On path factors of $(3, 4)$-biregular bigraphs

Armen S. Asratian*, Carl Johan Casselgren†

Abstract. A $(3, 4)$-biregular bigraph G is a bipartite graph where all vertices in one part have degree 3 and all vertices in the other part have degree 4. A path factor of G is a spanning subgraph whose components are nontrivial paths. We prove that a simple $(3, 4)$-biregular bigraph always has a path factor such that the endpoints of each path have degree three. Moreover we suggest a polynomial algorithm for the construction of such a path factor.

Keywords: path factor, biregular bigraph, interval edge coloring

1 Introduction

We use [9] and [7] for terminology and notation not defined here and consider finite loop-free graphs only. $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. A *proper edge coloring* of a graph G with colors $1, 2, 3, \dots$ is a mapping $f : E(G) \to \{1, 2, 3, \dots\}$ such that $f(e_1) \neq f(e_2)$ for every pair of adjacent edges e_1 and e_2. A bipartite graph with bipartition (Y, X) is called an (a, b)-*biregular* bigraph if every vertex in Y has degree a and every vertex in X has degree b. A *path factor* of a graph G is a spanning subgraph whose components are nontrivial paths. Some results on different types of path factors can be found in [1, 2, 17, 18, 20, 23]. In particular, Ando et al [2] showed that a claw-free graph with minimum degree d has a path factor whose components are paths of length at least d. Kaneko [17] showed that every cubic graph has a path factor such that each component is a path of length $2, 3$ or 4. It was shown in [18] that a 2-connected cubic graph has a path factor whose components are paths of length 2 or 3.

In this paper we investigate the existence of path factors of $(3, 4)$-biregular bigraphs such that the endpoints of each path have degree three. Our investigation is motivated by a problem on interval colorings. A proper edge coloring of a graph G with colors $1, 2, 3, \dots$ is called an *interval* (or *consecutive*) coloring if the colors received by the edges incident with each vertex of G form an interval of integers. The notion of interval colorings was introduced in 1987 by Asratian and Kamalian [5] (available in English as [6]). Generally, it is an $\mathcal{NP}$-complete problem to determine whether a given bipartite graph has an interval coloring [22]. Nevertheless, trees, regular and

*Linköping University, Linköping Sweden, arasr@mai.liu.se.
†Umeå University, Umeå , Sweden, carl-johan.casselgren@math.umu.se.

complete bigraphs [13, 16], doubly convex bigraphs [16], grids [12] and all outerplanar bigraphs [8, 11] have interval colorings. Hansen [13] proved that every $(2, \beta)$-biregular bigraph admits an interval coloring if β is an even integer. A similar result for $(2, \beta)$-biregular bigraphs for odd β was given in [14, 19]. Only a little is known about $(3, \beta)$-biregular bigraphs. It follows from the result of Hanson and Loten [15] that no such a graph has an interval coloring with fewer than $3 + b - \gcd(3, b)$ colors, where gcd denotes the greatest common divisor. We showed in [3] that the problem to determine whether a $(3, \beta)$-biregular bigraph has an interval coloring is $\mathcal{NP}$-complete in the case when 3 divides β.

It is unknown whether all $(3, 4)$-biregular bigraphs have interval colorings. Pyatkin [21] showed that such a graph G has an interval coloring if G has a 3-regular subgraph covering the vertices of degree four. Another sufficient condition for the existence of an interval coloring of a $(3, 4)$-biregular bigraph G was obtained in [4, 10]: G admits an interval coloring if it has a path factor where every component is a path of length not exceeding 8 and the endpoints of each path have degree three. It was conjectured in [4] that every simple $(3, 4)$-biregular bigraph has such a path factor. However this seems difficult to prove.

In this note we prove a little weaker result. We show that a simple $(3, 4)$-biregular bigraph always has a path factor such that the endpoints of each path have degree three. Moreover, we suggest a polynomial algorithm for the construction of such a path factor.

