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Fuzzy transform based approximation method for solving fractional semi-explicit differential–algebraic equations

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Abstract

We present an efficient numerical method to approximate the solution of a system of fractional-order linear semi-explicit differential-algebraic equations with variable coefficients. The method is based on the use of the direct and inverse fuzzy transforms ($\mathcal{F}$-transforms). By employing this method, we obtain an analytical approximate solution to the main problem in terms of flexible basic functions. The nonlocal property of fuzzy transforms helps us to have an efficient method for problems involving non-singular kernels. The error analysis and convergence evaluation of the method is demonstrated in detail. We give some examples to illustrate the significant features of the method.

Keywords: Fractional differential equations, differential–algebraic equations, Fuzzy partition, Fuzzy transforms

1 Introduction

A fractional differential equation (FDE) is a generalization of an ordinary one with an operator of a non-integer (fractional) order. The most important property of fractional operators is that they consider the entire history of the phenomena. Thus they are excellent tools to describe the memory and hereditary properties of phenomena and processes. The mathematical models of real-world problems are fractional-order systems in general. Some applications of fractional derivatives in continuum and statistical mechanics are given by Mainardi [9]. In many cases, when modeling real-world physical problems with hereditary effects, the states of the physical systems have in some ways constraints, for instance, by conservation laws such as Kirchhoff's laws in electrical networks, or by position constraints such as the movement of mass points on a surface. Then the corresponding mathematical models contain algebraic equations to describe these constraints and the fractional differential equations that describe the dynamics of the system. Such systems, comprising of both fractional differential and algebraic equations, are called fractional differential-algebraic equations (FDAEs). Thus DAEs and FDAEs are a natural way to model dynamical systems subject to constraints and hereditary effects(see [2]). Recently, fractional-order differential-algebraic equations have been studied by researchers in [3, 6, 23, 24]. Most of the FDAEs do not have exact solutions, then numerical techniques must be used to get approximate solutions for these types of equations; however, the numerical treatment of FDAEs may be more complicated than the numerical treatment of classical DAEs. By the authors' knowledge, there are only a few numerical methods for solving FDAEs (for example [6, 18, 24, 26]). So introducing an efficient numerical method for solving a system of FDEs is the subject of this paper. In this study, we are interested in using the technique of fuzzy transforms ($\mathcal{F}$-transforms) to give a simple structure and accurate approximate solution to the following initial value problem for the linear semi-explicit differential-algebraic equations with fractional-order and variable coefficients

$$\begin{cases} D_0^\alpha x(t) = E(t)x(t) + F(t)y(t) + Q_1(t), \\ 0 = G(t)x(t) + H(t)y(t) + Q_2(t), \\ x(0) = X_0, \quad y(0) = Y_0, \quad t \in I = [0, b] \end{cases} \tag{1}$$

where $m \in \mathbb{N}$, $\alpha \in (0,1)$, $b > 0$, $E, F, G, H \in C(I, \mathbb{R}^{m,m})$, $Q_j(t) \in C(I, \mathbb{R}^m)$ for $j = 1, 2$, $x(t) = [x_1, ..., x_m]^T$, $y(t) = [y_1, ..., y_m]^T$, and X_0, Y_0 are given as constant vectors. The operator D_0^α denotes the fractional operator of order α in Caputo sense [4, 12]. We also suppose that the consistency condition

$$0 = G(0)X_0 + H(0)Y_0 + Q_2(0)$$

is verified. Irina Perfilieva firstly introduced the $\mathcal{F}$-transform in 2006 and it received significant attention because of its strong connection with real-world problems such as the construction of approximate models, filtering, solving differential equations, application in signal processing, decompression of images, and data compression [5, 10, 13–16, 19–21]. The approximation property of $\mathcal{F}$-transforms and the effect of the shapes of basic functions on the approximation quality were described in [1, 13]. One interesting feature of $\mathcal{F}$-transform is its significant performance in noisy problems in which the inputs of the problem possess some disturbing noise. It has been demonstrated that the $\mathcal{F}$-transform acts as a filter and removes effectively the noises (see [27]). Another interesting feature of $\mathcal{F}$-transform is that in contrary to the traditional methods, which result in discrete solutions in the grid points, it gives continuous and even differentiable solutions. The approximate solution is as smooth as the basic functions $A_k(x)$. It means that if we need a smooth approximation, then we have to utilize the smooth basic functions. The most important properties of the $\mathcal{F}$-transforms technique can be summarized as follow:

- the error bound depends only on the modulus of continuity of the solution;
- the method is flexible in implementation;
- it gives sufficiently smooth piecewise best approximation in small support;
- it doesn't require any starting point or auxiliary function for starting.
- since the support of basic functions is compact, the computational cost deceases
- it is as accurate as the most of existing numerical methods.
- it can be generalized to the $\mathcal{F}^m$-transform based method which is more accurate.

The structure of this contribution is the following: a brief review of $\mathcal{F}$-transforms is given in Sections 2. The new technique is introduced in Section 3. The solvability of the corresponding algebraic system is investigated in Section 4. Illustrative examples are given in the final section.

2 Review of the fuzzy transforms

The method of fuzzy transforms is a well-known soft computing method applied to many practical problems. In this section, we recall some definitions and results from literature that will be used throughout the paper.

Definition 1 Let $[a, b]$ be an interval on $\mathbb{R}$, $n \geq 2$, and let $t_1, \ldots, t_n$ be fixed nodes within $[a, b]$, such that $a = t_1 < \ldots < t_n = b, t_0 = t_1, t_{n+1} = t_n$. We say that the fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1(t), \ldots, A_n(t)$ defined on $[a, b]$, form a fuzzy partition of $[a, b]$ if they verify the following conditions for $k = 1, \ldots, n$,

1. $A_k : [a, b] \to [0, 1]$ is continuous and $A_k(t_k) = 1$;
2. $A_1(t) > 0$ if $t \in [t_1, t_2)$, $A_k(t) > 0$ if $t \in (t_{k-1}, t_{k+1})$, $A_n(t) > 0$ if $t \in (t_{n-1}, t_n]$;
3. for all $t \in [a, b]$, $\sum_{k=1}^{n} A_k(t) = 1$;
4. for all $k = 2, \ldots, n$, $A_k(t)$ strictly increases on $[t_{k-1}, t_k]$ and for all $k = 1, \ldots, n-1$ strictly decreases on $[t_k, t_{k+1}]$.

The membership functions $A_1, \dots, A_n$ are called basic functions.

