Quantifying quantum coherence based on the Tsallis relative operator entropy

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Coherence is a fundamental ingredient in quantum physics and a key resource in quantum information processing. The quantification of quantum coherence is of great importance. We present a family of coherence quantifiers based on the Tsallis relative operator entropy. Shannon inequality and its reverse one in Hilbert space operators derived by Furuta [Linear Algebra Appl. 381 (2004) 219] are extended in terms of the parameter of the Tsallis relative operator entropy These quantifiers are shown to satisfy all the standard criteria for a well-defined measure of coherence, and include some existing coherence measures as special cases. Detailed examples are given to show the relations among the measures of quantum coherence.

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I. INTRODUCTION

Quantum coherence is one of the most fundamental physical resources in quantum mechanics, which can be used in quantum optics [1], quantum information and quantum computation [2], thermodynamics [3, 4] and low temperature thermodynamics [5–8]. Coherent quantification is one of the most important ingredient not only in quantum theory but also in practical applications. Recently, resource theory of coherence based on positive operator valued measurement (POVM) has been studied in [9–11]. This approach provide us to understand coherence in more fundamental way as POVMs are the most general kind of quantum measurements. In Ref. [12], the author established a consistent framework of resource theory to quantify coherence. In this theory, the coherence describes the superposition of quantum states relative to fixed orthogonal bases. Since then, a lot of work has been done to enrich this theory [13–18]. This framework has some important limitations on the measurement of coherence. Different coherence measures may reflect different physical aspects of quantum systems [19–25].

Let $\mathcal{H}$ be a finite dimensional Hilbert space with an orthogonal basis $\{|i\rangle\}_{i=1}^{d}$. In this basis, the diagonal density matrices are free states [26], which are also called incoherent states. We label the set of incoherent quantum states as $\mathcal{I}$,

$$\mathcal{I} = \{\sigma \mid \sigma = \sum_{i=1}^{d} \lambda_i |i\rangle\langle i|\}.$$

Free operation in coherence theory is a completely positive and trace preservation (CPTP) mapping, which admits an incoherent Kraus representation. Namely, there always exists a set of Kraus operators $\{K_i\}$ such that

$$\frac{K_i \sigma K_i^{\dagger}}{\text{Tr} K_i \sigma K_i^{\dagger}} \in \mathcal{I},$$

for each i and any incoherent state σ. These operations are also called incoherent operations and we label them by Φ.

Similar to the quantification of entanglement [27–30], any measure of coherence C should satisfy the following axioms [12]:

(C_1) Faithfulness: $C(\rho) \geq 0$, for all quantum states ρ, and $C(\rho) = 0$ if and only if $\rho \in \mathcal{I}$;

(C_2) Monotonicity: C does not increase under incoherent completely positive and trace preserving maps (ICPTP) Φ, i.e.,

$$C(\Phi(\rho)) \leq C(\rho);$$

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(C_3) Strong monotonicity: C does not increase on average under selective incoherent operations, i.e.,

$$\sum_i p_i C(\sigma_i) \leq C(\rho),$$

with probabilities $p_i = \mathrm{Tr} K_i \rho K_i^\dagger$, post-measurement states $\sigma_i = K_i \rho K_i^\dagger / p_i$, and incoherent operators K_i;

(C_4) Convexity: Non-increasing under mixing of quantum states, i.e.,

$$\sum_i p_i C(\rho_i) \geq C(\sum_i p_i \rho_i),$$

for any set of states $\{\rho_i\}$ and $p_i \geq 0$ with $\sum_i p_i = 1$.

In [31] the authors show that the conditions (C_3) and (C_4) are equivalent to the following additivity of coherence for block-diagonal states,

(C_5)

$$C(p\rho_1 \oplus (1-p)\rho_2) = pC(\rho_1) + (1-p)C(\rho_2),$$

for any $p \in [0,1]$, $\rho_i \in \varepsilon(\mathcal{H}_i)$, $i = 1, 2$, and $p\rho_1 \oplus (1-p)\rho_2 \in \varepsilon(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where $\varepsilon(\mathcal{H})$ denotes the set of density matrices on the Hilbert space $\mathcal{H}$.

Other frameworks for quantifying coherence have been further investigated [32–34]. So far, various quantities have been proposed to serve as a coherence quantifier, however the available candidates are still quite limited. Up to now, many coherence measures have been proposed based on different applications and backgrounds, such as the relative entropy of coherence [12], the l_1 norm of coherence [12], geometric coherence [35], coherence measures based on Tsallis relative entropy [36–38] and so on. The Tsallis relative entropy lays the foundation to the non-extensive thermo-statistics and have important applications in the information theory. But it was shown to violate the strong monotonicity, even though it can unambiguously distinguish the coherent and the incoherent states with the monotonicity. Here we establish a class of coherence quantifiers which are closely related to the Tsallis relative α entropy. It proves that this family of quantifiers satisfy all the standard criteria and particularly cover several typical coherence measures. Therefore, it is important to study the properties of the coherence measures based on the Tsallis relative entropy, as well as studying the properties before taking a trace, that is, the Tsallis relative operator entropy, which is a parametric extension of the relative operator entropy.