Note that $(3, 4)$-biregular bigraphs with multiple edges need not have path factors with the required property. For example, consider the graph G formed from three triple-edges by adding a claw; that is, the pairs $x_i y_i$ have multiplicity three for $i \in \{1, 2, 3\}$, and there is an additional vertex y_0 with neighborhood $\{x_1, x_2, x_3\}$. Clearly, there is no path factor of G such that the endpoints of each path have degree 3.

2 The result

A *pseudo path factor* of a $(3, 4)$-biregular bigraph G with bipartition (Y, X) is a subgraph F of G, such that every component of F is a path of even length and $d_F(x) = 2$ for every $x \in X$. Let $V_F = \{y \in Y : d_F(y) > 0\}$.

Theorem 1. *Every simple $(3, 4)$-biregular bigraph has a pseudo path factor.*

Proof. Let G be a simple $(3, 4)$-biregular bigraph with bipartition (Y, X). The algorithm below constructs a sequence of subgraphs F_0,F_1,$F_2, \ldots$ of G, where $V(F_0) = V(G)$, $\emptyset = E(F_0) \subset E(F_1) \subset E(F_2) \subset \ldots$ and each component of F_j is a path, for every $j \geq 0$. At each step $i \geq 1$ the algorithm constructs F_i by adding to F_{i-1} one or two edges until the condition $d_{F_j}(x) = 2$ holds for all $x \in X$, where $j \geq 1$. Then $F = F_j$ is a pseudo path factor of G. Parallelly the algorithm constructs a sequence of subgraphs $U_0, U_1, U_2, \ldots$ of G, where $V(U_0) = V(G)$, $\emptyset = E(U_0) \subset E(U_1) \subset E(U_2) \subset \cdots \subset E(U_j)$. The edges of each U_i will not be in the final pseudo path factor F. The algorithm is based on Properties 1-4. During the algorithm the vertices in the set Y are considered to be unscanned or scanned. Initially all vertices in Y are unscanned. At the beginning of each step $i \geq 1$ we have a current vertex x_i. The algorithm selects an unscanned vertex y_i, adjacent to x_i, and determines which edges incident with y_i will be in F_i and which ones in U_i. If $d_{F_i}(v) = 2$ for each $v \in X$, the algorithm stops. Otherwise the algorithm selects a new current vertex and goes to the next step.

Algorithm

Initially $F_0 = (V(G), \emptyset)$, $U_0 = (V(G), \emptyset)$ and all vertices in Y are unscanned.

Step 0. Select a vertex $y_0 \in Y$. Let x_0, x_1, w be the vertices in X adjacent to y_0 in G. Put $F_1 = F_0 + \{wy_0, y_0x_0\}$ and $U_1 = U_0 + y_0x_1$. Consider the vertex y_0 to be scanned. Go to step 1 and consider the vertex x_1 as the current vertex for step 1.

Step i $(i \geq 1)$**.** Suppose that a vertex x_i with $d_{F_{i-1}}(x_i) \leq 1$ was selected at step $(i-1)$ as the current vertex. By Property 4 (see below), $d_{U_{i-1}}(x_i) \leq 2$. Therefore there is an edge x_iy_i with $y_i \in Y$ which neither belongs to F_{i-1}, nor to U_{i-1}. Then, by Property 3, the vertex y_i is an unscanned vertex and therefore the subgraph $F_{i-1}+x_iy_i$ does not contain a cycle. Since $d_G(y_i) = 3$, the vertex y_i, besides x_i, is adjacent to two other vertices, $w_1^{(i)}$ and $w_2^{(i)}$.