Remark 1 A fuzzy partition $A_1, \dots, A_n$, $n \geq 2$, is called $h-$uniform, if the nodes $t_1, \dots, t_n$ are $h-$equidistant, i.e., $t_k = a + h(k-1)$, $k = 1, \dots, n$, where $h = \frac{b-a}{n-1}$, and the following two additional properties are verified:

1. for all $k = 2, \dots, n-1$ and for all $t \in [0, h]$, $A_k(t_k - t) = A_k(t_k + t)$;
2. for all $k = 2, \dots, n-1$ and $t \in [t_k, t_{k+1}]$, $A_k(t) = A_{k-1}(t-h)$ and for all $k = 3, \dots, n$ and $t \in [t_{k-1}, t_k]$, $A_k(t) = A_{k-1}(t-h)$.

The uniform fuzzy partitions constructed by the triangular and sinusoidal membership functions as defined in Examples 1, 2, are the famous fuzzy partitions for a given interval $[a, b]$.

Example 1 An h-uniform fuzzy partition $A_1, \dots, A_n$ by triangular shaped basic functions is defined by

$$A_k(t) = \begin{cases} \frac{t - t_{k-1}}{h}, & t \in [t_{k-1}, t_k], \\ \frac{t_{k+1} - t}{h}, & t \in [t_k, t_{k+1}], \\ 0, & t \notin [t_{k-1}, t_{k+1}], \end{cases} \tag{2}$$

for $k = 1, \dots, n$, where $t_0 = t_1, \quad t_{n+1} = t_n$. This partition illustrated in Fig. 1 for the interval $[0, 1]$.

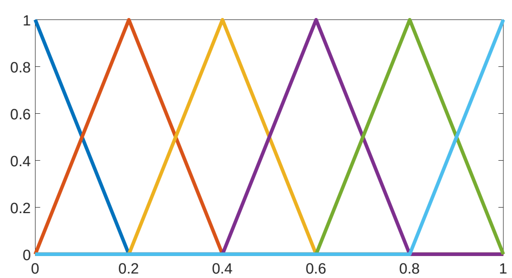

Fig. 1 A uniform fuzzy partition of $[0, 1]$ by triangular membership function.

Example 2 An h-uniform fuzzy partition $A_1, \dots, A_n$ by sinusoidal shaped basic functions is defined by

$$A_k(t) = \begin{cases} \frac{1}{2}(\cos \frac{\pi}{h}(t - t_k) + 1), & t \in [t_{k-1}, t_{k+1}], \\ 0, & o.w. \end{cases} \tag{3}$$

for $k = 1, \dots, n$, where $t_0 = t_1, \quad t_{n+1} = t_n$. See Fig. 2 for the illustration of this partition.

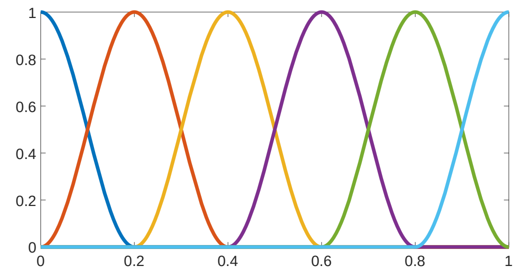

Fig. 2 A uniform fuzzy partition of $[0, 1]$ by sinusoidal membership function.

Definition 2 (Direct $\mathcal{F}$-transform). Let $A_1, \dots, A_n$ be basic functions, which form a fuzzy partition of $[a, b]$, and f be any continuous function on $[a, b]$. The n-tuple $[\mathcal{F}_1(f), \dots, \mathcal{F}_n(f)]$ of real numbers with the components given by

$$\mathcal{F}_k(f) = \frac{\int_a^b f(t) A_k(t) dt}{\int_a^b A_k(t) dt}, \quad k = 1, \dots, n,$$

is called the $\mathcal{F}$-transform of f with respect to $A_1, \dots, A_n$, and is denoted by $\mathcal{F}(f)$.

Definition 3 Let $f : [a, b] \to \mathbb{R}$ be a given function, and let $[\mathcal{F}_1(f), \mathcal{F}_2(f), \dots, \mathcal{F}_n(f)]$ be the $\mathcal{F}$-transform of f with respect to $A_1, \dots, A_n$. The function $\mathcal{F}_f^{-1} : [a, b] \to \mathbb{R}$ defined by

$$\mathcal{F}_f^{-1}(t) = \check{f}_n(t) = \sum_{k=1}^{n} \mathcal{F}_k(f) A_k(t),$$

is called the inverse $\mathcal{F}$-transform of f.

Remark 2 (see[8, 17]) Let f be a twice differentiable function on $[a, b]$ and $A_1, \dots, A_n$ be basic functions which form a fuzzy partition of $[a, b]$. Then for each $k = 1, \dots, n$, by applying the trapezoidal rule with the nodes t_{k-1}, t_k, t_{k+1} to numeric integration of $\frac{1}{h} \int_{t_{k-1}}^{t_{k+1}} f(t) A_k(t) dt$, we obtain

$$\mathcal{F}_k(f) = f(t_k) - \frac{h^2}{12} f''(\mu), \quad \mu \in (t_{k-1}, t_{k+1}).$$

Therefore $\mathcal{F}_k(f) = f(t_k) + O(h^2)$.

The following results hold true for the $\mathcal{F}$-transform of f (see[11]):

(A) Let f be a given continuous function on $[a, b]$. Then the kth component of the $\mathcal{F}$-transform of f minimizes the function

$$\phi_k(y) = \int_a^b (f(t) - y)^2 A_k(t) dt$$

on $\mathbb{R}$.

(B) $\mathcal{F}$ is linear; i.e., for all $f, g \in C[a,b]$, and for all $\alpha, \beta \in \mathbb{R}$,

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g).$$

We are now ready to give two theorems that play crucial roles in our discussion.

Theorem 1 *(see[25]) Let $A_1, \ldots, A_n$ be an $h-$uniform fuzzy partition of $[a,b]$. Let f be a continuous function on $[a,b]$ and $f(x) = \check{f}_n(x) + r_n(x) = \sum_{k=1}^{n} \mathcal{F}_k(f) A_k(x) + r_n(x)$. Then $\mathsf{r\_n(x)} \leq \lambda \omega(f,h)$ for $x \in [a,b]$, where $\omega(f,h)$ is the modulus of continuity of f and λ is a constant independent of n.*