In this paper, we provide a class of coherence measures based on the Tsallis relative operator entropy. Tsallis relative operator entropy was defined as a parametric extension of relative operator entropy [39]. It is meaningful to study the properties of Tsallis relative operator entropy for the development of the noncommutative statistical physics and nonadditive quantum information theory, therefore, we think it is indispensable to study the coherence of Tsallis relative operator entropy as a versatile resource for quantum information protocols. This paper is organized as follows. We first introduce the coherence measure and the Tsallis relative operator entropy. Then we present the family of coherence quantifier satisfy all the standard criteria for well-defined measures of coherence, then we study the maximal coherence and include some existing coherence measures as special cases. Detailed examples are given to show the relations among the measures of quantum coherence. Finally, we finish the paper by the conclusion.

II. COHERENCE QUANTIFICATION

We first recall the Tsallis relative operator entropy. As we all know, a bounded linear operator T on a Hilbert space H is said to be positive (denoted by $T \geq 0$) if the inner product $(Tx, x) \geq 0$ for all $x \in H$, and an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive. As "T" is a linear and Hermitian operator we have that range T is simply a subspace spanned by all eigenvectors of T belonging to nonzero eigenvalues. The Tsallis relative operator entropy is defined by

$$T_q(\rho||\sigma) = \frac{\rho^{\frac{1}{2}} \left(\rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}} \right)^{1-q} \rho^{\frac{1}{2}} - \rho}{1-q}, \tag{1}$$

for arbitrary two invertible positive operators ρ and σ on Hilbert space, and any real number $q \in [0, 1)$. For convenience, one writes $T_q(\rho||\sigma)$ as [42],

$$T_q(\rho||\sigma) = \rho^{\frac{1}{2}} \ln_{1-q}(\rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}}) \rho^{\frac{1}{2}},$$

where $\ln_{1-q} X \equiv \frac{X^{1-q}-1}{1-q}$ for the positive operator $X = \rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}$. As for any strictly positive real number x and $q \in [0, 1)$, the following inequalities hold:

$$1 - \frac{1}{x} \leq \ln_{1-q} x \leq x - 1, \tag{2}$$

we have the following conclusion.

Lemma 1 *For any positive invertible operators ρ and σ and $q \in [0,1)$,*

$$\rho - \rho\sigma^{-1}\rho \leq T_q(\rho||\sigma) \leq \sigma - \rho.$$

Moreover, $T_q(\rho||\sigma) = 0$ if and only if $\rho = \sigma$.

[Proof] According to (2), we have

$$\mathrm{I} - \rho^{\frac{1}{2}}\sigma^{-1}\rho^{\frac{1}{2}} \leq \ln_{1-q}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}) \leq -\mathrm{I} + \rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}}.$$

Multiplying $\rho^{\frac{1}{2}}$ on both sides of the terms in above inequality, one gets

$$\rho^{\frac{1}{2}}(\mathrm{I} - \rho^{\frac{1}{2}}\sigma^{-1}\rho^{\frac{1}{2}})\rho^{\frac{1}{2}} \leq \rho^{\frac{1}{2}}\ln_{1-q}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})\rho^{\frac{1}{2}} \leq \rho^{\frac{1}{2}}(-\mathrm{I} + \rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})\rho^{\frac{1}{2}}.$$

Hence, $\rho - \rho\sigma^{-1}\rho \leq T_q(\rho||\sigma) \leq \sigma - \rho$. Moreover, suppose $T_q(\rho||\sigma) = 0$, then $\rho - \rho\sigma^{-1}\rho \leq 0 \leq \sigma - \rho$, which implies that $\rho \geq \sigma$ and $\rho \leq \sigma$, namely, $\rho = \sigma$. If $\rho = \sigma$, one can easily verify that $T_q(\rho||\sigma) = 0$. $\square$

In addition, $T_q(\rho||\sigma)$ satisfies the following properties [42]:

(I) (homogeneity) $T_q(\alpha\rho||\alpha\sigma) = \alpha T_q(\rho||\sigma)$ for any positive number α.

(II) (monotonicity) If $\sigma \leq \tau$, then $T_q(\rho||\sigma) \leq T_q(\rho||\tau)$.

(III) (superadditivity) $T_q(\rho_1 + \rho_2||\sigma_1 + \sigma_2) \geq T_q(\rho_1||\sigma_1) + T_q(\rho_2||\sigma_2)$.

(IV) (joint concavity) $T_q(\alpha\rho_1 + \beta\rho_2||\alpha\sigma_1 + \beta\sigma_2) \geq \alpha T_q(\rho_1||\sigma_1) + \beta T_q(\rho_2||\sigma_2)$.

(V) For any unitary operator $T_q(U\rho U^\dagger||U\sigma U^\dagger) = T_q(\rho||\sigma)$.