Case 1. $d_{F_{i-1}}(w_1^{(i)}) = 2 = d_{F_{i-1}}(w_2^{(i)})$.
Put $F_i = F_{i-1} + x_iy_i$ and $U_i = U_{i-1} + \{y_iw_1^{(i)}, y_iw_2^{(i)}\}$. Consider the vertex y_i to be scanned. If $d_{F_i}(v) = 2$ for every vertex $v \in X$ then Stop. Otherwise select an arbitrary vertex $x_{i+1} \in X$ with $d_{F_i}(x_{i+1}) \leq 1$, go to step $(i+1)$ and consider x_{i+1} as the current vertex for step $(i+1)$.

Case 2. $d_{F_{i-1}}(w_1^{(i)}) = 2$ and $d_{F_{i-1}}(w_2^{(i)}) \leq 1$.
Put $F_i = F_{i-1} + x_iy_i$, $U_i = U_{i-1} + \{y_iw_1^{(i)}, y_iw_2^{(i)}\}$ and consider the vertex y_i to be scanned. Furthermore put $x_{i+1} = w_2^{(i)}$, go to step $(i+1)$ and consider the vertex x_{i+1} as the current vertex for step $(i+1)$.

Case 3. $d_{F_{i-1}}(w_1^{(i)}) \leq 1$ and $d_{F_{i-1}}(w_2^{(i)}) \leq 1$.
Subcase 3a. $d_{F_{i-1}}(w_1^{(i)}) = 0$ or $d_{F_{i-1}}(w_2^{(i)}) = 0$.
We assume that $d_{F_{i-1}}(w_1^{(i)}) = 0$. Put $F_i = F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$, $U_i = U_{i-1} + y_iw_2^{(i)}$ and consider the vertex y_i to be scanned. Furthermore put $x_{i+1} = w_2^{(i)}$, go to step $(i+1)$ and consider the vertex x_{i+1} as the current vertex for step $(i+1)$.
Subcase 3b. $d_{F_{i-1}}(w_1^{(i)}) = 1 = d_{F_{i-1}}(w_2^{(i)})$.
Since y_i is an unscanned vertex and $F_{i-1} + x_iy_i$ does not contain a cycle, the vertex y_i is an endvertex of only one path in $F_{i-1} + x_iy_i$. Then at least one of the graphs $F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$ and $F_{i-1} + \{x_iy_i, y_iw_2^{(i)}\}$ does not contain a cycle. Assume, for example, that $F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$ does not contain a cycle. Then put $F_i = F_{i-1} + \{x_iy_i, y_iw_1^{(i)}\}$, $U_i = U_{i-1} + y_iw_2^{(i)}$ and consider the vertex y_i to be scanned. Furthemore put $x_{i+1} = w_2^{(i)}$, go to step $(i+1)$ and consider the vertex x_{i+1} as

the current vertex for step $(i+1)$.

Now we will prove the correctness of the algorithm. At the beginning of step i we have that x_i is the current vertex, y_i is an unscanned vertex adjacent to x_i and $w_1^{(i)}$, $w_2^{(i)}$ are the two other vertices adjacent to y_i. The following two properties are evident.

Property 1. The algorithm determines which edges incident with y_i will be in F_i and which edges will be in U_i. The vertex y_i is then considered to be scanned and the algorithm will never consider y_i again.

Property 2. The current vertex x_{i+1} for step $(i+1)$ is selected among the vertices $w_1^{(i)}$ and $w_2^{(i)}$, except the case $d_{F_i}(w_1^{(i)}) = d_{F_i}(w_2^{(i)}) = 2$ when an arbitrary vertex $x_{i+1} \in X$ with $d_{F_i}(x_{i+1}) \leq 1$ is selected as the current vertex.

Properties 1 and 2 imply the next property:

Property 3. If $x \in X$, $y \in Y$ and the edge xy neither belongs to F_{i-1}, nor to U_{i-1}, then the vertex y is unscanned at the beginning of step i.

Property 4. If $x \in X$ and $d_{F_{i-1}}(x) \leq 1$ then $d_{U_{i-1}}(x) \leq 2$.

