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SIGNIFICANCE PROBABILITIES OF THE WILCOXON TEST

By Evelyn Fix[1] and J. L. Hodges, Jr.

University of California, Berkeley

0. Summary. Tables are presented from which exact values of the Wilcoxon distribution may be obtained when the smaller sample size m does not exceed 12. The Edgeworth approximation to terms of order $1/m^2$ is given and its accuracy investigated.

1. Introduction. We are interested in the problem of obtaining significance probabilities for the Wilcoxon unpaired two-sample test [1], [2]. Let $m \leqq n$ be positive integers, and let $R_1 < R_2 < \cdots < R_m$ represent a random sample of size m drawn without replacement from the first $m + n$ positive integers. Let $S_i = R_i - i$ and $U = S_1 + S_2 + \cdots + S_m$, and let $\pi(u, m, n)$ denote the distribution function of U. It is the values of the function π that are required in the Wilcoxon test. [Wilcoxon actually considered $W = R_1 + \cdots + R_m = U + \frac{1}{2}m(m + 1)$].

Mann and Whitney [2] have tabled[2] π to $3D$ for $n \leqq 8$, and have shown that π, suitably normalized, tends to the normal as $m, n \to \infty$. White [3] has tabled the largest value of u for which $\pi(u, m, n) \leqq 0.005, 0.025$ for $m + n \leqq 30$. Auble [4] has published a similar table for $m, n \leqq 20$, and significance levels 0.001, 0.005, 0.01, 0.02, 0.025, 0.04, 0.05, 0.1. These tables and the normal approximation serve most ordinary needs in hypothesis testing. For some purposes (such as relative efficiency studies, in which it is the relative error that matters) the normal approximation is not sufficiently precise, and the restriction $n \leqq 8$ of the Mann-Whitney table is confining. The White and Auble tables give significance probabilities in most cases with even less accuracy than the normal approximation. [[3] contains several errors of one, apparently due to rounding.]

The connection of π with a partition function is well known. If $A(u, m, n)$ denotes the number of ways (without regard to order) in which it is possible to choose exactly m nonnegative integral summands, none greater than n, whose sum does not exceed u [or, equivalently, the number of ways in which it is possible to choose exactly m positive distinct integral summands, none greater than $m + n$, whose sum does not exceed $u + \frac{1}{2}m(m + 1)$], then

$$\pi(u, m, n) = A(u, m, n) \Big/ \binom{m + n}{m}.$$

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[1] This investigation was supported (in part) by research grant from The National Institutes of Health, Public Health Service.

[2] According to a review in *Mathematical Tables and Other Aids to Computation*, Vol. 6 (1952), p. 157, this table has been extended to $n = 10$ with $7D$ by H. R. van der Vaart. His table does not seem to be widely available in this country.

Simple recursion formulas permit the ready tabulation of A, but the problem of publication is formidable. The usefulness of a triple-entry table of exact values of A over the range of interest would scarcely justify the many pages it would require.[3]

Wilcoxon [1] presented without proof a formula which, for small values of u, permits one to obtain values of A from those of the double-entry quantity $A_0(u, m) = A(u, m, \infty)$. This function A_0 was studied and tabulated by Euler [5]. [More precisely, Euler tabled $a_0(u, m) = A_0(u, m) - A_0(u - 1, m)$, which is the number of ways of partitioning the exact value u into m parts.] In Section 2 we derive an identity similar in nature to that of Wilcoxon, but valid for all values of u. We also present tables of A_0 and of a related quantity A_2, from which values of A are readily obtained.

Our tables may be used provided $m \leqq 12$. This requirement that the smaller sample size not exceed 12 is considerably less restrictive than that the larger sample size not exceed 8, but still will leave many situations of interest uncovered. We turn therefore to approximations, and develop in Section 3 a polynomial expression for the sixth central moment of U. This permits us to obtain simple formulas for the coefficients of the Edgeworth series for π to terms of order $1/m^2$. A numerical investigation indicates this series to be reliable to about $4D$ when $m = 12$.