Theorem 2 *(see[25]) Let $h(t,s) = h_s(t)$ be integrable function on $[a,b]^2$ and $g(t) := \int_a^t h(t,s)ds$, and $A_1, \ldots, A_n$ be an $h-$uniform fuzzy partition of $[a,b]$ with the nodes $t_1 = a, \ldots, t_n = b$. If $\mathcal{F}(h_s) = [\mathcal{F}_1(h_s), \mathcal{F}_2(h_s), \ldots, \mathcal{F}_n(h_s)]$ and $\mathcal{F}(g) = [\mathcal{F}_1(g), \mathcal{F}_2(g), \ldots, \mathcal{F}_n(g)]$ are the $\mathcal{F}$-transforms of h_s and g respectively, then*

$$\begin{aligned}
\mathcal{F}_1(g) &= \int_{t_1}^{t_2} \mathcal{F}_1(h_s u_s) ds, \\
\mathcal{F}_k(g) &= \int_{t_1}^{t_{k-1}} \mathcal{F}_k(h_s) ds \\
&\quad + \int_{t_{k-1}}^{t_{k+1}} \mathcal{F}_k(h_s u_s) ds, k = 2, \ldots, n-1 \\
\mathcal{F}_n(g) &= \int_{t_1}^{t_{n-1}} \mathcal{F}_n(h_s) ds + \int_{t_{n-1}}^{t_n} \mathcal{F}_n(h_s u_s) ds,
\end{aligned}$$

in which

$$u_s(t) = \begin{cases} 0, & 0 \leq t < s \\ 1, & t \geq s. \end{cases}$$

3 The method of fuzzy transforms

In this section, we employ a numerical method based on fuzzy transforms to (1). For the convenience of notations and without loss of generality, we concentrate on the case $m = 1$ and $b = 1$. Recalling the problem with $m = 1$, we have

$$\begin{cases} D^\alpha x(t) = E(t)x(t) + F(t)y(t) + Q_1(t), \\ \qquad 0 = G(t)x(t) + H(t)y(t) + Q_2(t), \\ \quad x(0) = X_0, \quad y(0) = Y_0, \quad t \in I = [0,1], \end{cases} \tag{4}$$

where D^α is the Caputo-type fractional derivative of order α defined as ([4, 12])

$$D^\alpha x(t) = I^{1-\alpha}(x'(t)),$$

in which

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

is the Riemann-Liouville-type fractional integral operator of order α and $\Gamma(\alpha)$ denotes the Gamma function. We will use the following relations

$$I^\alpha(D^\alpha x(t)) = x(t) - x(0), \tag{5}$$

$$D^\alpha t^\beta = \begin{cases} \frac{\gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{k-\alpha}, & \beta - \alpha > -1, \\ 0, \quad k = 0. \end{cases} \tag{6}$$

We are going to present a new method based on the $\mathcal{F}$-transform to approximate the solution of (4). Since $H(t)$ is invertible, without loss of generality we assume that $H(t) = 1$. Hence, we have

$$D^\alpha x(t) = E(t)x(t) + F(t)(-Q_2(t) - G(t)x(t)) + Q_1(t). \tag{7}$$

Let $n \in \mathbb{N}, n \geq 2$ and A_k, $k = 1, \ldots, n$ be a uniform fuzzy partition of interval [0,1] with the step size $h = \frac{1}{n-1}$, $t_1 = 0$, $t_n = 1$. We first apply the operator I^α to both sides of (7). Then using (5) and linearity of I^α, we obtain

$$\begin{aligned}
\Gamma(\alpha)(x(t) - x(0)) &= \int_0^t (t-s)^{\alpha-1} E(s)x(s) ds \\
&+ \int_0^t (t-s)^{\alpha-1} K_1(s) ds + \int_0^t (t-s)^{\alpha-1} K_2(s)x(s) ds \\
&+ \int_0^t (t-s)^{\alpha-1} Q_1(s) ds,
\end{aligned} \tag{8}$$

where $K_1(s) = -F(s)Q_2(s)$, $K_2(s) = -F(s)G(s)$. By applying $\mathcal{F}$-transform on both sides of (8) and Theorem 2, we deduce

$$\begin{aligned}
\Gamma(\alpha)\left(\mathcal{F}_1(x) - \mathcal{F}_1(X_0)\right) &= \int_{t_1}^{t_2} E(s)x(s)\bar{P}_1(s) ds \\
&+ \int_{t_1}^{t_2} K_1(s)\bar{P}_1(s) ds + \int_{t_1}^{t_2} K_2(s)x(s)\bar{P}_1(s) ds \\
&+ \int_{t_1}^{t_2} Q_1(s)\bar{P}_1(s) ds,
\end{aligned} \tag{9}$$

$$\Gamma(\alpha)\left(\mathcal{F}_k(x)-\mathcal{F}_k(X_0)\right)=\int_{t_1}^{t_{k-1}} E(s)x(s)P_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}} E(s)x(s)\bar{P}_k(s)ds+\int_{t_1}^{t_{k-1}} K_1(s)P_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}} K_1(s)\bar{P}_k(s)ds+\int_{t_1}^{t_{k-1}} K_2(s)x(s)P_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}} K_2(s)x(s)\bar{P}_k(s)ds+\int_{t_1}^{t_{k-1}} Q_1(s)P_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}} Q_1(s)\bar{P}_k(s)ds, \qquad (10)$$

for $k=2,\dots,n-1$, and

$$\Gamma(\alpha)\left(\mathcal{F}_n(x)-\mathcal{F}_n(X_0)\right)=\int_{t_1}^{t_{n-1}} E(s)x(s)P_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n} E(s)x(s)\bar{P}_n(s)ds+\int_{t_1}^{t_{n-1}} K_1(s)P_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n} K_1(s)\bar{P}_n(s)ds+\int_{t_1}^{t_{n-1}} K_2(s)x(s)P_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n} K_2(s)x(s)\bar{P}_n(s)ds+\int_{t_1}^{t_{n-1}} Q_1(s)P_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n} Q_1(s)\bar{P}_n(s)ds, \qquad (11)$$

where

$$P_k(s)=\mathcal{F}_k((t-s)^{\alpha-1})=\frac{\int_{t_{k-1}}^{t_{k+1}}(t-s)^{\alpha-1}A_k(t)dt}{\int_{t_{k-1}}^{t_{k+1}}A_k(t)dt}$$
$$\bar{P}_k(s)=\mathcal{F}_k\left((t-s)^{\alpha-1}u_s(t)\right)=\frac{\int_{t_{k-1}}^{t_{k+1}}(t-s)^{\alpha-1}u_s(t)A_k(t)dt}{\int_{t_{k-1}}^{t_{k+1}}A_k(t)dt}=\frac{\int_{s}^{t_{k+1}}(t-s)^{\alpha-1}A_k(t)dt}{\int_{t_{k-1}}^{t_{k+1}}A_k(t)dt},$$