(VI) For a unital positive linear map Φ from the set of the bounded linear operators on Hilbert space to itself, one has $\Phi(T_q(\rho||\sigma)) \leq T_q(\Phi(\rho)||\Phi(\sigma))$.

The Tsallis relative α entropy is a special case of the quantum f-divergences [38]. Moreover, in order to make our definition correspond to the definition of the relative operator entropy defined, we change the sign of the original Tsallis relative α entropy. For two density matrices ρ and σ, the Tsallis relative α entropy is defined by,

$$\tilde{D}_q(\rho||\sigma) = \frac{1}{q-1}\left(\mathrm{Tr}\rho^q\sigma^{1-q} - 1\right),$$

for $q \in (0, 2]$. $\tilde{D}_q(\rho||\sigma)$ can also be reformulated as

$$\tilde{D}_q(\rho||\sigma) = \frac{1}{q-1}\left(\tilde{f}_q(\rho,\sigma) - 1\right),$$

where $\tilde{f}_q(\rho,\sigma) = \mathrm{Tr}\rho^q\sigma^{1-q}$.

Based on the Tsallis relative α entropy $\tilde{D}_q(\rho||\sigma)$, the coherence in the fixed reference basis $\{|j\rangle\}$ can be characterized by [38]

$$\tilde{C}_q(\rho) = \min_{\delta\in\mathcal{I}} \tilde{D}_q(\rho||\delta).$$

Nevertheless, $\tilde{C}_q(\rho)$ violates the strong monotonicity condition of a coherence measure, even though it can unambiguously distinguish the coherent states from the incoherent ones with the monotonicity.

In the following, we define a generalized Tsallis relative operator entropy,

$$D_q(\rho||\sigma) = \frac{1}{q-1}(f_q^{\frac{1}{q}}(\rho,\sigma) - 1), \tag{3}$$

where

$$f_q(\rho,\sigma) = \mathrm{Tr}[\rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{1-q}\rho^{\frac{1}{2}}]. \tag{4}$$

$f_q(\rho,\sigma)$ has the following properties.

Lemma 2 *For any quantum states ρ and σ with $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$, we have $f_q(\Phi(\rho), \Phi(\sigma)) \geq f_q(\rho, \sigma)$ for any CPTP map Φ, where $q \in [0,1)$.*

[Proof] Due to properties (VI), we get

$$\Phi[\rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{1-q}\rho^{\frac{1}{2}} - \rho] \leq \Phi(\rho^{\frac{1}{2}})(\Phi(\rho^{-\frac{1}{2}})\Phi(\sigma)\Phi(\rho^{-\frac{1}{2}}))^{1-q}\Phi(\rho^{\frac{1}{2}}) - \Phi(\rho). \tag{5}$$

For any CPTP map Φ, we have

$$\operatorname{Tr}\left[\Phi[\rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{1-q}\rho^{\frac{1}{2}} - \rho]\right] = \operatorname{Tr}[\rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{1-q}\rho^{\frac{1}{2}}] - \operatorname{Tr}\rho, \tag{6}$$

and

$$\operatorname{Tr}\left[\Phi(\rho^{\frac{1}{2}})(\Phi(\rho^{-\frac{1}{2}})\Phi(\sigma)\Phi(\rho^{-\frac{1}{2}}))^{1-q}\Phi(\rho^{\frac{1}{2}}) - \Phi(\rho)\right] = f_q(\Phi(\rho), \Phi(\sigma)) - \operatorname{Tr}\rho. \tag{7}$$

According to (4), (5), (6) and (7), we get $f_q(\rho, \sigma) \leq f_q(\Phi(\rho), \Phi(\sigma))$. $\square$

Next, we give a lemma about the function $f_q(\rho, \sigma)$, which is important in deriving our main results. Similar to the Lemma 1 in Ref. [48], we have

Lemma 3 *Suppose both ρ and σ simultaneously undergo a TPCP map $\Phi := \left\{K_n : \sum_n K_n^\dagger K_n = \mathcal{I}_H\right\}$ which transforms the states ρ and σ into the ensemble $\{p_n, \rho_n\}$ and $\{q_n, \sigma_n\}$, respectively. We have*

$$f_q(\rho_H, \delta_H) \leq \sum_n p_n^q q_n^{1-q} f_q(\rho_n, \sigma_n).$$

[Proof] Any TPCP map can be achieved by unitary operations and local projection measurements on the composite system [2]. Let A be an auxiliary system. For a TPCP map $\Phi := \left\{K_n : \sum_n K_n^\dagger K_n = \mathcal{I}_H\right\}$, we can always find a unitary operation U_{H_A} and a set of projectors $\left\{\Pi_n^A = |n\rangle_A \langle n|\right\}$ such that

$$K_n \rho_H K_n^\dagger \otimes \Pi_n^A = \left(\mathcal{I}_H \otimes \Pi_n^A\right) U_{H_A} \left(\rho_H \otimes \Pi_0^A\right) U_{H_A}^\dagger \left(\mathcal{I}_H \otimes \Pi_n^A\right). \tag{8}$$