2. A combinatorial identity. To simplify notation, we adopt the conventions that $A(u, m, n)$ and $A_0(u, m)$ are 0 when $u < 0$, and that all variables of summation are integers. We observe

$$A(u, m, n) = A_0(u, m) - \sum_{t>n} A(u - t, \quad m - 1, \quad t). \tag{1}$$

This formula may be verified by observing that $A_0(u, m)$ counts the partitions $S_1 + S_2 + \cdots + S_m \leqq u$ where $0 \leqq S_1 \leqq \cdots \leqq S_m$, while $A(u, m, n)$ counts those of these partitions satisfying the additional restriction $S_m \leqq n$. Since $A(u - t, \quad m - 1, \quad t)$ is the number of the partitions with $S_m = t$, the sum in (1) represents just the number of partitions counted by $A_0(u, m)$ but not counted by $A(u, m, n)$.

We now apply (1) to itself repeatedly, obtaining the development

$$\begin{aligned} A(u, m, n) = A_0(u, m) &- \sum_{t_1>n} A_0(u - t_1, \quad m - 1) \\ &+ \sum_{t_2>t_1>n} A_0(u - t_1 - t_2, \quad m - 2) \\ &- \sum_{t_3>t_2>t_1>n} A_0(u - t_1 - t_2 - t_3, \quad m - 3) + \cdots . \end{aligned} \tag{2}$$

This formula may now be simplified by the change of summation variable $s_i = t_i - n - i$. If we write $u - kn - \frac{1}{2}k(k + 1) = w$, the $(k + 1)$st term on

[3] Auble has attacked this problem by placing a table covering $m, n \leqq 20$ on file with the American Documentation Institute, from whom it may be purchased for \$4.25 (microfilm) or \$12.50 (photostat). See [4], p. 14 for details.

the right side of (2) may be written

$$(-1)^k \sum_{0 \leq s_1 \leq \cdots \leq s_k} A_0[w - (s_1 + \cdots + s_k), \quad m - k].$$

In this sum, the term $A_0(w - v, \quad m - k)$ occurs as many times as there are ways of partitioning v into just k nonnegative integers, that is, $a_0(v, k)$ times. The $(k + 1)$st term of (2) is thus equal to

$$(-1)^k \sum_{v \geq 0} a_0(v, k) \, A_0(w - v, \quad m - k).$$

If we now write

$$A_k(u, m) = \sum_{v \geq 0} a_0(v, k) \, A_0(u - v, \quad m), \tag{3}$$

we can present (2) in the form

$$\begin{aligned} A(u, m, n) &= \sum_{k \geq 0} (-1)^k A_k(u - kn - \tfrac{1}{2}k(k + 1), \quad m - k) \\ &= A_0(u, m) - A_1(u - n - 1, \quad m - 1) \\ &\quad + A_2(u - 2n - 3, \quad m - 2) - A_3(u - 3n - 6), \quad m - 3) + \cdots . \end{aligned} \tag{4}$$

The series is extended until the first argument becomes negative. Formulas (3) and (4) express the restricted partition function A in terms of the unrestricted partition function A_0.

We present in Table I the values of $A_0(u, m)$ for $m \leq 12$ and $u \leq 100$ [Euler's table of a_0 covers $m \leq 20$ and $u \leq 59$]. These values were computed with the aid of the familiar recursion relation

$$A_0(u, m) = A_0(u, \quad m - 1) + A_0(u - m, \quad m),$$

together with the boundary values $A_0(0, m) = 1$ and $A_0(u, 1) = u + 1$. Values of A_k for $k > 0$ can be computed from the relation

$$\begin{aligned} A_k(u, \quad m - k) &= \sum_{v=0}^{u} A_0(v, k) \, A_0(u - v, \quad m - k) \\ &\quad - \sum_{v=0}^{u} A_0(v - 1, \quad k) \, A_0(u - v, \quad m - k), \end{aligned}$$

but for convenience we also give in Table II the values of $A_2(u, m)$ for $m \leq 11$ and $u \leq 75$. Table II was computed with the aid of

$$A_2(u, m) = A_2(u, \quad m - 1) + A_2(u - m, \quad m), \qquad A_2(0, m) = 1,$$

which follow from (3). In the use of (4) for the range covered by our tables, one often needs A_1 and occasionally A_3. These quantities are readily obtained from Table II with the aid of

$$\begin{aligned} A_1(u, m) &= A_2(u, m) - A_2(u - 2, \quad m) \\ A_3(u, m) &= A_2(u, m) + A_2(u - 3, \quad m) + A_2(u - 6, \quad m) + \cdots . \end{aligned} \tag{5}$$