$k=1,\dots,n$. Using X_k instead of $\mathcal{F}_k(x)$ for $k=1,\dots,n$ and inserting

$$x(s)=\mathcal{F}_x^{-1}(s)+r_n(s)=\check{x}_n(s)+r_n(s)=\sum_{i=1}^{n}X_iA_i(s)+r_n(s)$$

into the equations (9)-(11), we have

$$\Gamma(\alpha)(X_1-X_0)=$$
$$\int_{t_1}^{t_2} E(s)\sum_{i=1}^{n}X_iA_i(s)\bar{P}_1(s)ds$$
$$+\int_{t_1}^{t_2} K_2(s)\sum_{i=1}^{n}X_iA_i(s)\bar{P}_1(s)ds+$$
$$\int_{t_1}^{t_2}\big(K_1(s)+Q_1(s)\big)\bar{P}_1(s)ds$$
$$+\int_{t_1}^{t_2}\big(E(s)+K_2(s)\big)\bar{P}_1(s)r_n(s)ds$$
$$=X_1\Big(\int_{t_1}^{t_2}\big(E(s)+K_2(s)\big)A_1(s)\bar{P}_1(s)ds\Big)$$
$$+X_2\Big(\int_{t_1}^{t_2}\left(E(s)+K_2(s)\right)A_2(s)\bar{P}_1(s)ds\Big)$$
$$+\int_{t_1}^{t_2}\big(K_1(s)+Q_1(s)\big)\bar{P}_1(s)ds+\int_{t_1}^{t_2}\big(E(s)$$
$$+K_2(s)\big)\bar{P}_1(s)r_n(s)ds,$$

$$\Gamma(\alpha)(X_k-X_0)=$$
$$\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)\sum_{i=1}^{n}X_iA_i(s)P_k(s)ds$$
$$+\int_{t_1}^{t_{k-1}}\big(K_1(s)+Q_1(s)\big)P_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)\sum_{i=1}^{n}X_iA_i(s)\bar{P}_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}}\big(K_1(s)+Q_1(s)\big)\bar{P}_k(s)ds$$
$$+\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)P_k(s)r_n(s)ds+$$
$$\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)\bar{P}_k(s)r_n(s)ds$$
$$=\sum_{i=1}^{k-1}X_i\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)A_i(s)P_k(s)ds$$
$$+\int_{t_1}^{t_{k-1}}\big(K_1(s)+Q_1(s)\big)P_k(s)ds$$
$$+\sum_{i=k-1}^{k+1}X_i\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)A_i(s)\bar{P}_k(s)ds$$

$$+\int_{t_{k-1}}^{t_{k+1}}\big(K_1(s)+Q_1(s)\big)\bar{P}_k(s)ds$$
$$+\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)P_k(s)r_n(s)ds+$$
$$\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)\bar{P}_k(s)r_n(s)ds,$$

for $k=2,\ldots,n-1$, and

$$\Gamma(\alpha)(X_n-X_0)=$$
$$\int_{t_1}^{t_{n-1}}\big(E(s)+K_2(s)\big)\sum_{i=1}^{n}X_iA_i(s)P_n(s)ds$$
$$+\int_{t_1}^{t_{n-1}}\big(K_1(s)+Q_1(s)\big)P_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n}\big(E(s)+K_2(s)\big)\sum_{i=1}^{n}X_iA_i(s)\bar{P}_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n}\big(K_1(s)+Q_1(s)\big)\bar{P}_n(s)ds$$
$$+\int_{t_1}^{t_{n-1}}\big(E(s)+K_2(s)\big)P_n(s)r_n(s)ds+$$
$$\int_{t_{n-1}}^{t_n}\big(E(s)+K_2(s)\big)\bar{P}_n(s)r_n(s)ds$$
$$=\sum_{i=1}^{n-1}X_i\int_{t_1}^{t_{n-1}}\big(E(s)+K_2(s)\big)A_i(s)P_n(s)ds$$
$$+\int_{t_1}^{t_{n-1}}\big(K_1(s)+Q_1(s)\big)P_n(s)ds$$
$$+\sum_{i=n-1}^{n}X_i\int_{t_{n-1}}^{t_n}\big(E(s)+K_2(s)\big)A_i(s)\bar{P}_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n}\big(K_1(s)+Q_1(s)\big)\bar{P}_n(s)ds$$
$$+\int_{t_1}^{t_{n-1}}\big(E(s)+K_2(s)\big)P_n(s)r_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n}\big(E(s)+K_2(s)\big)\bar{P}_n(s)r_n(s)ds.$$

In the matrix form, we have

$$(I-\frac{1}{\Gamma(\alpha)}M)X=\frac{-1}{\Gamma(\alpha)}(B+R), \qquad (12)$$

where I is the identity matrix, and

$$X=\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix},\ B=\begin{bmatrix}B_1\\ \vdots\\ B_n\end{bmatrix},\ R=\begin{bmatrix}R_1\\ \vdots\\ R_n\end{bmatrix},$$

$$M=\begin{bmatrix} m_{11} & m_{12} & 0 & 0 & \ldots & 0\\ m_{21} & m_{22} & m_{23} & 0 & \ldots & 0\\ m_{31} & m_{32} & m_{33} & m_{34} & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ m_{n-11} & m_{n-12} & m_{n-13} & m_{n-14} & \ldots & m_{n-1n}\\ m_{n1} & m_{n2} & m_{n3} & m_{n4} & \ldots & m_{nn}\end{bmatrix}$$

in which

$$m_{kj}=\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)A_j(s)P_k(s)ds,$$
$$k=2,\ldots,n, j=1,\ldots,k-2,$$

$$m_{kk-1}=\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)A_{k-1}(s)P_k(s)ds$$
$$+\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)A_{k-1}(s)\bar{P}_k(s)ds,$$
$$k=1,\ldots,n-1,$$

$$m_{kj}=\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)A_j(s)\bar{P}_k(s)ds,$$
$$k=1,\ldots,n-1, j=k,k+1,$$

$$m_{nn-1}=\int_{t_1}^{t_{n-1}}\big(E(s)+K_2(s)\big)A_{n-1}(s)P_n(s)ds$$
$$+\int_{t_{n-1}}^{t_n}\big(E(s)+K_2(s)\big)A_{n-1}(s)\bar{P}_n(s)ds,$$
$$m_{nn}=\int_{t_{n-1}}^{t_n}\big(E(s)+K_2(s)\big)A_n(s)\bar{P}_n(s)ds,$$

and

$$R_1=-\int_{t_1}^{t_2}\big(E(s)+K_2(s)\big)\bar{P}_1(s)r_n(s)ds,$$
$$R_k=-\int_{t_1}^{t_{k-1}}\big(E(s)+K_2(s)\big)P_k(s)r_n(s)ds$$
$$-\int_{t_{k-1}}^{t_{k+1}}\big(E(s)+K_2(s)\big)\bar{P}_k(s)r_n(s)ds,$$

