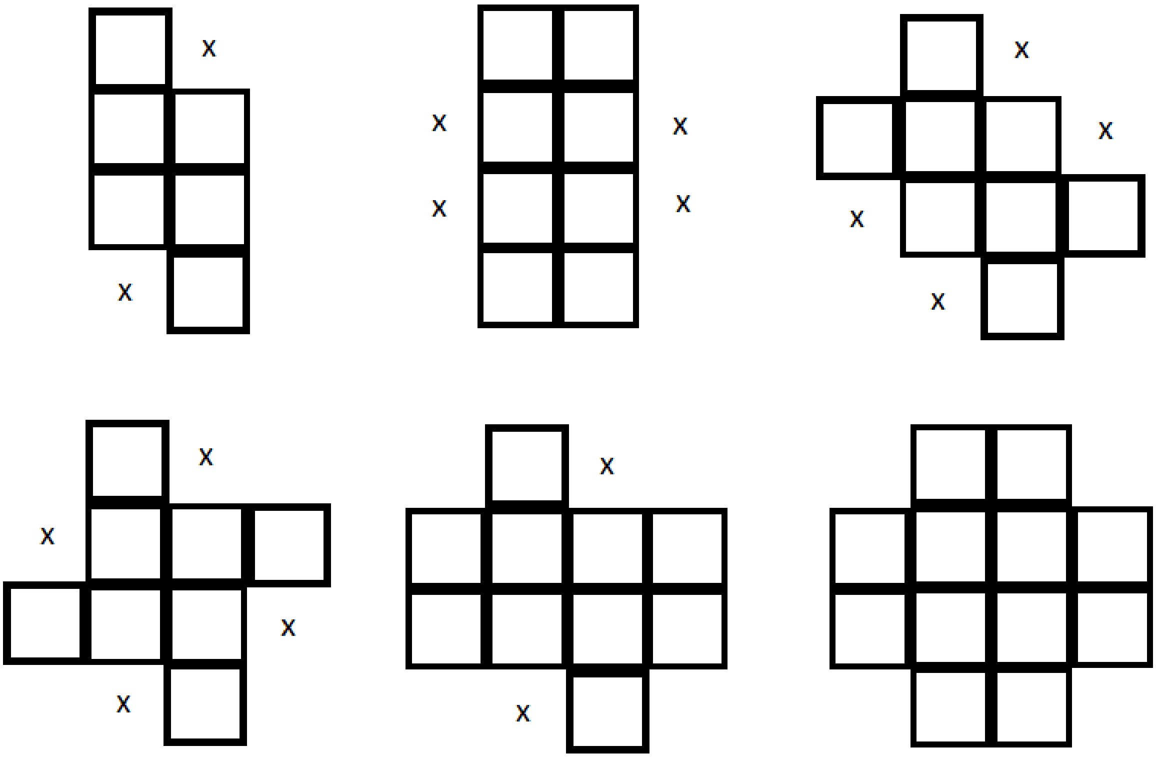
Any R180C polyomino (except for the very smallest) about such a ring may be generated from one of the following formations, which are the starting templates for phases A, B, C and D. In the diagram, an “x” indicates an unoccupiable position.