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TABLE I

$A_0(u, m)$

u	m = 2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1	1	1	1	1	1	1	1
1	2	2	2	2	2	2	2	2	2	2	2
2	4	4	4	4	4	4	4	4	4	4	4
3	6	7	7	7	7	7	7	7	7	7	7
4	9	11	12	12	12	12	12	12	12	12	12
5	12	16	18	19	19	19	19	19	19	19	19
6	16	23	27	29	30	30	30	30	30	30	30
7	20	31	38	42	44	45	45	45	45	45	45
8	25	41	53	60	64	66	67	67	67	67	67
9	30	53	71	83	90	94	96	97	97	97	97
10	36	67	94	113	125	132	136	138	139	139	139
11	42	83	121	150	169	181	188	192	194	195	195
12	49	102	155	197	227	246	258	265	269	271	272
13	56	123	194	254	298	328	347	359	366	370	372
14	64	147	241	324	388	433	463	482	494	501	505
15	72	174	295	408	498	564	609	639	658	670	677
16	81	204	359	509	634	728	795	840	870	889	901
17	90	237	431	628	797	929	1025	1092	1137	1167	1186
18	100	274	515	769	996	1177	1313	1410	1477	1522	1552
19	110	314	609	933	1231	1477	1665	1803	1900	1967	2012
20	121	358	717	1125	1513	1841	2099	2291	2430	2527	2594
21	132	406	837	1346	1844	2277	2624	2889	3083	3222	3319
22	144	458	973	1601	2235	2799	3262	3621	3890	4085	4224
23	156	514	1123	1892	2689	3417	4026	4508	4874	5145	5340
24	169	575	1292	2225	3221	4150	4945	5584	6078	6448	6720
25	182	640	1477	2602	3833	5010	6035	6875	7533	8034	8106
26	196	710	1683	3029	4512	6019	7332	8424	9294	9964	10469
27	210	785	1908	3509	5353	7191	8859	10269	11406	12295	12972
28	[illegible]			[illegible]	[illegible]	[illegible]	[illegible]	12463	13940	15107	16008
				[illegible]	[illegible]	[illegible]	12764	15055	16955	[illegible]	[illegible]

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23	156	511	1123	1892	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5145	[illegible]
24	169	575	1292	2225	3221	[illegible]	[illegible]	5584	6078	6148	6720

25	182	640	1477	2602	3833	5010	6035	6875	7533	8034	8406
26	196	710	1683	3029	4542	6019	7332	8424	9294	9964	10169
27	210	785	1908	3509	5353	7194	8859	10269	11406	12295	12972
28	225	865	2157	4049	6284	8561	10660	12463	13940	15107	16008
29	240	950	2427	4652	7311	10140	12764	15055	16955	18477	19663
30	256	1041	2724	5326	8547	11964	15226	18115	20545	22512	24064
31	272	1137	3045	6074	9907	14057	18083	21704	24787	27314	29326
32	289	1239	3396	6905	11447	16457	21402	25910	29800	33022	35616
33	306	1347	3774	7823	13176	19195	25230	30814	35688	39773	43092
34	324	1461	4185	8837	15121	22315	29647	36522	42600	47745	51969
35	342	1581	4626	9952	17293	25854	34713	43137	50670	57118	62458
36	361	1708	5104	11178	19725	29865	40525	50794	60088	68122	74842
37	380	1841	5615	12520	22427	34391	47155	59618	71024	80988	89394
38	400	1981	6166	13989	25436	39493	54719	69774	83714	96009	1 06478
39	420	2128	6754	15591	28767	45224	63307	81422	98377	1 13484	1 26456
40	441	2282	7386	17338	32459	51654	73056	94760	1 15305	1 33782	1 49790
41	462	2443	8058	19236	36529	58844	84074	1 09984	1 34771	1 57283	1 76946
42	484	2612	8778	21298	41023	66877	96524	1 27338	1 57138	1 84452	2 08516
43	506	2788	9542	23531	45958	75823	1 10536	1 47058	1 82746	2 15768	2 45094
44	529	2972	10358	25949	51385	85776	1 26301	1 69438	2 12038	2 51811	2 87427
45	552	3164	11222	28560	57327	96820	1 43975	1 94769	2 45439	2 93184	3 36276
46	576	3364	12142	31378	63837	1 09061	1 63780	2 23398	2 83186	3 40604	3 92573
47	600	3572	13114	34412	70941	1 22595	1 85902	2 55676	3 26700	3 94822	4 57280
48	625	3789	14147	37678	78701	1 37545	2 10601	2 92023	3 75737	4 56725	5 31567
49	650	4014	15236	41185	87143	1 54020	2 38094	3 32854	4 31231	5 27240	6 16634
50	676	4248	16390	44950	96335	1 72158	2 68682	3 78666	4 93971	6 07455	7 13933
51	702	4491	17605	48983	1 06310	1 92086	3 02622	4 29960	5 64731	6 98513	8 24969
52	729	4743	18890	53302	1 17139	2 13959	3 40260	4 87318	6 44456	8 01739	9 51529
53	756	5004	20240	57918	1 28859	2 37920	3 81895	5 51333	7 34079	9 18531	10 95477
54	784	5275	21665	62850	1 41551	2 64146	4 27926	6 22695	8 34733	10 50501	12 59017