According to Lemma 1 and the property (V), for any two states ρ_H and σ_H we have

$$f_q(\rho_H, \delta_H) = f_q\left(U_{H_A}\left(\rho_H \otimes \Pi_0^A\right)U_{H_A}^\dagger, U_{H_A}\left(\sigma_H \otimes \Pi_0^A\right)U_{H_A}^\dagger\right).$$

Denote $\rho_{H_f} = \Phi_{H_A}\left[U_{H_A}\left(\rho_H \otimes \Pi_0^A\right)U_{H_A}^\dagger\right]$ and $\sigma_{H_f} = \Phi_{H_A}\left[U_{H_A}\left(\sigma_H \otimes \Pi_0^A\right)U_{H_A}^\dagger\right]$. Due to Lemma 2 we obtain

$$f_q(\rho_H, \delta_H) \leq f_q\left(\rho_{H_f}, \sigma_{H_f}\right). \tag{9}$$

Let the TPCP map be given by $\Phi_{H_A} := \left\{\mathcal{I}_H \otimes \Pi_n^A\right\}$. According to Eq. (8), ρ_{H_f} and σ_{H_f} can be replaced in Eq. (9), respectively, by

$$\rho_{H_f} \to \tilde{\rho}_{H_f} = \sum_n K_n \rho_H K_n^\dagger \otimes \Pi_n^A$$

and

$$\sigma_{H_f} \to \tilde{\sigma}_{H_f} = \sum_n K_n \sigma_H K_n^\dagger \otimes \Pi_n^A.$$

Thus, we have

$$\begin{aligned}
f_q(\rho_H, \delta_H) &\leq f_q\left(\tilde{\rho}_{H_f}, \tilde{\sigma}_{H_f}\right) \\
&= \sum_n f_q\left(K_n \rho_H K_n^\dagger \otimes \Pi_n^A, K_n \sigma_H K_n^\dagger \otimes \Pi_n^A\right) \\
&= \sum_n f_q\left(K_n \rho_H K_n^\dagger, K_n \sigma_H K_n^\dagger\right) \\
&= \sum_n p_n^q q_n^{1-q} f_q(\rho_n, \sigma_n),
\end{aligned}$$

which comletes the proof. $\square$

Based on the above results, we have the following main theorem.

Theorem 1 *The coherence $C_q(\rho)$ of a quantum state ρ given by*

$$C_q(\rho) = \min_{\sigma \in \mathcal{I}} D_q(\rho||\sigma), \tag{10}$$

defines a well-defined measure of coherence for $q \in (0,1)$.

Proof From (3), (4) and (10), for $0 < q < 1$, we have

$$C_q(\rho) = \min_{\sigma \in \mathcal{I}} \frac{1}{q-1} \left(f_q^{\frac{1}{q}}(\rho, \sigma) - 1 \right).$$

From Lemma 1, we have $C_q(\rho) \geq 0$, and $C_q(\rho) = 0$ if and only if $\rho = \sigma$.

Next we prove that $C_q(\rho)$ satisfies (C_3)—strong monotonicity. Let δ^o be the optimal incoherent state to the minimal value of $f_q(\rho, \delta)$, i.e., $f_q(\rho, \delta^o) = \max_{\delta \in \mathcal{I}} f_q(\rho, \delta)$. Let $\Phi = \{K_n\}$ be the incoherent selective quantum operations given by Kraus operators $\{K_n\}$, with $\sum_n K_n^\dagger K_n = I$, where I is the identity operator on H. Under the operation Φ on a state ρ, the post-measurement ensemble is given by $\{p_n, \rho_n\}$ with $p_n = \text{Tr} K_n \rho K_n^\dagger$ and $\rho_n = K_n \rho K_n^\dagger / p_n$. Hence the average coherence is

$$\sum_n p_n C_q(\rho_n) = \min_{\delta_n \in \mathcal{I}} \frac{1}{q-1} \left(\sum_n p_n f_q^{\frac{1}{q}}(\rho_n, \delta_n) - 1 \right). \tag{11}$$

Since the incoherent operation cannot generate coherence from an incoherent state, for the optimal incoherent state δ^o, we have $\delta_n^o = K_n \delta^o K_n^\dagger / q_n \in \mathcal{I}$ with $q_n = \text{Tr} K_n \delta^o K_n^\dagger$ for any incoherent operation K_n. Due to $q \in (0,1)$ and $C_q(\rho) \geq 0$, $C_q(\rho)$ is the smallest when $f_q^{\frac{1}{q}}(\rho, \delta)$ is maximum. Therefore, one immediately finds that $\max_{\delta \in \mathrm{I}} f_q^{\frac{1}{q}}(\rho, \delta) \geq f_q^{\frac{1}{q}}(\rho_n, \delta_n^o)$. Therefore, Eq. (11) can be rewritten as

$$\sum_n p_n C_q(\rho_n) \leq \frac{1}{q-1} \left(\sum_n p_n f_q^{\frac{1}{q}}(\rho_n, \delta_n^o) - 1 \right). \tag{12}$$