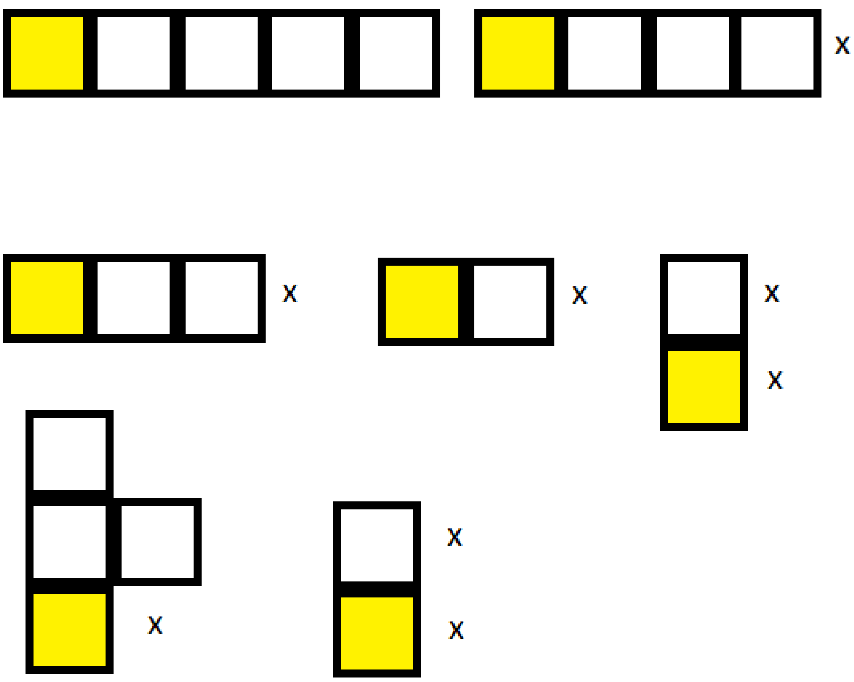
- R180 2x1 ring



- R180 2x2 ring



- M90C. The yellow square is the lowest square on the vertical reflective axis. The squares to the left are not shown.



3 INTERMEDIATE RESULTS.

These tables show the intermediate results of the activity. The individual columns are not of interest in themselves but are included as they could be useful for comparison with any future activity that follows the same approach. All referenced sequences will be proposed for update in the OEIS, including those not listed in this chapter.

R180C:

	A	B	C	D	3x3	3x5	Other rings	A351615
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5[7]	0	0	0	0	0	0	0	1
6	0	0	0	0	0	0	0	0
7	0	0	3	1	0	0	0	4
8	0	0	0	0	0	0	0	0
9	3	2	10	4	0	0	0	19
10	0	0	0	0	1	0	0	1
11	12	9	36	16	0	0	0	73
12	0	0	0	0	4	0	0	4
13	50	40	132	61	0	0	0	283
14	0	0	0	0	22	3	0	25
15	203	158	494	235	0	0	0	1090
16	0	0	0	0	88	17	1	106
17	789	623	1871	900	0	0	0	4183
18	0	0	0	0	357	82	24	463
19	3064	2421	7150	3470	0	0	0	16105
20	0	0	0	0	1390	349	153	1892
21	11838	9405	27506	13421	0	0	0	62170
22	0	0	0	0	5436	1442	874	7752
23	45882	36514	106371	52140	0	0	0	240907
24	0	0	0	0	21103	5800	4309	31212
25	178065	142036	413116	203230	0	0	0	936447
26	0	0	0	0	82204	23101	20304	125609
27	693280	553562	1610156	794620	0	0	0	3651618
28	0	0	0	0	320310	91419	91436	503165
29	2705653	2162285	6294613	3114968	0	0	0	14277519
30	0	0	0	0	1251358	360933	402535	2014826
31	10586810	8464089	24671223	12238996	0	0	0	55961118
32	0	0	0	0	4897394	1423386	1738010	8058790
33	41515621	33199714	96914243	48183986	0	0	0	219813564
34	0	0	0	0	19207904	5613059	7410934	32231897
35	163139380	130467704	381453173	190031719	0	0	0	865091976
36	0	0	0	0	75471479	22143430	31282338	128897247
37	642243613	513599078	1504021317	750634438	0	0	0	3410498446
38	0	0	0	0	297067521	87416522	131070724	515554767
39	2532595494	2025039166	5939429035	2969198892	0	0	0	13466262587
40	0	0	0	0	1171156174	345382419	545906470	2062445063
41	10001916720	7996009899	23487932469	11759736322	0	0	0	53245595410
42	0	0	0	0	4623955969	1365827557	2263084550	8252868076
43	39554296662	31614825927	93002750945	46628559789	0	0	0	210800433323
44	0	0	0	0	18280744613	5406137046	9346064396	33032946055
45	156618454956	125152206695	368678178183	185079424893	0	0	0	835528264727
46	0	0	0	0	72361836687	21417599907	38478104961	132257541555
47	620847564746	495990984221	1463039730698	735322158088	0	0	0	3315200437753
48	0	0	0	0	286758622300	84925184710	158010966276	529694773286
49	2463647402752	1967698110611	5811440662114	2923997564400	0	0	0	13166783739877
50					1137565465756	337031309843	647497468513	2122094244112

[7] A351615(5) is forced to 1 as the specific polyomino is not generated by the formations indicated above.