$$k = 2, \dots, n$$

$$R_n = -\int_{t_1}^{t_{n-1}} \big(E(s) + K_2(s)\big) P_n(s) r_n(s) ds$$
$$- \int_{t_{n-1}}^{t_n} \big(E(s) + K_2(s)\big) \bar{P}_n(s) r_n(s) ds,$$

and

$$B_1 = -\int_{t_1}^{t_2} \big(K_1(s) + Q_1(s)\big) \bar{P}_1(s) ds - X_0 \Gamma(\alpha),$$
$$B_k = -\int_{t_1}^{t_{k-1}} \big(K_1(s) + Q_1(s)\big) P_k(s) ds$$
$$- \int_{t_{k-1}}^{t_{k+1}} \big(K_1(s) + Q_1(s)\big) \bar{P}_k(s) ds - X_0 \Gamma(\alpha),$$
$$k = 2, \dots, n$$
$$B_n = -\int_{t_1}^{t_{n-1}} \big(K_1(s) + Q_1(s)\big) P_n(s) ds$$
$$- \int_{t_{n-1}}^{t_n} \big(K_1(s) + Q_1(s)\big) \bar{P}_n(s) ds - X_0 \Gamma(\alpha).$$

We now consider the system

$$(I - \frac{1}{\Gamma(\alpha)} M) \hat{X} = \frac{-1}{\Gamma(\alpha)} B. \tag{13}$$

By solving (13), the unknown values $\hat{X}_1, \dots, \hat{X}_n$ are obtained. The solutions of this system, $\hat{X}_1, \dots, \hat{X}_n$, are the approximate values of $X_1, \dots, X_n$ which are the components of $\mathcal{F}-$transform of $x(t)$. Then the approximate solution of the problem (1) is given by the inverse $\mathcal{F}$-transform, i.e.

$$\hat{x}_n(t) = \sum_{k=1}^{n} \hat{X}_k A_k(t). \tag{14}$$

4 Solvability and convergence

In this section, we first show that under some sufficient conditions the system (13) is solvable. Then we investigate the convergence of the approximate solution to the exact solution of problem (1). For solvability of the system (13), by applying the geometric series theorem, it suffices to show $\|\frac{1}{\Gamma(\alpha)} M\| < 1$. This is done in the following theorem.

Theorem 3 *Let sets $A_k, k = 1, \dots, n$ form a fuzzy partition to $[0,1]$ for $n \in \mathbb{N}$ and M be the matrix given in the system (13). Let $\sup_{[0,1]} \mathsf{E(s)+K\_–2''(s)} \le C$. If C is sufficiently small, then $I - \frac{1}{\Gamma(\alpha)} M$ is invertible.*

Proof We have to compute the norm of the coefficient matrix M. To do this, we first notice that for $k = 2, \dots, n$ with $m_{n,n+1} = 0$, we have

$$\sum_{j=1}^{k+1} \mid m_{kj} \mid =$$
$$\sum_{j=1}^{k-1} \mid \int_{t_1}^{t_{k-1}} \big(E(s) + K_2(s)\big) A_j(s) P_k(s) ds \mid$$
$$+ \sum_{j=k-1}^{k+1} \mid \int_{t_{k-1}}^{t_{k+1}} \big(E(s) + K_2(s)\big) A_j(s) \bar{P}_k(s) ds \mid$$
$$\le \int_{t_1}^{t_2} A_0(\frac{s - t_1}{h}) \mid \big(E(s) + K_2(s)\big) P_k(s) \mid ds$$
$$+ \sum_{j=2}^{k-2} \int_{t_{j-1}}^{t_{j+1}} A_0(\frac{s - t_j}{h}) \mid \big(E(s) + K_2(s)\big) P_k(s) \mid ds$$
$$+ \int_{t_{k-2}}^{t_{k-1}} A_0(\frac{s - t_{k-1}}{h}) \mid \big(E(s) + K_2(s)\big) P_k(s) \mid ds$$
$$+ \int_{t_{k-1}}^{t_k} A_0(\frac{s - t_{k-1}}{h}) \mid \big(E(s) + K_2(s)\big) \bar{P}_k(s) \mid ds$$
$$+ \int_{t_{k-1}}^{t_{k+1}} A_0(\frac{s - t_k}{h}) \mid \big(E(s) + K_2(s)\big) \bar{P}_k(s) \mid ds$$
$$+ \int_{t_k}^{t_{k+1}} A_0(\frac{s - t_{k+1}}{h}) \mid \big(E(s) + K_2(s)\big) \bar{P}_k(s) \mid ds$$
$$\le hC \int_0^1 A_0(u) P_k(t_1 + uh) du$$
$$+ hC \sum_{j=2}^{k-2} \int_{-1}^{1} A_0(u) p_k(t_j + uh) du$$
$$+ hC \int_{-1}^{0} A_0(u) P_k(t_{k-1} + uh) du$$
$$+ hC \int_0^1 A_0(u) \bar{P}_k(t_{k-1} + uh) du$$
$$+ hC \int_{-1}^{1} A_0(u) \bar{P}_k(t_k + uh) du +$$
$$hC \int_{-1}^{0} A_0(u) \bar{P}_k(t_{k+1} + uh) du$$
$$\le hC [\int_0^1 P_k(t_1 + uh) du + \sum_{j=2}^{k-2} \int_{-1}^{1} P_k(t_j + uh) du$$
$$+ \int_{-1}^{0} P_k(t_{k-1} + uh) du + \int_0^1 \bar{P}_k(t_{k-1} + uh) du +$$

$$\int_{-1}^{1} \bar{P}_k(t_k + uh)du + \int_{-1}^{0} \bar{P}_k(t_{k+1} + uh)du\Big],$$

where C is the upper bound of $\mid E(s) + K_2(s) \mid$ $\big(i.e., \mid (E(s) + K_2(s)) \mid \leq C,\ \forall s \in [0,1]\big)$. Now, we prove that $\sum_{j=2}^{k-2} \int_{-1}^{1} \mid P_k(t_j + uh) \mid du$ is bounded:

$$h \sum_{j=2}^{k-2} P_k(t_j + uh) =$$

$$h \sum_{j=2}^{k-2} \mid \frac{\int_{t_{k-1}}^{t_{k+1}} A_k(t)(t - t_j - uh)^{\alpha-1} dt}{\int_{t_{k-1}}^{t_{k+1}} A_k(t)dt} \mid$$

$$\leq h \sum_{j=2}^{k-2} \frac{\int_{t_{k-1}}^{t_{k+1}} A_0(\frac{t-t_k}{h})(t - t_j - uh)^{\alpha-1} dt}{\int_{t_{k-1}}^{t_{k+1}} A_0(\frac{t-t_k}{h})dt}$$

$$= h \sum_{j=2}^{k-2} \frac{\int_{-1}^{1} A_0(v)(t_k + vh - t_j - uh)^{\alpha-1} dv}{\int_{-1}^{1} A_0(v)dv}$$

$$\leq h^{\alpha} \sum_{j=2}^{k-2} \frac{\int_{-1}^{1} (k - j + v - u)^{\alpha-1} dv}{\int_{-1}^{1} A_0(v)dv}$$

$$= h^{\alpha} \sum_{j=2}^{k-2} \int_{-1}^{1} (j + v - u)^{\alpha-1} dv$$

$$= \frac{h^{\alpha}}{\alpha} \sum_{j=2}^{k-2} (j - u + 1)^{\alpha} - (j - u - 1)^{\alpha}. \tag{15}$$