In addition, consider the Hölder inequality

$$\sum_{k=0}^{d} a_k b_k \leq \left(\sum_{k=0}^{d} a_k^n \right)^{\frac{1}{n}} \left(\sum_{k=0}^{d} b_k^m \right)^{\frac{1}{m}},$$

for $\frac{1}{n} + \frac{1}{m} = 1$ and $n > 1$. The equality holds if and only if $\frac{a_k^n}{\sum_{k=0}^{d} a_k^n} = \frac{b_k^n}{\sum_{k=0}^{d} b_k^n}$, and the inequality is reversed for $n \in (0,1)$. By using the Hölder inequality we obtain

$$\left[\sum_n q_n \right]^{1-q} \left[\sum_n p_n f_q^{\frac{1}{q}}(\rho_n, \delta_n^o) \right]^q \geq \sum_n p_n^q q_n^{1-q} f_q(\rho_n, \delta_n^o), \tag{13}$$

where $q \in (0,1)$. Therefore, Eq. (12) becomes

$$\sum_n p_n C_q(\rho_n) \leq \frac{1}{q-1} \left(\sum_n p_n f_q^{\frac{1}{q}}(\rho_n, \delta_n^o) - 1 \right) \tag{14}$$

$$\leq \frac{1}{q-1} \left(\left[\sum_n p_n^q q_n^{1-q} f_q(\rho_n, \delta_n^o) \right]^{\frac{1}{q}} - 1 \right)$$

$$\leq \frac{1}{q-1} \left(f_q^{\frac{1}{q}}(\rho, \delta^o) - 1 \right)$$

$$= C_q(\rho),$$

where the first inequality is due to Eq. (12), and from Eq. (13) we get the second inequality. The third inequality is due to Lemma 2. Eq. (14) shows the strong monotonicity. The monotonicity is directly given by the convexity of $C_q(\rho)$, $C_q(\rho) \geq C_q(\sum_n p_n \rho_n) = C_q(\Phi(\rho))$.

Finally we prove that $C_q(\rho)$ satisfies condition (C_5). Suppose ρ is block-diagonal in the reference basis $\{|j\rangle\}_{j=1}^d$, $\rho = p_1\rho_1 \oplus p_2\rho_2$ with $p_1 \geq 0$, $p_2 \geq 0$, $p_1 + p_2 = 1$, where ρ_1 and ρ_2 are density operators. Let $\sigma = q_1\sigma_1 \oplus q_2\sigma_2$ with σ_1, σ_2 the diagonal states having the same rows (columns) as ρ_1, ρ_2, respectively, $q_1 \geq 0$, $q_2 \geq 0$, $q_1 + q_2 = 1$. It follows that

$$\max_{\sigma \in \mathcal{I}} \text{Tr}\left(\rho^{\frac{1}{2}} (\rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}})^{1-q} \rho^{\frac{1}{2}} \right) \tag{15}$$

$$= \max_{\sigma \in \mathcal{I}} \text{Tr}\left\{ (p_1\rho_1 + p_2\rho_2)^{\frac{1}{2}} \left[(p_1\rho_1 + p_2\rho_2)^{-\frac{1}{2}} (q_1\sigma_1 + q_2\sigma_2)(p_1\rho_1 + p_2\rho_2)^{-\frac{1}{2}} \right]^{1-q} (p_1\rho_1 + p_2\rho_2)^{\frac{1}{2}} \right\}$$

$$= \max_{\sigma \in \mathcal{I}} \text{Tr}\left[(p_1^{\frac{1}{2}}\rho_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\rho_2^{\frac{1}{2}})(p_1^{-\frac{1}{2}} q_1 p_1^{-\frac{1}{2}} \rho_1^{-\frac{1}{2}} \sigma_1 \rho_1^{-\frac{1}{2}} + p_2^{-\frac{1}{2}} q_2 p_2^{-\frac{1}{2}} \rho_2^{-\frac{1}{2}} \sigma_2 \rho_2^{-\frac{1}{2}})^{1-q}(p_1^{\frac{1}{2}}\rho_1^{\frac{1}{2}} + p_2^{\frac{1}{2}}\rho_2^{\frac{1}{2}}) \right]$$

$$= \max_{\sigma \in \mathcal{I}} \text{Tr}\left[p_1^q q_1^{1-q} \rho_1^{\frac{1}{2}} (\rho_1^{-\frac{1}{2}} \sigma_1 \rho_1^{-\frac{1}{2}})^{1-q} \rho_1^{\frac{1}{2}} + p_2^q q_2^{1-q} \rho_2^{\frac{1}{2}} (\rho_2^{-\frac{1}{2}} \sigma_2 \rho_2^{-\frac{1}{2}})^{1-q} \rho_2^{\frac{1}{2}} \right]$$

$$= \max_{q_1,q_2}\{p_1^q q_1^{1-q} t_1 + p_2^q q_2^{1-q} t_2\},$$

where we denoted

$$t_1 = \max_{\sigma_1} \rho_1^{\frac{1}{2}} \left(\rho_1^{-\frac{1}{2}} \sigma_1 \rho_1^{-\frac{1}{2}} \right)^{1-q} \rho_1^{\frac{1}{2}},$$

$$t_2 = \max_{\sigma_2} \rho_2^{\frac{1}{2}} \left(\rho_2^{-\frac{1}{2}} \sigma_2 \rho_2^{-\frac{1}{2}} \right)^{1-q} \rho_2^{\frac{1}{2}}.$$