TABLE—I *Continued*

$A_0(u, m)$

u	$m = 2$	3	4	5	6	7	8	9	10	11	12
55	812	5555	23160	68110	1 55253	2 92798	4 78700	7 02098	9 47537	11 99348	14 44442
56	841	5845	24735	73718	1 70053	3 24073	5 34674	7 90350	10 73836	13 67020	16 54447
57	870	6145	26385	79687	1 85997	3 58155	5 96249	8 88272	12 14972	15 55576	18 91852
58	900	6455	28120	86038	2 03177	3 95263	6 63945	9 96799	13 72536	17 67358	21 59931
59	930	6775	29935	92785	2 21644	4 35603	7 38225	11 16891	15 48122	20 04847	24 62127
60	961	7106	31841	99951	2 41502	4 79422	8 19682	12 49642	17 43613	22 70853	28 02420
61	992	7447	33832	1 07550	2 62803	5 26940	9 08844	13 96162	19 60893	25 68348	31 84982
62	1024	7799	35919	1 15606	2 85659	5 78457	10 06383	15 57716	22 02172	29 00685	36 14618
63	1056	8162	38097	1 24135	3 10132	6 34205	11 12905	17 35600	24 69679	32 71418	40 96387
64	1089	8536	40377	1 33162	3 36339	6 94494	12 29108	19 31266	27 65999	36 84530	46 36059
65	1122	8921	42753	1 42704	3 64348	7 59611	13 55860	21 46210	30 93747	41 44248	52 39725
66	1156	9318	45237	1 52787	3 94289	8 29892	14 93837	23 82109	34 55945	46 55293	59 14310
67	1190	9726	47823	1 63429	4 26232	9 05654	16 43879	26 40678	38 55650	52 22670	66 67112
68	1225	10146	50523	1 74658	4 60317	9 87266	18 06948	29 23839	42 96375	58 51951	75 06398
69	1260	10578	53331	1 86493	4 96625	10 75082	19 83026	32 33568	47 81690	65 49048	84 40900
70	1296	11022	56259	1 98963	5 35302	11 69507	21 75800	35 72052	53 15665	73 20512	94 80443
71	1332	11478	59301	2 12088	5 76436	12 70930	23 83835	39 41551	59 02444	81 73297	106 35424
72	1369	11947	62470	2 25809	6 20188	13 79799	26 08967	43 44567	65 46739	91 15087	119 17507
73	1406	12428	65759	2 40417	6 66649	14 96541	28 52401	47 83667	72 53346	101 54031	133 39013
74	1444	12922	69181	2 55674	7 15991	16 21645	31 15482	52 61602	80 27691	112 99109	149 13727
75	1482	13429	72730	2 71693	7 68318	17 55584	33 99463	57 81572	88 75319	125 59849	166 56236
76	1521	13949	76419	2 88507	8 23809	18 98891	37 05839	63 46517	98 02462	139 46710	185 82769
77	1560	14482	80241	3 06140	8 82576	20 52083	40 36009	69 59848	108 15498	154 70791	207 10516
78	1600	15029	84210	3 24627	9 44815	22 15745	43 91635	76 25203	119 21578	171 44248	230 58558
79	1640	15589	88319	3 43993	10 10642	23 90441	47 74276	83 46328	131 28018	189 79969	256 47081
80	1681	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	91 27325	[illegible]	[illegible]	284 98436

78	1600	15029	84210	3 24627	9 44815	22 15745	43 91635	76 25203	119 21578	171 44218	230 58558
79	1640	15589	88319	3 43093	10 10642	23 [illegible]	47 74276	83 46328	131 28018	189 79969	256 47081