For $j \geq 2$ and $-1 \leq u \leq 1$, using the mean value theorem there exists $-1 < \delta < 1$ such that

$$\frac{(j - u + 1)^{\alpha} - (j - u - 1)^{\alpha}}{1 - (-1)} = \alpha(j - u + \delta)^{\alpha-1} \tag{16}$$

From (15), (16) and substituting $h = \frac{1}{n}$, we have

$$h \sum_{j=2}^{k-2} P_k(t_j + uh) \leq 2 \sum_{j=2}^{k-2} \frac{1}{n} (\frac{j - u + \delta}{n})^{\alpha-1}$$

$$\leq 2 \sum_{j=2}^{k-2} \frac{1}{n} (\frac{j - u - 1}{n})^{\alpha-1} \leq 2 \sum_{j=2}^{k-2} \frac{1}{n} (\frac{j - 2}{n})^{\alpha-1}$$

$$= 2 \sum_{J=0}^{k-4} \frac{1}{n} (\frac{J}{n})^{\alpha-1} \leq 2 \sum_{J=0}^{n} \frac{1}{n} (\frac{J}{n})^{\alpha-1}$$

$$\leq 2 \int_{0}^{1} x^{\alpha-1} dx = \frac{2}{\alpha}. \tag{17}$$

By a similar calculation, we can prove that

$$hC \Big[\int_{0}^{1} P_k(t_1 + uh)du + \sum_{j=2}^{k-2} \int_{-1}^{1} p_k(t_j + uh)du$$

$$+ \int_{-1}^{0} p_k(t_{k-1} + uh)du + \int_{0}^{1} \bar{P}_k(t_{k-1} + uh)du$$

$$+ \int_{-1}^{1} \bar{P}_k(t_k + uh)du + \int_{-1}^{0} \bar{P}_k(t_{k+1} + uh)du \Big]$$

$$\leq CC_1, \tag{18}$$

where C_1 is a constant independent from n. By (15) and (18), we conclude

$$\sum_{j=1}^{k+1} \text{m\_-kj''} \leq CC_1, \tag{19}$$

for $k = 2, \ldots, n$. In a similar way, we can show that there exists a constant C_2 such that

$$m_{11} + m_{12} \leq CC_2.$$

Thus

$$\left\| \frac{1}{\Gamma(\alpha)} M \right\| = \frac{1}{\Gamma(\alpha)} \|M\| = \frac{1}{\Gamma(\alpha)} \max_{k} \sum_{j=1}^{n} \mid m_{kj} \mid$$

$$\leq \frac{1}{\Gamma(\alpha)} C \max\{C_1, C_2.\}$$

Finally, for sufficiently small C we conclude that $\frac{C \max\{C_1, C_2\}}{\Gamma(\alpha)} < 1$ which implies that $(I - \frac{1}{\Gamma(\alpha)} M)$ is invertible. □

The theorem above guarantees that the system (13) has a unique solution. At this stage, we intend to prove the convergence of the proposed method for Problem (1).

Theorem 4 *Assume the hypothesis of Theorem 3. Let* $\hat{x}_n(t) = \sum_{k=1}^{n} \hat{X}_k A_k(t)$, *where* $\hat{X}_k$ *are the solutions of the System* (13) *and* $x(t)$ *be the exact solution of the initial value problem* (1). *Then*

$$\lim_{n \to \infty} \|x - \hat{x}_n\| = \lim_{n \to \infty} \sup_{t \in [0,1]} \mid x(t) - \hat{x}_n(t) \mid = 0.$$

Proof Let $\check{x}_n(t) = \mathcal{F}_x^{-1}(t) = \sum_{k=1}^{n} X_k A_k(t)$ be the inverse fuzzy transform of $x(t)$, $t \in [0,1]$ and $\hat{x}_n(t)$ be as in (14). Then

$$\mid x(t) - \hat{x}_n(t) \mid \leq \mid x(t) - \check{x}_n(t) \mid + \mid \check{x}_n(t) - \hat{x}_n(t) \mid, \quad t \in [0,1].$$

From Theorem 1 we have

$$\lim_{n \to \infty} \|x - \check{x}_n\| = 0.$$

On the other hand,

$$\mid \check{x}_n(t) - \hat{x}_n(t) \mid \leq \sum_{k=1}^{n} \mid X_k - \hat{X}_k \mid A_k(t), t \in [0,1],$$

where $[X_1, \ldots, X_n]$ and $[\hat{X}_1, \ldots, \hat{X}_n]$ are the solutions of the systems

$$(I - \frac{1}{\Gamma(\alpha)} M)X = \frac{-1}{\Gamma(\alpha)}(B + R),$$

and

$$(I - \frac{1}{\Gamma(\alpha)} M)\hat{X} = \frac{-1}{\Gamma(\alpha)} B,$$

respectively. Hence

$$(I - \frac{1}{\Gamma(\alpha)}M)(X - \hat{X}) = \frac{-1}{\Gamma(\alpha)}R.$$

We notice that $(I - \frac{1}{\Gamma(\alpha)}M)^{-1}$ exists. Thus,

$$(X - \hat{X}) = -\frac{1}{\Gamma(\alpha)}(I - \frac{1}{\Gamma(\alpha)}M)^{-1}R,$$

and consequently,

$$\begin{aligned}\left\|X - \hat{X}\right\| &= \max_k \mid X_k - \hat{X}_k \mid \\ &= \left\|\frac{1}{\Gamma(\alpha)}(I - \frac{1}{\Gamma(\alpha)}M)^{-1}\right\| \|R\|, \qquad (20)\end{aligned}$$