Proof. The statement is evident if $d_{U_{i-1}}(x) = 0$. Suppose that $d_{U_{i-1}}(x) \geq 1$ and j is the minimum number such that $j < i$ and an edge incident with x was included in U_{j-1} at step $(j-1)$. Then the statement of Property 4 is evident if $j = i - 1$.

Now we consider the case $j < i - 1$. Clearly, $d_{F_{j-1}}(x) \leq 1$ because $F_{j-1} \subset F_{i-1}$ and $d_{F_{j-1}}(x) \leq d_{F_{i-1}}(x) \leq 1$. Let xy_{j-1} be the edge included in U_{j-1} at step $(j-1)$. Since $d_{U_{j-1}}(x) = 1$ and $d_{F_{j-1}}(x) \leq 1$, there is an edge xy_j with $y_j \in Y$ which neither belongs to F_{j-1} , nor to U_{j-1}. Then, by Property 3, the vertex y_j is an unscanned vertex and therefore the subgraph $F_{j-1} + xy_j$ does not contain a cycle. According to the description of the algorithm, the edge xy_j will be in any case included in F_j at step j, that is, $d_{F_j}(x) \geq 1$. Then $d_{F_k}(x) = 1$ for every k, $j \leq k \leq i-1$, because $F_j \subset F_k \subset F_{i-1}$ and $1 \leq d_{F_j}(x) \leq d_{F_k}(x) \leq d_{F_{i-1}}(x) \leq 1$. Now we will show that $d_{U_{k-1}}(x) = 1$ for each k, $j \leq k < i-1$. Suppose to the contrary that $d_{U_{k-2}}(x) = 1$ and $d_{U_{k-1}}(x) = 2$ for some k, $j < k < i-1$, that is, another edge incident with x was included in U_{k-1} at step $(k-1)$. Then the conditions $d_{U_{k-1}}(x) = 2$ and $d_{F_{k-1}}(x) = 1$ imply that there is an edge $e \neq y_j x$ incident with x which neither belongs to F_{k-1} , nor to U_{k-1}. Using a similar argument as above we obtain that the edge e should be included in F_k at step k. But then $d_{F_{i-1}}(x) \geq d_{F_k}(x) = 2$, which contradicts our assumption $d_{F_{i-1}}(x) \leq 1$. Thus $d_{U_{k-1}}(x) = 1$ for each k, $j \leq k < i-1$. It is possible that an edge incident with x will be included in U_{i-1} at step $(i-1)$. Therefore $d_{U_{i-1}}(x) \leq 2$. □

The description of the algorithm and Properties 1-4 show that the algorithm will stop at step i only when $d_{F_i}(x) = 2$ for every $x \in X$, that is, when F_i is a pseudo path factor of G. The proof of Theorem 1 is complete. □

Now we will prove that every pseudo path factor of a $(3,4)$-biregular bigraph G can be transformed into a path factor of G, such that the endpoints of each path have degree 3.

Lemma 2. *Let G be a $(3,4)$-biregular bigraph with bipartition (Y,X). Then $|X| = 3k$ and $|Y| = 4k$, for some positive integer k.*

This is evident because $|E(G)| = 4|X| = 3|Y|$.

Lemma 3. *Let F be a pseudo path factor of a $(3,4)$-biregular bigraph G with bipartition (Y,X). Then F has a component which is a path of length at least four.*

Proof. By Lemma 2 we have that $|X| = 3k$ and $|Y| = 4k$ for some integer k. We also have that $d_F(x) = 2$ for each vertex $x \in X$. If the length of all paths in F is two, then $|Y| \geq 2|X| = 6k$ which contradicts $|Y| = 4k$. Therefore F has a component which is a path of length at least four. □

Theorem 4. *Let F be a pseudo path factor of a simple $(3,4)$-biregular bigraph G with bipartition (Y,X). If $V_F \neq Y$ and y_0 is a vertex with $d_F(y_0) = 0$, then there is a pseudo path factor F' with $V_{F'} = V_F \cup \{y_0\}$, such that no path in F' is longer than the longest path in F.*

Proof. Let $y_0 \in Y$ and $d_F(y_0) = 0$. We will describe an algorithm which will construct a special trail T with origin y_0.

