

A solution for fractional PDE constrained optimization problems using reduced basis method

A. Rezazadeh^a and M. Mahmoudi^a and M. Darehmiraki^b

^aDepartment of Mathematics, University of Qom, Qom 37161466711, Iran;

^bDepartment of Mathematics, Behbahan Khatam Alanbia University of Technology, Behbahan, Khuzestan, Iran

ARTICLE HISTORY

Compiled October 21, 2019

ABSTRACT

In this paper, we employ a reduced basis method for solving the PDE constrained optimization problem governed by a fractional parabolic equation with the fractional derivative in time from order $\beta \in (0, 1)$ is defined by Caputo fractional derivative. Here we use optimize-then-discretize method to solve it. In order to this, First, we extract the optimality conditions for the problem, and then solve them by reduced basis method. To get a numerical technique, the time variable is discretized using a finite difference plan. In order to show the effectiveness and accuracy of this method, some test problem are considered, and it is shown that the obtained results are in very good agreement with exact solution.

KEYWORDS

Optimal control, Reduced basis method, Fractional time derivative, Parametrized PDE .

AMS CLASSIFICATION

Primary 43A62; Secondary 42C15.

1. Introduction

Fractional calculus appear in many settings across engineering and science disciplines, it can be referred to Sun et al. (2018) for more information. Also, many real-life examples are modeled by PDE constrained optimization, such as the optimal control of tumor invasion, the optimal strategy of a thermal treatment in cancer therapy, and the medical image analysis Mang et al. (2018); Quiroga et al. (2018, 2015). Thanks to the increasing use of fractional derivative and fractional calculus in ordinary and partial differential equations and related problems, there is a interest for presenting efficient and reliable solutions for them.

The reduced basis method is a model order reduction effective technique for approximating the solution of parameterized partial differential equations which have been used in the past few decades. Firstly, the RB method was proposed to solve the non-linear analysis of structures in the late 1970s and has subsequently been further investigated and developed more broadly Balmes (1996); Grepl and Patera (2005).

The main idea of the RB method for parameter dependent PDEs is to approximate its solution by a linear combination of few global basis functions, obtained from a set of FE solutions corresponding to different parameter values. The RB method was repeatedly used to solve the PDE constrained optimization problem (PCOP). A reduced basis surrogate model was proposed for PCOP in Dihlmann and Haasdonk (2015). Authors in Qian et al. (2017) used RB method in conjunction with a trust region optimization framework to accelerate PCOP. RB method was applied to solve quadratic multi-objective optimal control problems governed by linear parameterized variational equations in Lapichino et al. (2017). Authors in Aquino et al. (2018) employed a locally adapted RB method for solving risk-averse PCOP.

1.1. Literature review

Many researchers pay attention to finding the numerical solution for solving fractional PDE constrained optimization problem. Antil and Otarola (2015) use finite element method (FEM) for an Optimal Control Problem of Fractional Powers of Elliptic Operators. In Otarola (2017) a piecewise linear FEM was proposed for an optimal control problem involving the fractional powers of a symmetric and uniformly elliptic second order operator. A spectral method for optimal control problems governed by the time fractional diffusion equation with control constraints was presented in Ye and Xu (2014). Authors in Bhrawy et al. (2016) provided a space-time Legendre spectral tau method for the two-sided space-time Caputo fractional diffusion-wave equation. A new spectral collocation algorithm for solving time-space fractional partial differential equations with subdiffusion was reported in Bhrawy (2016). A hybrid meshless method was proposed for FODCP in Darehmiraki et al. (2016). Zacky and Mchado Zaky and Machado (2017) provided a solution for FODCP by pseudo-spectral method. As well as there exist variety of solutions for various fractional optimal control problems that the some of them can be seen in Bai et al. (2018); Salati et al. (2018).

1.2. The main aim of this paper

Our main motivation in this paper is to apply reduced basis method for the rapid and reliable solution of fractional PDE constrained optimization problem. There exists, to the best of our knowledge, no previous work providing reduced order approximations for fractional PDE constrained optimization problem. It is known that the reduced basis method has been an efficient tool for computing numerical solutions of differential equations because of its high-order accuracy. We demonstrate the efficiency of the proposed schemes by considering several numerical examples.