R180V:

	A	B	C	D	E	F	Other rings	A351616[8]
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6[9]	0	0	0	0	0	0	0	1
7	0	0	0	0	0	0	0	0
8	2	0	0	0	0	0	0	2
9	0	0	0	0	0	0	0	0
10	6	2	1	2	0	0	0	12
11	0	0	0	0	0	0	0	0
12	20	6	5	7	5	0	0	43
13	0	0	0	0	0	0	0	0
14	71	25	19	32	21	1	5	174
15	0	0	0	0	0	0	0	0
16	259	90	73	122	84	5	24	657
17	0	0	0	0	0	0	0	0
18	964	349	282	485	330	26	135	2571
19	0	0	0	0	0	0	0	0
20	3642	1322	1084	1871	1286	102	604	9911
21	0	0	0	0	0	0	0	0
22	13909	5096	4193	7305	4998	421	2711	38633
23	0	0	0	0	0	0	0	0
24	53536	19631	16258	28382	19428	1638	11564	150437
25	0	0	0	0	0	0	0	0
26	207301	76144	63256	110755	75620	6449	49019	588544
27	0	0	0	0	0	0	0	0
28	806531	296076	246779	432492	294873	25133	204320	2306204
29	0	0	0	0	0	0	0	0
30	3149911	1155500	965317	1693234	1152091	98345	846781	9061179
31	0	0	0	0	0	0	0	0
32	12340600	4520858	3784547	6639885	4510172	384562	3484963	35665587
33	0	0	0	0	0	0	0	0
34	48474514	17732291	14868236	26088415	17689735	1507304	14288215	140648710
35	0	0	0	0	0	0	0	0
36	190835296	69695580	58520066	102668106	69506223	5915075	58373831	555514177
37	0	0	0	0	0	0	0	0
38	752727853	274455938	230714186	404680407	273556594	23254031	237909552	2197298561
39	0	0	0	0	0	0	0	0
40	2974025222	1082590109	910953290	1597367489	1078297717	91551679	967653940	8702439446
41	0	0	0	0	0	0	0	0
42	11767717269	4276739583	3601727460	6313567461	4256449229	360980722	3929718492	34506900216
43	0	0	0	0	0	0	0	0
44	46624012325	16918111176	14258219354	24984619820	16823878275	1425224454	15938801976	136972867380
45	0	0	0	0	0	0	0	0
46	184942931605	67008222924	56508461262	98982927171	66578041314	5634252469	64583076171	544237912916
47	0	0	0	0	0	0	0	0
48	734389831778	265700975850	224189202047	392554120542	263768122565	22299647314	261475193280	2164377093376
49	0	0	0	0	0	0	0	0
50	2918999041879	1054643877594	890293224419	1558323827780	1046077323346	88355761446	1057937247973	8614630304437

[8] The sequence is defined for 2n instead of n, thus eliminating all the zero cells.

[9] A351616(6) and A351616(10) are forced up by 1 as 2 specific polyominoes are not generated by the formations indicated above.

R180M:

	A	B	C	4*3	Other rings	A234008[10]
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	1
5	0	0	0	0	0	0
6	3	0	1	0	0	4
7	0	0	0	0	0	0
8	10	2	4	0	0	16
9	0	0	0	0	0	0
10	36	8	16	0	0	60
11	0	0	0	0	0	0
12	133	34	61	3	0	231
13	0	0	0	0	0	0
14	498	129	235	14	1	877
15	0	0	0	0	0	0
16	1884	500	898	64	16	3362
17	0	0	0	0	0	0
18	7189	1910	3454	258	94	12905
19	0	0	0	0	0	0
20	27611	7343	13320	1038	513	49825
21	0	0	0	0	0	0
22	106595	28269	51604	4080	2455	193003
23	0	0	0	0	0	0
24	413316	109321	200595	16013	11336	750581
25	0	0	0	0	0	0
26	1608482	424069	782330	62627	50284	2927792
27	0	0	0	0	0	0
28	6279163	1650552	3059559	245137	218760	11453171
29	0	0	0	0	0	0
30	24578557	6442342	11995277	960030	935647	44911853
31	0	0	0	0	0	0
32	96434206	25211164	47130505	3765104	3958626	176499605
33	0	0	0	0	0	0
34	379143747	98885713	185537755	14786356	16600845	694954416
35	0	0	0	0	0	0
36	1493392314	388658744	731655483	58156049	69168667	2741031257
37	0	0	0	0	0	0
38	5891913861	1530383756	2889672903	229057932	286699528	10827727980
39	0	0	0	0	0	0
40	23279809506	6036020099	11428562812	903430468	1183532610	42831355495
41	0	0	0	0	0	0
42	92104647495	23842327123	45256282672	3567846953	4869657966	169640762209
43	0	0	0	0	0	0
44	364846148198	94304857958	179416033587	14107401252	19982777168	672657218163
45	0	0	0	0	0	0
46	1446831847051	373468449414	712026441828	55844690719	81819306141	2669990735153
47	0	0	0	0	0	0
48	5743362085023	1480682918338	2828434534633	221298397637	334398130445	10608176066076
49	0	0	0	0	0	0
50	22820166423307	5876487697402	11245491621256	877818903261	1364614408777	42184579054003

[10] The sequence is defined for 2n instead of n, thus eliminating all the zero cells.

M90C:

	A1	A2	A3	B	E	F	G	A349328[11]
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	1	0	1	0	1
5	0	0	0	3	0	0	1	2
6	0	0	1	2	1	2	2	4
7	0	0	5	7	2	0	4	9
8	0	1	5	9	3	6	8	16
9	0	7	16	27	9	1	16	38
10	1	7	19	35	14	19	29	62
11	9	28	63	92	35	6	61	147
12	10	33	75	134	55	68	107	241
13	55	123	221	337	137	27	228	564
14	62	140	290	499	213	249	399	926
15	291	465	818	1232	525	113	852	2148
16	321	572	1109	1879	818	926	1497	3561
17	1328	1791	3003	4590	2012	469	3197	8195
18	1518	2277	4221	7104	3153	3477	5650	13700
19	5783	6771	11245	17216	7694	1907	12082	31349
20	6799	8944	16133	27051	12172	13151	21466	52858
21	24261	25816	42303	65170	29538	7714	45912	120357
22	29392	34915	61850	103523	47135	50061	82012	204444
23	99962	98490	160582	248198	113717	31024	175451	463712
24	124122	136128	238155	398170	182985	191499	314913	792986
25	406495	377667	612877	950745	439270	124440	673670	1792582
26	515894	530740	920294	1537649	712144	735855	1214362	3083469
27	1640306	1452850	2352077	3658537	1701814	498065	2597509	6950579
28	2120984	2071622	3568497	5959671	2777788	2838103	4700123	12018394
29	6582928	5610238	9065185	14134910	6611320	1990972	10051465	27023509
30	8652621	8097152	13877881	23169758	10857233	10982837	18249336	46943409
31	26326692	21731017	35072079	54795222	25746874	7951709	39017839	105320716
32	35104077	31698138	54114286	90321953	42514899	42623511	71054026	183715445
33	105033489	84420543	136124579	213044897	100489437	31741046	151874145	411364068
34	141844425	124280234	211498605	352925698	166759628	165844021	277320913	720236762
35	418388044	328790107	529828932	830440217	392984074	126659704	592582778	1609836928
36	571456346	488000364	828324533	1381889199	655091473	646748944	1084693371	2828102115
37	1664884671	1283480202	2067272850	3244320057	1539591812	505339726	2317065858	6310977588
38	2297217824	1918886824	3250052527	5420754503	2577007685	2527298270	4250675289	11120946461
39	6620885859	5020519142	8083533253	12700030723	6041409058	2016067062	9077193747	24779819422
40	9219707521	7555301586	12773048108	21298969023	10150357566	9893993629	16685873127	43788625280
41	26320534954	19674808430	31669355482	49803465088	23741576483	8043407051	35620804450	97436975969
42	36958135966	29784255727	50273699496	83810314650	40027081005	38797717515	65600351387	172625777873
43	104618356468	77231625491	124286662004	195618325178	93425176027	32093358229	139998234045	383635868721
44	148019510420	117548286995	198138171771	330229221010	158013405958	152368174094	258265712508	681291241378
45	415832363043	303623597919	488519670423	769464000223	368089118687	128070770785	550993130778	1512296325929
46	592445409481	464413146489	781849900428	1302747273264	624404947237	599210465810	1018067265961	2691569204335
47	1652995515659	1195286859321	1922862578350	3030670432614	1451891328938	511162395102	2171306751138	5968087930561
48	2370165067872	1836616447265	3088593384231	5145010774120	2469676080526	2359456137930	4017800659990	10643659275967
49	6572051530373	4711439371300	7578236541736	11951197661123	5732839181320	2040573111963	8566454408683	23576395903249
50	9479201061630	7269893158828	12213501948294	20340164547782	9776583050890	9301401440352	15873090970916	42126918089346