80	1681	16163	92582	3 64275	10 80266	25 76807	51 85774	91 27325	144 42990	209 92038	284 98436
81	1722	16751	96992	3 85499	11 53817	27 75462	56 27863	99 72430	158 74874	231 95386	316 36286
82	1764	17353	1 01563	4 07703	12 31512	29 87096	61 02578	108 86245	174 32984	256 06281	350 86724
83	1806	17969	1 06288	4 30915	13 13491	32 12382	66 11815	118 73537	191 26883	282 41970	388 77394
84	1849	18600	1 11182	4 55175	13 99990	34 52073	71 57912	129 39484	209 67175	311 21206	430 38713
85	1892	19245	1 16237	4 80512	14 91154	37 06899	77 42908	140 89425	229 64744	342 63853	476 02866
86	1936	19905	1 21468	5 06967	15 87233	39 77674	83 69309	153 29157	251 31619	376 91468	526 05195
87	1980	20580	1 26868	5 34571	16 88388	42 65195	90 39471	166 64674	274 80172	414 26882	580 83118
88	2025	21270	1 32452	5 63367	17 94879	45 70341	97 56115	181 02443	300 24021	454 94812	640 77581
89	2070	21975	1 38212	5 93387	19 06878	48 93974	105 21837	196 49162	327 77180	499 21428	706 31944
90	2116	22696	1 44164	6 24676	20 24666	52 37048	113 39626	213 12056	357 55046	547 35015	777 93573
91	2162	23432	1 50300	6 57267	21 48421	56 00494	122 12339	230 98584	389 73458	599 65496	856 12577
92	2209	24184	1 56636	6 91207	22 78440	59 85339	131 43251	250 16788	424 49772	656 45158	941 43594
93	2256	24952	1 63164	7 26531	24 14919	63 92593	141 35501	270 74985	462 01868	718 08149	1034 44435
94	2304	25736	1 69900	7 63287	25 58166	68 23361	151 92670	292 82095	502 49270	784 91240	1135 77964
95	2352	26536	1 76836	8 01512	27 08390	72 78731	163 18202	316 47359	546 12103	857 33309	1246 10703
96	2401	27353	1 83989	8 41256	28 65922	77 59896	175 16011	341 80685	593 12304	935 76157	1366 14870
97	2450	28186	1 91350	8 82557	30 30978	82 68026	187 89863	368 92306	643 72478	1020 63946	1496 66812
98	2500	29036	1 98936	9 25467	32 03907	88 04401	201 44027	397 93189	698 17210	1112 44092	1638 49287
99	2550	29903	2 06739	9 70026	33 84945	93 70284	215 82623	428 94679	756 71859	1211 66671	1792 49789
100	2601	30787	2 14776	10 16288	35 74454	99 67047	231 10298	462 08882	819 63928	1318 85356	1959 62937

end

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TABLE II

$A_2(u, m)$

u	$m = 1$	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1
1	3	3	3	3	3	3	3	3	3	3	3
2	7	8	8	8	8	8	8	8	8	8	8
3	13	16	17	17	17	17	17	17	17	17	17
4	22	30	33	34	34	34	34	34	34	34	34
5	34	50	58	61	62	62	62	62	62	62	62
6	50	80	97	105	108	109	109	109	109	109	109
7	70	120	153	170	178	181	182	182	182	182	182
8	95	175	233	267	284	292	295	296	296	296	296
9	125	245	342	403	437	454	462	465	466	466	466
10	161	336	480	594	656	690	707	715	718	719	719
11	203	448	681	851	959	1021	1055	1072	1080	1083	1084
12	252	588	930	1197	1375	1484	1546	1580	1597	1605	1608
13	308	756	1245	1648	1932	2113	2222	2284	2318	2335	2343
14	372	960	1641	2235	2672	2964	3146	3255	3317	3351	3368
15	444	1200	2130	2981	3637	4091	4386	4568	4677	4739	4773
16	525	1485	2730	3927	4886	5576	6038	6334	6516	6625	6687
17	615	1815	3156	5104	6479	7500	8207	8672	8968	9150	9259
18	715	2200	4330	6565	8497	9981	11036	11751	12217	12513	12695
19	825	2610	5370	8351	11023	13136	14682	15754	16472	16938	17234
20	946	3146	6602	10529	14166	17130	19352	20932	22012	22731	23197
21	1078	3718	8048	13152	18038	22129	25275	27559	29156	30239	30958
22	1222	4368	9738	16303	22782	28358	32744	35999	38317	39922	41006
23	1378	5096	11698	20049	28546	36046	42084	46652	49969	52304	53912
24	1547	5915	13963	24492	35515	45496	53703	60037	64714	68065	70408
25	172[illegible]	6825	16563	29715	43881	57017	68053	76725	83241	87980	91348
	[illegible]	[illegible]	19[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	97442	1 06410	1 13035	1 17808
	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	1 22989	1 35206	1 44356	[illegible]