Step 1. Select an edge $y_0x_1 \notin E(F)$. Since $d_F(x_1) = 2$, there are two edges of F, x_1y_1 and x_1u_1, which are incident with x_1.

Case 1. $d_F(y_1) = 2$ or $d_F(u_1) = 2$.

Suppose, for example, that $d_F(y_1) = 2$. Then put $T = y_0 \to x_1 \to y_1$ and Stop.

Case 2. $d_F(y_1) = 1 = d_F(u_1)$.

Put $T = y_0 \to x_1 \to y_1$ and go to Step 2.

Step i $(i \geq 1)$**.** Suppose that we have already constructed a trail $T = y_0 \to x_1 \to y_1 \to \cdots \to x_i \to y_i$ which satisfies the following conditions:

(a) All edges in T are distinct and $y_{j-1}x_j \notin E(F)$, $x_jy_j \in E(F)$ for $j = 1, \ldots, i$.

(b) The vertices $y_1, \ldots, y_i$ are distinct.

(c) A component of F containing the vertex x_j is a path of length 2, for $j = 1, \ldots, i$.

Select an edge $e \in E(G) \setminus E(F)$ which is incident with y_i. The existence of such an edge follows from the conditions (a), (b) and (c). Moreover, the condition (b) implies that $e \notin T$. Let $e = y_ix_{i+1}$. Then $d_F(x_{i+1}) = 2$ because F is a pseudo path factor of G. Since $e \notin E(T)$, the conditions (a), (b) and (c) imply that at least one of the edges of F incident with x_{i+1}, does not belong to T.

Case 1. x_{i+1} lies on a component of F which is a path of length two.

Select a vertex y_{i+1} such that $x_{i+1}y_{i+1} \in E(F) \setminus E(T)$, add the edge $x_{i+1}y_{i+1}$ and the vertex y_{i+1} to T and go to step $(i+1)$. Now $T = y_0 \to x_1 \to y_1 \to \cdots \to x_{i+1} \to y_{i+1}$.

Case 2. x_{i+1} lies on a component of F which is a path of length at least four.
There is a vertex y_{i+1} such that $x_{i+1}y_{i+1} \in E(F) \setminus E(T)$ and $d_F(y_{i+1}) = 2$. Add the edge $x_{i+1}y_{i+1}$ and the vertex y_{i+1} to T and Stop. We have now that $T = y_0 \to x_1 \to y_1 \to \cdots \to x_{i+1} \to y_{i+1}$.

By Lemma 3, F has a component which is a path of length at least four. Therefore the algorithm will stop after a finite number of steps. Let the trail $T = y_0 \to x_1 \to y_1 \to \cdots \to x_{i+1} \to y_{i+1}$, be the result of the algorithm, where $i \geq 0$, the vertex x_j lies on a component of F which is a path of length two for each $j \leq i$, the vertex x_{i+1} lies on a component of F which is a path of length at least 4, and $d_F(y_{i+1}) = 2$. We define a new pseudo path factor F' by setting $V(F') = V(F)$ and

$$E(F') = (E(F) \setminus \{x_jy_j : j = 1, \ldots, i, i+1\}) \cup \{y_{j-1}x_j : j = 1, \ldots, i, i+1\}.$$

Clearly, $V_{F'} = V_F \cup \{y_0\}$ and the proof of Theorem 4 is complete. □

Theorems 1 and 4 imply the following theorem:

Theorem 5. *Every simple* $(3, 4)$*-biregular bigraph has a path factor such that the endpoints of each path have degree 3.*

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