[11] Column = half the sum of the component columns

4 PRINCIPAL RESULTS

This table shows the principal results. The figures not in bold had been calculated well before 2021. Of these, the higher values were calculated by Toshihiro Shirakawa. The figures in bold for sizes 46 and 48 were calculated by myself in late 2021, and for sizes 49 and 50 during the activity here described.

Size	Free	One-sided
1	1	1
2	1	1
3	2	2
4	5	7
5	12	18
6	35	60
7	108	196
8	369	704
9	1285	2500
10	4655	9189
11	17073	33896
12	63600	126759
13	238591	476270
14	901971	1802312
15	3426576	6849777
16	13079255	26152418
17	50107909	100203194
18	192622052	385221143
19	742624232	1485200848
20	2870671950	5741256764
21	11123060678	22245940545
22	43191857688	86383382827
23	168047007728	336093325058
24	654999700403	1309998125640
25	2557227044764	5114451441106
26	9999088822075	19998172734786
27	39153010938487	78306011677182
28	153511100594603	307022182222506
29	602621953061978	1205243866707468
30	2368347037571252	4736694001644862
31	9317706529987950	18635412907198670
32	36695016991712879	73390033697855860
33	144648268175306702	289296535756895985
34	570694242129491412	1141388483146794007
35	2253491528465905342	4506983054619138245
36	8905339105809603405	17810678207278478530
37	35218318816847951974	70436637624668665265
38	139377733711832678648	278755467406691820628
39	551961891896743223274	1103923783758183428889
40	2187263896664830239467	4374527793263174673335
41	8672737591212363420225	17345475182286431485513
42	34408176607279501779592	68816353214298169362691
43	136585913609703198598627	273171827218863802383383
44	542473001706357882732070	1084946003411691009916361
45	2155600091107324229254415	4311200182212516601049225
46	**8569720333296834568434605**	**17139440666589637839781602**
47	34085105553123831158180217	68170211106239275354867268
48	**135629410647775553284438364**	**271258821295535228672142075**
49	**539916438668093786698843965**	1079832877336154538674417465
50	**2150182610161041739167164220**	**4300365220322020871043392169**

5 SEQUENCE UPDATE

These are some of the sequences that may be extended according to the results of the activity. They are ordered according to their mutual dependency: each sequence is listed only after those sequences that appear in its formula. Each formula can be found in the OEIS page relative to the corresponding sequence.