21	1078	3718	8048	13152	18038	22129	25275	27559	29156	30239	30958
22	1222	4368	9738	16303	22782	[illegible]	32744	35999	38317	39922	41006
23	1378	5096	11698	20049	28546	[illegible]	42084	46652	49969	52304	53912
24	1547	5915	13963	24492	35515	45496	53703	60037	64714	68065	70408
25	1729	6825	16563	29715	43881	57017	68053	76725	83241	87980	91348
26	1925	7840	19538	35841	53879	71009	85691	97442	1 06410	1 13035	1 17808
27	2135	8960	22923	42972	65754	87883	1 07235	1 22989	1 35206	1 44356	1 51043
28	2360	10200	26763	51255	79801	1 08159	1 33434	1 54366	1 70838	1 83351	1 92610
29	2600	11560	31098	60813	96328	1 32374	1 65118	1 92677	2 14689	2 31627	2 44322
30	2856	13056	35979	71820	1 15701	1 61197	2 03281	2 39280	2 68436	2 91167	3 08401
31	3128	14688	41451	84423	1 38302	1 95319	2 49022	2 95674	3 33991	3 64230	3 87427
32	3417	16473	47571	98826	1 64580	2 35589	3 03642	3 63679	4 13648	4 53570	4 84528
33	3723	18411	54390	1 15203	1 95004	2 82887	3 68578	4 45303	5 10017	5 62321	6 03327
34	4047	20520	61971	1 33791	2 30119	3 38278	4 45513	5 42955	6 26196	6 94261	7 48173
35	4389	22800	70371	1 54794	2 70495	4 02869	5 36303	6 59292	7 65702	8 53682	9 24090
36	4750	25270	79660	1 78486	3 16788	4 77985	6 43103	7 97469	9 32675	10 45710	11 37058
37	5130	27930	89901	2 05104	3 69684	5 65003	7 68284	9 60961	11 31799	12 76155	13 93963
38	5530	30800	1 01171	2 34962	4 29966	6 65555	9 14577	11 53857	13 68546	15 51897	17 02940
39	5950	33880	1 13540	2 68334	4 98453	7 81340	10 84982	13 80656	16 49092	18 80719	20 73329
40	6391	37191	1 27092	3 05578	5 76073	9 14351	12 82929	16 46608	19 80599	22 71766	25 16088
41	6853	40733	1 41904	3 47008	6 63796	10 66665	15 12178	19 57481	23 71129	27 35359	30 43760
42	7337	44528	1 58068	3 93030	7 62714	12 40699	17 77002	23 19957	28 29974	32 83544	36 70971
43	7843	48576	1 75668	4 44002	8 73968	14 38971	20 82074	27 41366	33 67562	39 29883	44 14411
44	8372	52900	1 94804	5 00382	9 98835	16 64390	24 32674	32 30143	39 95845	46 90106	52 93433
45	8924	57500	2 15568	5 62576	11 38649	19 19989	28 34566	37 95527	47 28202	55 81884	63 30057
46	9500	62400	2 38068	6 31098	12 94894	22 09245	32 94227	44 48084	55 79883	66 25593	75 49683
47	10100	67600	2 62404	7 06406	14 69120	25 35785	38 18714	51 99370	65 67916	78 44071	89 81129
48	10725	73125	2 88693	7 89075	16 63043	29 03742	44 15920	60 62528	77 11620	92 63517	106 57480
49	11375	78975	3 17043	8 79619	18 78454	33 17425	50 94427	70 51908	90 32507	109 13226	126 16166
50	12051	85176	3 47580	9 78678	21 17327	37 81717	58 63791	81 83748	105 54877	128 26643	148 99972
51	12753	91728	3 80421	10 86827	23 81721	43 01710	67 34384	94 75750	123 05724	150 41083	175 57171
52	13482	98658	4 15701	12 04776	26 73896	48 83141	77 17707	109 47850	143 15412	175 98956	206 42716
53	14238	1 05966	4 53546	13 33165	29 96208	55 31993	88 26220	126 21747	166 17592	205 47475	242 18446
54	15022	1 13680	4 94101	14 72779	33 51233	62 54975	100 73689	145 21773	192 49975	239 40081	283 54492