According to the Hölder inequality with $0 < q < 1$, we have

$$p_1^q q_1^{1-q} t_1 + p_2^q q_2^{1-q} t_2 \leq \left(p_1 t_1^{\frac{1}{q}} + p_2 t_2^{\frac{1}{q}} \right)^q,$$

where the equality holds if and only if $q_1 = cp_1 t_1^{\frac{1}{\alpha}}$ and $q_2 = cp_2 t_2^{\frac{1}{\alpha}}$ with $c = \left[p_1 t_1^{\frac{1}{\alpha}} + p_2 t_2^{\frac{1}{\alpha}} \right]^{-1}$, i.e,

$$\max_{q_1,q_2} \left(p_1^q q_1^{1-q} t_1 + p_2^q q_2^{1-q} t_2 \right) = \left(p_1 t_1^{\frac{1}{q}} + p_2 t_2^{\frac{1}{q}} \right)^q. \tag{16}$$

Combining (15) and (16), we have

$$\max_{\sigma \in \mathcal{I}} f_q^{\frac{1}{q}}(\rho, \sigma) = p_1 \max_{\sigma_1 \in \mathcal{I}} f_q^{\frac{1}{q}}(\rho_1, \sigma_1) + p_2 \max_{\sigma_2 \in \mathcal{I}} f_q^{\frac{1}{q}}(\rho_2, \sigma_2).$$

Thus, C_q satisfies the additivity of coherence for block-diagonal states: $C_q(p_1\rho_1 \oplus p_1\rho_1) = p_1 C_q(\rho_1) + p_2 C_q(\rho_2)$. $\square$

III. MAXIMAL COHERENCE AND SEVERAL TYPICAL QUANTIFIERS

We show that the maximal coherence of $C_q(\rho)$ ($q \in (0,1)$) can be attained by the maximally coherent states. Based on the eigen-decomposition of a d-dimensional state $\rho = \sum_{j=1}^d \lambda_j |\varphi\rangle_j \langle\varphi|$, where λ_j and $|\varphi\rangle_j$ are the eigenvalues and

eigenvectors of ρ, respectively. We have

$$f_q^{\frac{1}{q}}(\rho,\sigma) = \left[\text{Tr}(\rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{1-q}\rho^{\frac{1}{2}})\right]^{\frac{1}{q}}$$

$$= \left\{\text{Tr}\left[\left(\sum_{k=1}^{d}\lambda_k|\varphi_k\rangle\langle\varphi_k|\right)^{\frac{1}{2}}\left[\left(\sum_{j=1}^{d}\lambda_j|\varphi_j\rangle\langle\varphi_j|\right)^{-\frac{1}{2}}\left(\sum_{i=1}^{d}\sigma_i|i\rangle\langle i|\right)\left(\sum_{j=1}^{d}\lambda_j|\varphi_j\rangle\langle\varphi_j|\right)^{-\frac{1}{2}}\right]^{1-q}\left(\sum_{k=1}^{d}\lambda_k|\varphi_k\rangle\langle\varphi_k|\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{q}}$$

$$= \left[\text{Tr}\left(\sum_{k=1}^{d}\lambda_k|\varphi_k\rangle\langle\varphi_k|\left(\sum_{i,j=1}^{d}\lambda_j^{-1}\sigma_i|\langle\varphi_j|i\rangle|^2|\varphi_j\rangle\langle\varphi_j|\right)^{1-q}\right)\right]^{\frac{1}{q}}$$

$$= \left[\text{Tr}\left(\sum_{i,j=1}^{d}\lambda_j^q\left(\sigma_i|\langle\varphi_j|i\rangle|^2\right)^{1-q}|\varphi_j\rangle\langle\varphi_j|\right)\right]^{\frac{1}{q}}$$

$$= \left[\sum_{i,j=1}^{d}\lambda_j^q\left(\sigma_i|\langle\varphi_j|i\rangle|^2\right)^{1-q}\right]^{\frac{1}{q}}$$

$$\geq \sum_{i=1}^{d}\left[\sum_{j=1}^{d}\lambda_j^q\left(\sigma_i|\langle\varphi_j|i\rangle|^2\right)^{1-q}\right]^{\frac{1}{q}}$$

$$\geq d^{\frac{q-1}{q}}\left[\sum_{i,j=1}^{d}\lambda_j^q\left(\sigma_i|\langle\varphi_j|i\rangle|^2\right)^{1-q}\right]^{\frac{1}{q}}$$

$$\geq d^{\frac{q-1}{q}}\left(\sum_{i,j=1}^{d}\lambda_j\sigma_i|\langle\varphi_j|i\rangle|^2\right)^{\frac{1}{q}}$$