<table>
<tr><th>Sequence</th><th>Content</th><th colspan="6">Formula</th></tr>
<tr><td>A234008</td><td>R180M</td><td colspan="6">Direct result of activity</td></tr>
<tr><td>A349328</td><td>M90C</td><td colspan="6">Direct result of activity</td></tr>
<tr><td>A351615</td><td>R180C</td><td colspan="6">Direct result of activity</td></tr>
<tr><td>A351616</td><td>R180V</td><td colspan="6">Direct result of activity</td></tr>
<tr><td>A006748</td><td>M45</td><td colspan="6">Direct result of activity</td></tr>
<tr><td rowspan="2">A006746</td><td rowspan="2">M90</td><td colspan="3">Even n</td><td colspan="3">Odd n</td></tr>
<tr><td colspan="3">A349328(n)+A349329(n/2)</td><td colspan="3">A349328(n)</td></tr>
<tr><td>A006747</td><td>R180</td><td colspan="6">A351615(n) + A234008(n/2) + A351616(n/2) for even n; A351615(n) for odd n</td></tr>
<tr><td>A000105</td><td>Free</td><td colspan="6">Formula 3</td></tr>
<tr><td>A259090</td><td>Symmetrical</td><td colspan="6">A006746(n) + A006748(n) + A006747(n) + A056877(n) + A056878(n) + A144553(n) + A142886(n)</td></tr>
<tr><td>A006749</td><td>ASYM</td><td colspan="6">A000105(n) – A259090(n)</td></tr>
<tr><td>A030227</td><td>Achiral</td><td colspan="6">A006746(n) + A006748(n) + A056877(n) + A056878(n) + A142886(n)</td></tr>
<tr><td>A030228</td><td>Chiral</td><td colspan="6">A006749(n) + A006747(n) + A144553(n)</td></tr>
<tr><td>A000988[12]</td><td>One-sided</td><td colspan="6">2*A000105(n) - A030227(n)</td></tr>
<tr><td rowspan="2">A001933</td><td rowspan="2">Chessboard</td><td colspan="2">n mod 2 == 1</td><td colspan="2">n mod 4 == 2</td><td colspan="2">n mod 4 == 0</td></tr>
<tr><td colspan="2">2*A000105(n)</td><td colspan="2">2*A000105(n) - (A234006(n/2) + A234008(n/2))</td><td colspan="2">2*A000105(n) - (A234006(n/2) + A234008(n/2) + A234007(n/4))</td></tr>
<tr><td>A006765</td><td>2-dimensional free</td><td colspan="6">A000105(n) - 1</td></tr>
<tr><td>A057766</td><td>Total area</td><td colspan="6">n*A000105(n)</td></tr>
<tr><td>A144554</td><td>At least R180</td><td colspan="6">A142886(n) + A056877(n) + A144553(n) + A056878(n) + A006747(n)</td></tr>
<tr><td>A349329</td><td>M90V</td><td colspan="6">2*(A006747(n) + A006748(n)) + A144553(n) + A056878(n) + A006746(n) + 4*A006749(n)</td></tr>
<tr><td>A056780</td><td>Free polyrects</td><td colspan="6">2*A006749(n) + 2*A006746(n) + A006748(n) + 2*A006747(n) + 2*A056877(n) + A056878(n) + A144553(n) + A142886(n)[13]</td></tr>
<tr><td>A151522</td><td>1-sided polyrhombs</td><td colspan="6">4*A006749(n) + 2*A006746(n) + 2*A006748(n) + 4*A006747(n) + 2*A056877(n) + 2*A056878(n) + 2*A144553(n) + A142886(n)</td></tr>
<tr><td>A056783</td><td>Diamond polyominoes</td><td colspan="6">2*A006749(n) + A006746(n) + 2*A006748(n) + 2*A006747(n) + A056877(n) + 2*A056878(n) + A144553(n) + A142886(n)</td></tr>
</table>

[12] Toshihiro Shirakawa has calculated a(49) so this sequence gives a useful check on the results of the current activity – “Enumeration of Polyominoes considering the symmetry” - April 2012 Toshihiro Shirakawa