The outline of this paper is as follows. After stating the exact form of the problem in Section 2, we present the weak form of the problem in Section 3. Necessary and sufficient conditions for the problem are introduced in section 4. Sections 5 review finite element method and reduced basis method. In section 6, we give some implementation details and present the numerical results to support the theoretical prediction. The error bound for proposed method is calculated in Section 7. Three numerical examples are presented in Section 8 to confirm our theoretical findings. In Section 9, we give some concluding remarks.

2. The exact form of the problem

This paper solves an optimal control problem with fractional PDE constraint. In this section, we present the parameterized optimal control problem.

Let $N \geq 1$ and Ω be a bounded open subset of \mathbb{R}^N with boundary $\partial\Omega$. For a time $T > 0$, we set $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. Given $\alpha \in (0, 1)$, and $D \subset \mathbb{R}^p$ be a p -dimensional compact set of parameters $\mu = (\mu_1, \dots, \mu_p)$ with $p \geq 1$. Let Y with $H_0^1(\Omega) \subset Y \subset H^1(\Omega)$ be a Hilbert space with inner product $(\cdot, \cdot)_Y$, $(v, w) = \int_{\Omega} v w d\Omega$ and associated norm $\|\cdot\|_Y = \sqrt{(\cdot, \cdot)_Y}$. We assume that the norm $\|\cdot\|_Y$ is equivalent to $H^1(\Omega)$ -norm and denote the dual space of Y by Y' . We also introduce the control Hilbert space U with its inner product $(\cdot, \cdot)_U$ and norm $\|\cdot\| = \sqrt{(\cdot, \cdot)_U}$ with associated dual space U' . We define the parameterized fractional optimal control problem:

$$\begin{aligned} \min \quad J(y, u) &= \frac{1}{2} \int_0^T \int_{\Omega} (y(\cdot; \mu) - y_d)^2 d\Omega dt + \frac{\gamma}{2} \int_0^T \int_{\Omega} u^2(\cdot; \mu) d\Omega dt, \\ &\text{subject to} \\ &-\partial_t^\alpha y + \mu \Delta y = u, && \text{in } Q, \\ &y = 0, && \text{on } \Sigma, \\ &y(\cdot, 0) = 0, && \text{in } \Omega, \end{aligned} \quad (1)$$

where $J(y, u)$ is the cost functional, $\gamma > 0$ is the so-called regularization parameter and y_d is the desired state. Here the variables $y \in H^1(Q)$ and $u \in L_2(Q)$ are named the state variable and the control variable, respectively. They are independent variables and should be determined.

The fractional derivative ∂_t^α for $\alpha \in (0, 1)$ is the left-sided Caputo fractional derivative of order α with respect to t and defined as:

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, \quad (2)$$

where Γ is the Gamma function. We consider ∂_t for $\alpha = 1$. The right-sided Caputo fractional is

$$\partial_{T-t}^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^T \frac{f'(\tau)}{(\tau-t)^\alpha} d\tau \quad \alpha \in (0, 1), \quad (3)$$

where $f \in L^1(0, T)$.

Definition 2.1. (Steeb, 1997; Steeb, 2006) Suppose $C = (c_{ij})_{m \times n}$ and D are two arbitrary matrices, then the matrix

$$C \otimes D = \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1n}D \\ c_{21}D & c_{22}D & \cdots & c_{2n}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1}D & c_{m2}D & \cdots & c_{mn}D \end{bmatrix}$$

is named Kronecker product of C and D .

Definition 2.2. Suppose $C = (c_{ij})_{m \times n}$ be a given matrix, then $vec(C)$ is a column vector made of the row of C stacked a top one another from left to right that the size of it is $m \times n$.

$$vec(C) = (c_{11}, c_{12}, \dots, c_{1n}, c_{21}, c_{22}, \dots, c_{m1}, \dots, c_{mn})^T.$$

3. Weak form of the problem

In this section, we consider the weak form of the problem (1).

$$\begin{aligned} \min \quad J(y, u) &= \frac{1}{2} \int_0^T \int_{\Omega} (y(\cdot; \mu) - y_d)^2 d\Omega dt + \frac{\gamma}{2} \int_0^T \int_{\Omega} u^2(\cdot; \mu) d\Omega dt, \\ \text{S.t} \quad & \\ -\partial_t^\alpha(y(t), v) - a(y(t), v; \mu) &= (u(t), v), & v \in Y, \\ y &= 0, & \text{on } \partial\Omega, \\ y(\cdot, 0) &= 0, & \text{in } \Omega. \end{aligned} \quad (4)$$