TABLE II—*Continued*

u	$m = 1$	2	3	4	5	6	7	8	9	10	11
55	15834	1 21800	5 37501	16 24328	37 41655	70 59080	114 75000	166 74370	222 54253	278 36137	331 29570
56	16675	1 30355	5 83901	17 88677	41 70398	79 52115	130 46542	191 09070	256 76986	323 02579	386 32636
57	17545	1 39345	6 33446	19 66611	46 40507	89 42217	148 06008	218 57916	295 69536	374 13607	449 63290
58	18445	1 48800	6 86301	21 59080	51 55288	100 38429	167 72813	249 56561	339 89068	432 52585	522 33714
59	19375	1 58720	7 42621	23 66949	57 18182	112 50175	189 67882	284 43632	389 98509	499 11735	605 69215
60	20336	1 69136	8 02582	25 91259	63 32914	125 87889	214 14109	323 61959	446 67683	574 94326	701 10492
61	21328	1 80048	8 66349	28 32960	70 03358	140 62438	241 36127	367 57874	510 73286	661 14369	810 14341
62	22352	1 91488	9 34109	30 93189	77 33696	156 85811	271 60811	416 82584	583 00176	758 99132	934 56303
63	23408	2 03456	10 06038	33 72987	85 28275	174 70492	305 17034	471 91404	664 41379	869 88854	1076 31570
64	24497	2 15985	10 82334	36 73593	93 91775	194 30204	342 36212	533 45282	755 99535	995 39616	1237 58062
65	25619	2 29075	11 63184	39 96144	103 29058	215 79233	383 52046	602 09962	858 86948	1137 23085	1420 77577
66	26775	2 42760	12 48798	43 41987	113 45345	239 33234	429 01116	678 57677	974 27213	1297 29702	1628 59362
67	27965	2 57040	13 39374	47 12361	124 46057	265 08495	479 22604	763 66236	1103 55304	1477 68911	1864 01547
68	29190	2 71950	14 35134	51 08727	136 37002	293 22813	534 58940	858 20899	1248 19408	1680 71993	2130 35283
69	30450	2 87490	15 36288	55 32432	149 24207	323 94699	595 55510	963 13384	1409 81067	1908 92802	2431 26516
70	31746	3 03696	16 43070	59 85057	163 14115	357 44319	662 61353	1079 43937	1590 17223	2165 11549	2770 80764
71	33078	3 20568	17 55702	64 68063	178 13408	393 92611	736 28853	1208 20257	1791 20433	2452 34802	3153 45294
72	34447	3 38143	18 74431	69 83158	194 29215	433 62449	817 14495	1350 59777	2015 01156	2774 00288	3584 14629
73	35853	3 56421	19 99491	75 31923	211 68925	476 77420	905 78536	1507 88498	2263 88033	3133 76887	4068 33190
74	37297	3 75440	21 31142	81 16199	230 40406	523 63219	1002 85823	1681 43500	2540 30448	3535 70064	4612 01634
75	38779	3 95200	22 69631	87 37694	250 51809	574 46508	1109 05448	1872 71684	2846 98807	3984 21982	5221 80044

These relations are so simple to use that tabulation of A_1 and A_3 is unnecessary. Values of A_k for $k > 3$ are seldom required. In general,

$$A_k(u, m) = \sum_{r \geq 0} A_{k-1}(u - rk, \quad m).$$

We illustrate the tables by computing $\pi(95, 12, 22)$. Using (5) and the tables, we find

$$\begin{aligned} A_0(95, 12) &= 124{,}610{,}703, \\ A_1(72, 11) &= 358{,}414{,}629 - 277{,}080{,}764 = 81{,}333{,}865, \\ A_2(48, 10) &= 9{,}263{,}517, \\ A_3(23, \ 9) &= 49{,}969 + 22{,}012 + 8{,}968 + 3{,}317 \\ &\qquad + 1{,}080 + 296 + 62 + 8 = 85{,}712. \end{aligned}$$

Combining, $A(95, 12, 22) = 52{,}454{,}643$; this is the exact number of partitions of $95 + \frac{1}{2}12 \cdot 13 = 173$ into just 12 distinct parts between 1 and 34 inclusive. Since $\binom{34}{12} = 548{,}354{,}040$, we find $\pi(95, 12, 22) = 0.095\ 658 \cdots$.