[13] This and the next few formulae from Andrew Howroyd

<table>
<tr><td>A151525</td><td>Poly-IH64-tiles</td><td colspan="6">4*A006749(n) + 3*A006746(n) + 2*A006748(n) + 2*A006747(n) + 2*A056877(n) + A056878(n) + A144553(n) + A142886(n)</td></tr>
<tr><td>A182645</td><td>Poly-IH68-tiles</td><td colspan="6">4*A006749(n) + 2*A006746(n) + 3*A006748(n) + 2*A006747(n) + A056877(n) + 2*A056878(n) + A144553(n) + A142886(n)</td></tr>
<tr><td>A343562</td><td>Polyominoes with at least M90 symmetry</td><td colspan="6">A056877(n) + A142886(n) + A006746(n)</td></tr>
<tr><td>A351191</td><td>BothM90V</td><td colspan="6">A144553(n) + A056877(n) + 2 * A006747(n) + 2 * A006746(n) + 4 * A006749(n) + A006748(n)</td></tr>
<tr><td>A346800</td><td>AllV</td><td colspan="6">2*A006748(n) + 2*A056878(n) + A142886(n)</td></tr>
<tr><td rowspan="2">A234006</td><td rowspan="2">At least M90V</td><td colspan="3">Odd n</td><td colspan="3">Even n</td></tr>
<tr><td colspan="3">A349329(n) + A346799(n)</td><td colspan="3">A349329(n) + A346799(n) + A346800(n/2) + A351191(n/2)</td></tr>
<tr><td>A234010</td><td>At least R180M</td><td colspan="6">A346799(n) + A234008(n)</td></tr>
<tr><td>A130866</td><td>Non-null polyominoes with <= n cells</td><td colspan="6">Sum_{k=1..n} A000105(k)</td></tr>
<tr><td>A173271</td><td>Polyominoes with <= n cells</td><td colspan="6">Sum_{k=0..n} A000105(k)</td></tr>
<tr><td>A210996</td><td>Even sized</td><td colspan="6">a(n)=A000105(2*n)</td></tr>
<tr><td>A210997</td><td>Odd sized</td><td colspan="6">a(n)=A000105(2*n - 1)</td></tr>
<tr><td rowspan="2">A121198</td><td rowspan="2">One-sided chessboard</td><td colspan="2">n mod 2 == 1</td><td colspan="2">n mod 4 == 2</td><td colspan="2">n mod 4 == 0</td></tr>
<tr><td colspan="2">2*A000105(n) + 2*A030228(n)</td><td colspan="2">2*A000105(n) + 2*A030228(n) - A346799(n/2) - 2*A234008(n/2)</td><td colspan="2">2*A000105(n) + 2*A030228(n) - A346799(n/2) - 2*A234008(n/2) - A234009(n/4) - A234007(n/4)</td></tr>
<tr><td>A283108</td><td>Fixed minus free</td><td colspan="6">A001168(n) - A000105(n)</td></tr>
<tr><td>A283109</td><td>Increment of free polyominoes</td><td colspan="6">A000105(n+1) - A000105(n)</td></tr>
<tr><td rowspan="2">A346799</td><td rowspan="2">BothM90M</td><td colspan="3">Odd n</td><td colspan="3">Even n</td></tr>
<tr><td colspan="3">A351127(n) + 2 * A351190(n) + 2 * A349328(n)</td><td colspan="3">A351127(n) + 2 * A351190(n) + A346799(n / 2) + 2 * A349328(n)</td></tr>
</table>

During the final phase of the activity, it was discovered that the previous b-files of some sequences published by myself on the OEIS site had incorrect values starting at position n=44. The new b-files corrected these errors. The sequences involved were: A259090, A006749, A349329, A056780, A151522, A056783, A151525, A234006, A182645.

6 CONCLUSIONS

The approach is heavily dependent on Jensen's calculation of Fixed(n). Thanks also to Robert Russell, whose data for some symmetries is included in the calculation of Free(50), and who introduced me to Redelmeier's inner rings.

In fact, the basic methods used in the activity described here are based on Redelmeier's paper, 43 years later.

More effort should have been devoted to predicting correctly the runtimes of the 27 jobs. It would then have been possible to schedule them more efficiently across the 10 CPU's available and so obtain a result in less calendar time, and spend less in server rental.
Two different methods of runtime prediction were used:

1. *A priori*: run the jobs for smaller sizes and then extrapolate to the required size.
2. *Runtime*: verify the throughput using the trace lines in the log file.

The following details must be taken into account:

- The log file traces polyominoes identified; some jobs require division of results for symmetry, so the numbers in the log file may need the same division.
- The log file should trace only polyominoes of the target size and target symmetry. Otherwise, it will be difficult to foresee process termination time.
- There is no guarantee that different cloud servers will have the same performance. Even worse, there is no guarantee that the same server will offer consistent performance. In one test, the same program on the same server ran in 107 seconds one day, and 71 seconds the next. With a job that runs for almost 2 months, such a difference could cause 2 weeks of discrepancy in runtime prediction.

The numbers of symmetric polyominoes increase at a rate of about 4 for a size increase of 2. Therefore, calculating Free(52) with the same algorithms, same code and same servers would cost about 4.5 CPU years.

For any further information, or to point out errors, please contact me at this email address: TheIllustratedPolyomino (at) gmail (dot) com.