We introduce the parameter-dependent bilinear form $a(\cdot, \cdot; \mu) : Y \times Y \rightarrow \mathbb{R}$ as

$$a(v, w; \mu) = \mu \int_{\Omega} \nabla v \nabla w d\Omega.$$

It is assumed that $a(\cdot, \cdot; \mu)$ is continuous

$$0 \leq \gamma_a(\mu) = \sup_{w \in Y/\{0\}} \sup_{v \in Y/\{0\}} \frac{a(w, v; \mu)}{\|w\|_Y \|v\|_Y} \leq \gamma_0^a < \infty, \quad \forall \mu \in D,$$

and coercive

$$\alpha(\mu) = \inf_{v \in Y/\{0\}} \frac{a(v, v; \mu)}{\|v\|_Y^2} \geq \alpha_0 > 0, \quad \forall \mu \in D. \quad (5)$$

The bilinear form is assumed to be dependent affinely on the parameter.

4. Optimality condition

Now, the first order necessary and sufficient optimality conditions for the fractional optimal control problem (4) are derived using the Lagrangian approach. Let p denote the adjoint variable, $\mu \in D$ is given, we define the Lagrangian functional related to Problem (4) as

$$L(y, u, p) = J(y, u) - \int_Q (-\partial_t^\alpha(y(t), v) - a(y(t), v; \mu) - (u(t), v)) p dx dt.$$

By derivative of L with respect to p , the state equation is as following:

$$\begin{cases} -\partial_t^\alpha(y^*(t), \phi) - a(y^*(t), \phi; \mu) = (u^*, \phi), & \phi \in Y, \\ y^*(\cdot, t) = 0, & \text{on } \partial\Omega, \\ y^*(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \quad (6)$$

By differentiating of L with respect to y , the adjoint equations are obtained

$$\begin{cases} -\partial_{T-t}^\alpha(p^*(t), \varphi) - a(p^*(t), \varphi; \mu) = (y_d(t) - y^*, \varphi), & \varphi \in Y, \\ p^*(\cdot, t) = 0, & \text{on } \partial\Omega, \\ p^*(\cdot, T) = 0, & \text{in } \Omega. \end{cases} \quad (7)$$

Finally, by derivative with respect to u , the gradient equation is obtained

$$\gamma u^* - p^* = 0, \quad \text{in } Q. \quad (8)$$

We replace $u^* = \frac{1}{\gamma}p^*$ in equation (6). Since the fractional state equation is left-sided Caputo fractional derivative and the fractional adjoint equation is right-sided, we apply change of variable $p(x, t) = \bar{p}(x, T - t)$ and then we have

$$\partial_{T-t}^\alpha p(x, t) = \partial_t^\alpha \bar{p}(x, T - t),$$

for more detail, see (Antil and Otarola , 2015). As a result, the right Caputo fractional derivative can be written as a left Caputo fractional derivative. Therefore, the equation (7), (8) change to

$$\begin{cases} -\partial_t^\alpha(y^*(t), \phi) - a(y^*(t), \phi; \mu) = \frac{1}{\gamma}(\bar{p}^*(T - t), \phi), & \phi \in Y, \\ -\partial_t^\alpha(\bar{p}^*(T - t), \varphi) - a(\bar{p}^*(T - t), \varphi; \mu) = (y_d - y^*, \varphi), & \varphi \in Y, \\ y^*(\cdot, t) = 0, \quad \bar{p}^*(\cdot, T - t) = 0, & \text{on } \partial\Omega, \\ y^*(\cdot, 0) = 0, \quad \bar{p}^*(\cdot, 0) = 0, & \text{in } \Omega. \end{cases} \quad (9)$$

For simplicity, we hereinafter eliminate the star icon in the upper index.