3. Approximations. Our tables provide values of $\pi(u, m, n)$ only for $m \leqq 12$ and $u \leqq 100$. As is shown below, the normal approximation at these limits is subject to sizable percentage errors. In the search of better approximations for $m > 12$, we turn to the Edgeworth series, which to terms of order $1/m^2$ is (taking advantage of the symmetry of U)

$$\pi(u, m, n) \doteq \Phi(x) + e_{m,n}^{(3)}\varphi^{(3)}(x) + e_{m,n}^{(5)}\varphi^{(5)}(x) + e_{m,n}^{(7)}\varphi^{(7)}(x), \tag{6}$$

where x is the normalized value of u. Using $E(U) = \frac{1}{2}mn$ and $\mu_2 = mn(m + n + 1) / 12$, and the usual continuity correction, we take

$$x = (u + \tfrac{1}{2} - \tfrac{1}{2}mn) / \sqrt{mn\ (m + n + 1) / 12}. \tag{7}$$

The Edgeworth coefficients are given by

$$e_{m,n}^{(3)} = \frac{1}{4!}\left(\frac{\mu_4}{\mu_2^2} - 3\right), \qquad e_{m,n}^{(5)} = \frac{1}{6!}\left(\frac{\mu_6}{\mu_2^3} - 15\frac{\mu_4}{\mu_2^2} + 30\right), \qquad e_{m,n}^{(7)} = \frac{35}{8!}\left(\frac{\mu_4}{\mu_2^2} - 3\right)^2, \tag{8}$$

where μ_k is the kth central moment of U. Mann and Whitney give

$$\mu_4 = \frac{mn(m + n + 1)}{240}\,[5(m^2n + mn^2) - 2(m^2 + n^2) + 3mn - 2(m + n)].$$

They show [their formula (14)] that

$$\mu_6 = \frac{mn(m + n + 1)}{4032}\,[35m^2n^2(m^2 + n^2) + 70m^3n^3 + P(m, n)] \tag{9}$$

where $P(m, n)$ is a symmetric polynomial of 5th degree in m and n. When $m = 1$ and $m = 2$, the distribution may be given explicitly and the moments determined. In this way it may be shown that

$$
\begin{aligned}
P(m, n) = & -42\, mn(m^3 + n^3) - 14\, m^2n^2(m + n) + 16(m^4 + n^4) \\
(10) \qquad & -52\, mn(m^2 + n^2) - 43\, m^2n^2 + 32(m^3 + n^3) \\
& +14\, mn(m + n) + 8(m^2 + n^2) + 16\, mn - 8(m + n).
\end{aligned}
$$

If we substitute (10), (9) and (8) into (6) we find after simplification

$$
\begin{aligned}
\pi(u, m, n) \doteq \Phi(x) & - \frac{m^2 + n^2 + mn + m + n}{20mn(m + n + 1)}\, \varphi^{(3)}(x) \\
(11) \qquad & + \frac{[2(m^4 + n^4) + 4mn(m^2 + n^2) + 6m^2n^2 + 4(m^3 + n^3) + 7mn(m + n) + (m^2 + n^2) + 2mn - (m + n)]}{210m^2n^2(m + n + 1)^2}\, \varphi^{(5)}(x) \\
& + \frac{(m^2 + n^2 + mn + m + n)^2}{800m^2n^2(m + n + 1)^2}\, \varphi^{(7)}(x).
\end{aligned}
$$

An appreciation of the accuracy of the Edgeworth approximations at the limit $m = 12$ of our tables may be gained from an examination of Table III. Column (a) gives the normal approximation; column (b) the first two terms of (11); column (c) the entire approximation (11); the last column gives the exact value.

TABLE III

m	n	u	(a)	(b)	(c)	$\pi(u, m, n)$
12	12	55	.17039	.17359	.17367	.17368
		45	.06301	.06380	.06388	.06384
		35	.01754	.01671	.01666	.01662
		25	.00363	.00285	.00280	.00278
12	24	100	.07218	.07303	.07308	.07307
		80	.01655	.01588	.01584	.01583
		60	.00254	.00202	.00199	.00198

It appears that (11) may be relied on to about $4D$ when $m = 12$, and its accuracy should improve with large values of m. The normal approximation (a) is subject to large percentage errors at the high significance levels, and is much improved by the use of the simple term in $\varphi^{(3)}(x)$.

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