where

$$\|R\| = \max_k \mid R_k \mid .$$

For $k = 2, \dots, n-1$

$$\begin{aligned}&\mid R_k \mid = \\ &\mid -\int_{t_1}^{t_{k-1}} \big(E(s) + K_2(s)\big)P_k(s)r_n(s)ds \\ &- \int_{t_{k-1}}^{t_{k+1}} \big(E(s) + K_2(s)\big)\bar{P}_k(s)r_n(s)ds \mid \\ &\le \mid \int_{t_1}^{t_{k-1}} \big(E(s) + K_2(s)\big)P_k(s)r_n(s)ds \mid \\ &+ \mid \int_{t_{k-1}}^{t_{k+1}} \big(E(s) + K_2(s)\big)\bar{P}_k(s)r_n(s)ds \mid \\ &\le \int_{t_1}^{t_{k-1}} \mid E(s) + K_2(s) \mid\mid r_n(s) \mid P_k(s)ds \\ &+ \int_{t_1}^{t_{k-1}} \mid E(s) + K_2(s) \mid\mid r_n(s) \mid \bar{P}_k(s)ds \\ &\le C(\int_{t_1}^{t_{k-1}} \mid r_n(s) \mid P_k(s)ds \\ &+ \int_{t_1}^{t_{k-1}} \mid r_n(s) \mid \bar{P}_k(s)ds) \\ &\le C\lambda\omega(x,h)\big(\int_{t_1}^{t_{k-1}} P_k(s)ds \\ &+ \int_{t_1}^{t_{k-1}} \bar{P}_k(s)ds\big) \\ &\le \lambda C\acute{C}\omega(x,h)(kh) \le \lambda C\acute{C}\omega(x,h), \qquad (21)\end{aligned}$$

where

$$\acute{C} = \max\{\max_{k\in\{2,\dots,n\}} \left\|P_k\right\|, \max_{k\in\{1,\dots,n\}} \left\|\bar{P}_k\right\|\},$$

with $\|P_k\| = \max_{t\in[0,1]} P_k(t)$ and $\|\bar{P}_k\| = \max_{t\in[0,1]} \bar{P}_k(t)$. It is easy to show that $\acute{C}$ is a constant independent of n. In a similar argument, we can obtain the same upper bound for R_1 and R_n. Thus, (20) and (21) result

$$\left\|X - \hat{X}\right\| \le \frac{1}{\Gamma(\alpha)}\lambda C\acute{C}\omega(x,h)\left\|(I - \frac{1}{\Gamma(\alpha)}M)^{-1}\right\|.$$

Since $\lim_{n\to\infty} \omega(x, \frac{1}{n}) = 0$, by recalling Theorem 3, we deduce the assertion. □

Table 1 The maximum absolute errors for the Example 3.

α	n=11	n=21	n=51
1	7.7000e-04	1.9536e-04	2.6984e-05
0.9	7.2760e-04	1.6495e-04	1.9150e-04
$\frac{\sqrt{2}}{2}$	6.7138e-04	1.6754e-04	1.5943e-05
0.5	6.0821e-04	1.3352e-04	5.8022e-05
$\frac{\sqrt{2}}{5}$	5.6629e-04	1.3326e-04	2.2273e-05

5 Examples

Example 3 Consider the fractional-order differential-algebraic equations

$$\begin{cases} D^{\alpha}x(t) = -x(t) + \frac{t^{4-\alpha}}{\Gamma(5-\alpha)}y(t) + Q_1(t), \\ \quad 0 = x(t) + (1+t^2)y(t) + Q_2(t), \\ \quad x(0) = 0, \quad t \in I = [0,1] \end{cases} \qquad (22)$$

where $Q_1(t) = 0$, $Q_2(t) = -t^4E_{\alpha,5}(-t^{\alpha}) - t^2 - 1$ and $\{x(t), y(t)\} = \left\{t^4E_{\alpha,5}(-t^{\alpha}), 1\right\}$ denotes the set of exact solutions, in which $E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}$ denotes the Mittag-Leffler function of two-parameters. Now we solve (22) by the method of Section 3, where we construct the basic functions $A_1, \dots, A_n$, as an h-uniform partition of the interval [0,1], by the triangular-shaped basic functions (2) and form the system

$$(I - \frac{1}{\Gamma(\alpha)}M)\hat{X} = \frac{-B}{\Gamma(\alpha)} \qquad (23)$$

to find the approximations $\hat{X}_1, \dots, \hat{X}_n$ to $X_1, \dots, X_n$. Then, by using the inverse $\mathcal{F}$-transform, we get the approximation

$$\hat{x} = \mathcal{F}_x^{-1}(t) = \sum_{k=1}^{n} \hat{X}_k A_k(t),$$

to the exact solution $x(t)$.

For $n = 11(h = 0.1), n = 21(h = 0.02), n = 51(h = 0.01)$ and different values of α, we report the maximum values of absolute errors in Table 1. The plots of these errors are shown in Figs. 3 and 4 for $\alpha = \frac{\sqrt{2}}{5}$, $n = 11$ and $n = 51$.

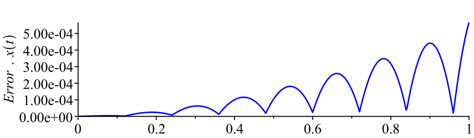

Fig. 3 The plots of absolute error for the Example 3($n = 11$).

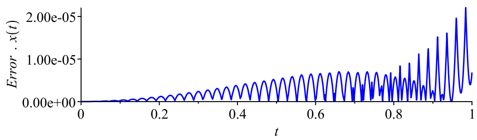

Fig. 4 The plots of absolute error the Example 3($n = 51$).

Table 2 The maximum absolute errors for the Example 4.

α	n=11	n=21	n=51
1	1.5987e-02	4.0045e-03	5.0199e-04
0.5	9.8993e-03	6.7818e-03	1.3934e-03
$\frac{\sqrt{3}}{2}$	1.4255e-02	3.4441e-03	3.9900e-04
$\frac{\sqrt{3}}{5}$	8.5122e-03	2.3331e-03	3.0215e-04
0.09	6.6778e-03	1.5668e-03	2.2932e-04

Example 4 Consider the fractional-order differential-algebraic equations

$$\begin{cases} D^{\alpha}x(t) = t^{\frac{1}{2}}x(t) + ty(t) + Q_1(t), \\ \quad 0 = x(t) + t^{\frac{1}{2}}y(t) + Q_2(t), \\ \quad x(0) = 0, \quad t \in I = [0,1] \end{cases} \tag{24}$$

where $Q_1(t) = \frac{\Gamma(\alpha+3)}{2}t^2$, $Q_2(t) = 0$ and the exact solution is $\{x(t), y(t)\} = \left\{t^{2+\alpha}, -t^{\frac{3}{2}+\alpha}\right\}$. By the same manner as we described in Example 3, we solve the system

$$(I - \frac{1}{\Gamma(\alpha)}M)\hat{X} = \frac{-B}{\Gamma(\alpha)}, \tag{25}$$

to find the approximations $\hat{X}_1, \ldots, \hat{X}_n$ to $X_1, \ldots, X_n$. Then, we use the inverse $\mathcal{F}$-transform to find the approximation

$$\hat{x} = \mathcal{F}_x^{-1}(t) = \sum_{k=1}^{n} \hat{X}_k A_k(t),$$

to the exact solution $x(t)$.