5. Finite element method

In this section, finite element method is used to discretize the problem. We divide the time interval $[0, T]$ into K sub-interval of equal length $\Delta t = \frac{T}{K}$ and $t^k = k\Delta t, 0 \leq k \leq K$ and $\mathbb{K} = \{0, 1, \dots, K\}$. We introduce the finite element spaces $Y_h \subseteq Y$ and $P_h \subseteq P$ of large dimension N_h . Then the following equations are gained

$$\begin{cases} -\partial_t^\alpha(y(t_k), \phi) - a(y(t_k), \phi; \mu) = \frac{1}{\gamma}(\bar{p}(T - t_k), \phi), & \phi \in Y_h, k \in \frac{\mathbb{K}}{\{0\}}, \\ -\partial_t^\alpha(\bar{p}(T - t_k), \varphi) - a(\bar{p}(T - t_k), \varphi; \mu) = (y_d(t_k) - y(t_k), \varphi), & \varphi \in Y_h, k \in \frac{\mathbb{K}}{\{K\}}, \\ y(\cdot, t_k) = 0, \quad \bar{p}(\cdot, T - t_k) = 0, & \text{on } \partial\Omega, \\ y(\cdot, 0) = 0, \quad \bar{p}(\cdot, T - t_K) = 0, & \text{in } \Omega. \end{cases} \quad (10)$$

Then, in algebraic formulation we have,

$$\left\{ \begin{array}{l} -M\partial_t^\alpha y(t_k) - \mu A y(t_k) - \frac{1}{\gamma} B \bar{p}(T - t_k) = 0, \\ -M' \partial_t^\alpha (\bar{p}(T - t_k) - \mu A' \bar{p}(T - t_k) + B' y(t_k)) = Y_d(t_k), \\ y(\cdot, t_k) = 0, \quad \bar{p}(\cdot, T - t_k) = 0, \\ y(\cdot, 0) = 0, \quad \bar{p}(\cdot, T - t_K) = 0, \end{array} \right. \quad \begin{array}{l} k \in \frac{\mathbb{K}}{\{0\}}, \\ k \in \frac{\mathbb{K}}{\{K\}}, \\ \text{on } \partial\Omega, \\ \text{in } \Omega, \end{array} \quad (11)$$

where

$$y_h(t_k) = \sum_{i=1}^{N_h} y_{hi}(t_k) \phi_i, \quad \bar{p}_h(T - t_k) = \sum_{i=1}^{N_h} \bar{p}_{hi}(T - t_k) \varphi_i,$$

ϕ_i and φ_i are the basis of finite element method. The matrices used in the state and adjoint equations are

$$\begin{aligned} (M)_{ij} &= (\phi_i, \phi_j), & (B)_{ij} &= (\varphi_i, \phi_j), \\ (M')_{ij} &= (\varphi_i, \varphi_j), & (B')_{ij} &= (\phi_i, \varphi_j), \\ Y_d(t_k) &= (y_d(t_k), \varphi_j), \end{aligned}$$

where M, M', B and B' are the mass matrix of the finite element method. Also, we have

$$(A)_{ij} = a(\phi_i, \phi_j), \quad (A')_{ij} = a(\varphi_i, \varphi_j), \quad 1 \leq i, j \leq N_h$$

where they are the same and are called the stiffness matrix of finite element method. Now, we express the following lemma for discretization of time fractional derivative.

Lemma 5.1. *Mohebbi et al. (2013). Suppose $0 \leq \alpha \leq 1$ and $g(t) \in C^2[0, t_k]$, it holds that*

$$\begin{aligned} \left| \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{g'(t)}{(t_n-t)^\alpha} dt - c [b_0 g(t_n) - \sum_{m=1}^{n-1} (b_{n-m-1} - b_{n-m}) g(t_m) - b_{n-1} g(t_0)] \right| \leq \\ \frac{1}{\Gamma(2-\alpha)} \left[\frac{1-\alpha}{12} + \frac{2^{2-\alpha}}{2-\alpha} - (1+2^{-\alpha}) \right] \max_{0 \leq t \leq t_k} |g''(t)| \tau^{2-\alpha}, \end{aligned}$$

where

$$b_m = (m+1)^{1-\alpha} - m^{1-\alpha}, \quad c = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}, \quad \tau = \Delta t.$$

Therefore, for $n = 1$,

$$\partial_t^\alpha g(t_1) \simeq c[g(t_1) - g(t_0)],$$

and for $2 \leq n \leq L$,

$$\partial_t^\alpha g(t_n) \simeq c[b_0 g(t_n) - \sum_{m=1}^{n-1} (b_{n-m-1} - b_{n-m})g(t_m) - b_{n-1}g(t_0)].$$