$$\geq d^{\frac{q-1}{q}},$$

where the first inequality is due to $(\sum_{i,j=1}^{d}a_ib_j)^{\frac{1}{q}} \geq \sum_{i=1}^{d}(\sum_{j=1}^{d}a_ib_j)^{\frac{1}{q}}$, with $a_i, b_i \geq 0$. The second inequality is due to that $\sum_{i=1}^{n}\lambda_ix_i^p \geq (\sum_{i=1}^{n}\lambda_i)^{1-p}(\sum_{i=1}^{n}\lambda_ix_i)^p$, $p > 1$, with $x_i = \sum_{j=1}^{d}\lambda_j^q(\sigma_i|\langle\varphi_j|i\rangle|^2)^{1-q} \geq 0$, $\lambda_i = 1$ $(i = 1, 2, ..., n)$ and $p = \frac{1}{q}$. The third inequality is due to $\sum_k a_k^q b_k^{1-q} \geq \sum_k a_k b_k$, where $a_k, b_k \in (0, 1)$. Then one can easily find that the upper bound of the coherence can be attained by the maximally coherent states $\rho_d = |\varphi\rangle\langle\varphi|$ with $|\varphi\rangle = \frac{1}{\sqrt{d}}\sum_j e^{i\phi_j}|j\rangle$. The corresponding coherence is given by

$$C_q(\rho_d) = \frac{1}{q-1}(d^{\frac{q-1}{q}} - 1).$$

$\square$

$C_q(\rho)$ actually defines a family of coherence measures related to the Tsallis relative operator entropy. Next, we introduce the special case of geometric coherence measures as the existing coherence measures, and give detailed examples to illustrate the relationship between quantum coherence measures. For $q = \frac{1}{2}$, one can also find that

$$C_{1/2}(\rho) = \min_{\sigma\in\mathcal{I}}2\left\{1 - \left[\text{Tr}(\rho^{\frac{1}{2}}(\rho^{-\frac{1}{2}}\sigma\rho^{-\frac{1}{2}})^{\frac{1}{2}}\rho^{\frac{1}{2}})\right]^2\right\}, \tag{17}$$

where

$$f_{\frac{1}{2}}^2(\rho,\sigma) = \left\{ \mathrm{Tr} \left[\rho^{\frac{1}{2}} (\rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}})^{\frac{1}{2}} \rho^{\frac{1}{2}} \right] \right\}^2 \tag{18}$$

$$\leq \left\{ \mathrm{Tr} \left[\rho(\rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}})\rho \right]^{\frac{1}{2}} \right\}^2$$

$$= \left[\mathrm{Tr}(\rho^{\frac{1}{2}} \sigma \rho^{\frac{1}{2}})^{\frac{1}{2}} \right]^2,$$

in which the inequality is due to the Araki-Lieb-Thirring inequality: for matrixes $A, B \geq 0$, $q \geq 0$ and for $0 \leq r \leq 1$, the following inequality holds [43], $tr(A^r B^r A^r)^q \leq tr(ABA)^{rq}$.

We consider now the relationship between $C_q(\rho)$ and the geometric measure of quantum coherence. The geometric measure of coherence $C_g(\rho)$ is defined by [41]

$$C_g(\rho) = 1 - \max_{\sigma \in \mathcal{I}} F(\rho,\sigma) \tag{19}$$

$$= 1 - \max_{\sigma \in \mathcal{I}} \left[\mathrm{Tr}(\rho^{1/2} \sigma \rho^{1/2})^{\frac{1}{2}} \right]^2,$$

where $F(\rho,\sigma)$ is the Uhlmann fidelity of two density operators ρ and σ. From Eq. (18), we have that $F(\rho,\sigma) \geq f_{\frac{1}{2}}^2(\rho,\sigma)$. Therefore, by the definition of geometric measure of coherence C_g, we get $C_{1/2}(\rho) \geq 2C_g(\rho)$.

As an example, let us consider a single-qubit state,

$$\rho = \frac{1}{2}(I_2 + \sum_i c_i \sigma_i),$$

where I_2 is the 2×2 identity matrix and σ_i $(i = 1, 2, 3)$ are Pauli matrices. Suppose that $\sigma = \sum_i p_i |i\rangle\langle i|$ with $p_1 + p_2 = 1$ and $0 \leq p_1, p_2 \leq 1$.