For $n = 11(h = 0.1), n = 21(h = 0.02), n = 51(h = 0.01)$ and different values of α, we report the maximum values of absolute errors in Table 2. The plots of these errors are shown in Figs. 5 and 6 for $\alpha = \frac{\sqrt{3}}{2}$, $n = 51$ and $n = 101$.

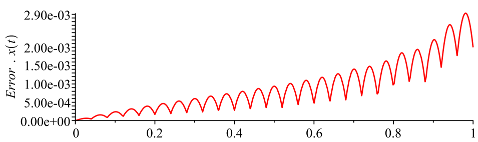

Fig. 5 The plots of absolute error for the Example 4($n = 51$).

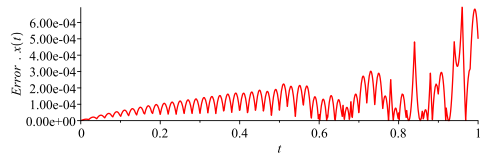

Fig. 6 The plots of absolute error for $x(t)$ in Example 4($n = 101$).

6 Conclusion

In this paper, we proposed a numerical method based on fuzzy transforms for solving the fractional-order linear semi-explicit differential-algebraic equations. We discussed the convergence analysis of the method and investigated the efficiency of the method by some illustrative examples. The implication of the method is fast and can be applied to (1) with arbitrary $\alpha \in (0, 1)$(rational or irrational one). However, some methods are restricted to only rational cases(see [6]). The method can be generalized to the problems with higher order.

Statements and Declarations

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- All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.
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- Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

- The authors declare that there is no conflict of interest.
- The authors declare that contributed equally to this work.
- All authors agree that authors list is correct in its content and order.

References

[1] R. Alikhani, M. Zeinali, F. Bahrami, S. Shahmorad, I. Perfilieva (2017), Trigonometric F^m-transform and its approximation properties, Soft Computing, 21 (13), 3567–3577.

[2] S. Campbell, A. Ilchmann, V. Mehrmann, T. Reis (2019), Applications of Differential-Algebraic Equations: Examples and Benchmarks. Springer.

[3] S. K. Damarla, M. Kundu (2015), Numerical solution of fractional order differential-algebraic equations using generalized triangular function operational matrices. Fractional Calculus and Applications, 6, 31–55.

[4] K. Diethelm (2010), The Analysis of Fractional Differential Equations: An application-oriented exposition using differential operators of Caputo type, Springer.

[5] F. Di Martino, V. Loia, I. Perfilieva, S. Sessa (2008), An image coding/decoding method based on direct and inverse fuzzy transforms, International Journal of Approximate Reasoning. 48 (1), 110–131.

[6] F. Ghanbari, K. Ghanbari, P. Mokhtary (2018), High-order Legendre collocation method for fractional-order linear semi-explicit differential algebraic equations. Electronic Transactions on Numerical Analysis, 48, 387–406.

[7] Feng Kang (1980), Differential vs. integral equations and finite vs. infinite elements. Math. Numer. Sinica, **2**:1 , 100-105.

[8] A. Khastan, I. Perfilieva, Z. Alijani (2016), A new fuzzy approximation method to Cauchy problems by fuzzy transform, Fuzzy Sets and Systems. 288, 75–95.

[9] F. Mainardi (1997), Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and fractional calculus in continuum mechanics. Springer-Verlag, New York, :291-348.

[10] V. Novák, I. Perfilieva, A. Dvořák, G. Chen, Q. Wei, P. Yan (2008), Mining pure linguistic associations from numerical data. International Journal of Approximate Reasoning, 48 (1) 4–22.

[11] I. Perfilieva, M. Daňková, B. Bede (2011), Towards a higher degree $\mathcal{F}$-transform, Fuzzy Sets and Systems, 180 (1), 3–19.

[12] I. Podlubny (1998), Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Academic Press, San Diego.

[13] I. Perfilieva (2006), Fuzzy transforms: Theory and applications, Fuzzy Sets and Systems, 157 (8), 993–1023.

[14] I. Perfilieva (2007), Fuzzy transforms: A challenge to conventional transforms. Advances in Imaging and Electron Physics, 147, 137–196.

[15] I. Perfilieva, H. De Meyer, B. De Baets, D. Piskova (2008), Cauchy problem with fuzzy initial condition and its approximate solution with the help of fuzzy transform. IEEE International Conference on Fuzzy Systems (IEEE World Congress on Computational Intelligence), 2285–2290.

[16] I. Perfilieva, M. Daňková (2008), Image fusion on the basis of fuzzy transforms. Computational Intelligence in Decision and Control, 471–476.

[17] I. Perfilieva (2004), Fuzzy transform: Application to the reef growth problem. Fuzzy Logic in Geology, 275–300.

[18] B. Shiri, D. Baleanu (2019), System of fractional differential-algebraic equations with applications. Chaos, Solitons & Fractals, 120, 203–212.

[19] S. Tomasiello (2017), An alternative use of fuzzy transform with application to a class of delay differential equations. International Journal of Computer Mathematics, 94(9) , pp. 1719-1726.

[20] P. Hurtik, S. Tomasiello (2019), A review on the application of fuzzy transform in data and image compression. Soft Computing, 23(23) , pp. 12641-12653

[21] S. Tomasiello (2021), Least-Squares Fuzzy Transforms and Autoencoders: Some Remarks and Application. IEEE Transactions on Fuzzy Systems, 29(1), 9134776, pp. 129-136.

[22] H. S. Wang, W. L. Jhu, C. F. Yung, P. F. Wang (2011), Numerical solutions of differential-algebraic equations and its applications in solving TPPC problems. Marine Science and Technology, 19 (1), 76–88.

[23] S. Westerlund, L. Ekstam (1994), Capacitor theory. IEEE Transactions on Dielectrics and Electrical Insulation, 1 (5), 826–839.

[24] L. Wang, N. Chen (2014), The predictor-corrector solution for fractional order differential-algebraic equation. ICFDA'14 International Conference on Fractional Differentiation and Its Applications, 1–6.

[25] M. Zeinali, R. Alikhani, F. Bahrami, S. Shahmorad, I. Perfilieva (2018), On the structural properties of F^m-transform with applications. Fuzzy Sets and Systems, 342, 32–52.

[26] M. Zurigat, S. Momani (2010), A. Alawneh, Analytical approximate solutions of systems of fractional algebraic–differential equations by homotopy analysis method. Computers & Mathematics with Applications, 59 (3), 1227-1235.