For the single-qubit pure state ρ, with $\sum_i c_i^2 = 1$, one has

$$\rho = \begin{pmatrix} \frac{1+c_3}{2} & \frac{c_1 - ic_2}{2} \\ \frac{c_1 + ic_2}{2} & \frac{1-c_3}{2} \end{pmatrix}. \tag{20}$$

The eigenvalues of ρ are 0 and 1. Then, we obtain

$$F(\rho,\sigma) = \left[\mathrm{Tr}(\rho^{1/2} \sigma \rho^{1/2})^{\frac{1}{2}} \right]^2$$

$$= \frac{1+c_3}{2} p_1 + \frac{1-c_3}{2} p_2. \tag{21}$$

Similar to the proof of (21), we obtain

$$f_{\frac{1}{2}}^2(\rho,\sigma) = \left\{ \mathrm{Tr} \left[\rho^{\frac{1}{2}} (\rho^{-\frac{1}{2}} \sigma \rho^{-\frac{1}{2}})^{\frac{1}{2}} \rho^{\frac{1}{2}} \right] \right\}^2$$

$$= \frac{1+c_3}{2} p_1 + \frac{1-c_3}{2} p_2.$$

Therefore, when ρ is a single-qubit pure state, $F(\rho,\sigma) = f_{\frac{1}{2}}^2(\rho,\sigma)$, i.e., $C_{1/2}(\rho) = 2C_g(\rho)$.

For the single-qubit state (20), with $\sum_i c_i^2 < 1$, the eigenvalues of ρ are given by

$$\lambda_1 = \frac{1 + \sqrt{c_1^2 + c_2^2 + c_3^2}}{2},$$

$$\lambda_2 = \frac{1 - \sqrt{c_1^2 + c_2^2 + c_3^2}}{2}.$$

Let $|\nu_1\rangle$ and $|\nu_2\rangle$ be the corresponding eigenvectors of ρ. We have $\rho = \lambda_1 |\nu_1\rangle\langle\nu_1| + \lambda_2 |\nu_2\rangle\langle\nu_2|$. Combining (4) and (19), we obtain

$$
\begin{aligned}
F(\rho,\sigma) =& \lambda_1^{-\frac{1}{2}} p_1^{\frac{1}{2}} |\langle\nu_1|1\rangle| + \lambda_1^{-\frac{1}{4}}\lambda_2^{-\frac{1}{4}} p_1^{\frac{1}{2}} |\langle\nu_2|1\rangle| + \lambda_1^{-\frac{1}{4}}\lambda_2^{-\frac{1}{4}} p_1^{\frac{1}{2}} (\langle\nu_1|1\rangle\langle 1|\nu_2\rangle)^{\frac{1}{2}} + \lambda_1^{-\frac{1}{2}} p_2^{\frac{1}{2}} |\langle\nu_1|2\rangle| + \\
& \lambda_1^{-\frac{1}{4}}\lambda_2^{-\frac{1}{4}} p_2^{\frac{1}{2}} |\langle\nu_2|2\rangle| + \lambda_2^{-\frac{1}{2}} p_1^{\frac{1}{2}} |\langle\nu_2|1\rangle| + \lambda_1^{-\frac{1}{4}}\lambda_2^{-\frac{1}{4}} p_2^{\frac{1}{2}} (\langle\nu_1|2\rangle\langle 2|\nu_2\rangle)^{\frac{1}{2}} + \lambda_2^{-\frac{1}{2}} p_2^{\frac{1}{2}} |\langle\nu_2|2\rangle|, \\
f_{\frac{1}{2}}^2(\rho,\sigma) =& \lambda_1^{\frac{1}{2}} p_1^{\frac{1}{2}} |\langle\nu_1|1\rangle| + \lambda_1^{-\frac{1}{4}}\lambda_2^{\frac{3}{4}} p_1^{\frac{1}{2}} |\langle\nu_2|1\rangle| + \lambda_1^{\frac{3}{4}}\lambda_2^{-\frac{1}{4}} p_1^{\frac{1}{2}} (\langle\nu_1|1\rangle\langle 1|\nu_2\rangle)^{\frac{1}{2}} + \lambda_1^{\frac{1}{2}} p_2^{\frac{1}{2}} |\langle\nu_1|2\rangle| + \\
& \lambda_1^{-\frac{1}{4}}\lambda_2^{\frac{3}{4}} p_2^{\frac{1}{2}} |\langle\nu_2|2\rangle| + \lambda_2^{\frac{1}{2}} p_1^{\frac{1}{2}} |\langle\nu_2|1\rangle| + \lambda_1^{\frac{3}{4}}\lambda_2^{-\frac{1}{4}} p_2^{\frac{1}{2}} (\langle\nu_1|2\rangle\langle 2|\nu_2\rangle)^{\frac{1}{2}} + \lambda_2^{\frac{1}{2}} p_2^{\frac{1}{2}} |\langle\nu_2|2\rangle|.
\end{aligned}
$$

Due to $0 < \lambda_1, \lambda_2 < 1$, we have $F(\rho,\sigma) > f_{\frac{1}{2}}^2(\rho,\sigma)$. Obviously $C_{1/2}(\rho) > 2C_g(\rho)$.

IV. CONCLUSION

In summary, we have proposed four types of coherent measures $C_q(\rho)$ based on the Tsallis relative operator entropy. It has been shown that these coherent measures meet all the necessary criteria for satisfactory coherence measures. Moreover, the connections between $C_q(\rho)$ and the geometric measure of quantum coherence have been investigated. Quantum coherence plays important roles in many quantum information processing. Our results may hight further researches on the characterization of quantum coherence.

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