Since $y(t_0) = 0$ and $\bar{p}(0) = \bar{p}(T - t_K) = 0$, we have the following estimation

$$\partial_t^\alpha y(t_n) \simeq c[b_0 y(t_n) - \sum_{m=1}^{n-1} (b_{n-m-1} - b_{n-m})g(t_m),$$

and

$$\partial_t^\alpha \bar{p}(T - t_n) \simeq c[b_0 \bar{p}(T - t_n) - \sum_{m=1}^{K-n-1} (b_{K-n-m-1} - b_{K-n-m})\bar{p}(T - t_{K-m})].$$

We define a matrix

$$D = \begin{bmatrix} cb_0 & 0 & 0 & \cdots & 0 \\ c(b_1 - b_0) & cb_0 & 0 & \cdots & 0 \\ c(b_2 - b_1) & c(b_1 - b_0) & cb_0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c(b_{K-1} - b_{K-2}) & c(b_{K-2} - b_{K-3}) & \cdots & c(b_1 - b_0) & cb_0 \end{bmatrix}$$

and

$$vec(y) = [y_1(t_1), \dots, y_{N_h}(t_1), y_1(t_2), \dots, y_{N_h}(t_2), \dots, y_1(t_K), \dots, y_{N_h}(t_K)],$$

$$vec(y_d) = [y_{d1}(t_0), \dots, y_{dN_h}(t_0), y_{d1}(t_1), \dots, y_{dN_h}(t_1), \dots, y_{d1}(t_{K-1}), \dots, y_{dN_h}(t_{K-1})],$$

$$vec(\bar{p}) = [\bar{p}_1(T - t_0), \dots, \bar{p}_{N_h}(T - t_0), \bar{p}_1(T - t_1), \dots, \bar{p}_{N_h}(T - t_1), \dots, \bar{p}_1(T - t_{K-1}), \dots, \bar{p}_{N_h}(T - t_{K-1})].$$

By using Lemma 5.1 the algebraic form of equation (11) is as following:

$$\begin{cases} -[D \otimes M + \mu I_K \otimes A]vec(y) - \frac{1}{\gamma}[I_b \otimes M]vec(\bar{p}) = 0, \\ [I_b^T \otimes M]vec(y) - [D^T \otimes M + \mu I_K \otimes A]vec(\bar{p}) = vec(Y_d), \end{cases} \quad (12)$$

where I_K is the $K \times K$ unitary matrix and I_b is a $K \times K$ matrix which the first diag of it is equal to one.

6. Reduced basis method

In general, the RBM constructs the reduced basis using the greedy algorithm and pre-compute the parameter independent parts of matrices at the off-line stage. We assemble

the matrices using the coefficients at new parameter, solve the system and compute the output at the on-line stage. In the whole process, we restrict the approximate space to the much smaller subspace chosen by the greedy algorithm and discard the unnecessary modes during the calculation of the basis.

6.1. Off-line stage

We employ the reduced basis method for the efficient solutions of equation (11). We first assume that a sample set $S_N = \{\mu_1, \dots, \mu_N\}$ is given, the reduced basis spaces are

$$Y_N = \text{span}\{\zeta_n^y, 1 \leq n \leq N\},$$

$$P_N = \text{span}\{\xi_n^p, 1 \leq n \leq N\}.$$

Here ζ_n^y and ξ_n^p , $1 \leq n \leq N$ are $(\cdot, \cdot)_Y$ -orthogonal basis functions derived by a Gram-Schmidt orthogonalization procedure and also $N \ll N_h$. We remark the greedy sampling procedure to construct S_N, Y_N, P_N in Section 5.2. The reduced optimal control problem is as following:

$$\begin{aligned} \min \quad & J(y_N, u_N; \mu) = \frac{1}{2} \int_Q \|y_N(\cdot; \mu) - y_{Nd}\|_{Y_N}^2 + \frac{\gamma}{2} \int_Q \|u_N(\cdot; \mu)\|_{U_N}^2, \quad (13) \\ \text{S.t} \quad & \\ & -\partial_t^\alpha(y_N, v) - a(y_N, v; \mu) = (u_N, v), \quad v \in Y_N, \\ & y_N(\cdot, t) = 0, \quad \text{on } \partial\Omega, \\ & y(\cdot, 0) = 0, \quad \text{in } \Omega, \end{aligned}$$

where y_{Nd} is the projection of y_d on the N dimensional space Y_N .

The first-order optimality system of (13) reads: Given $\mu \in D$, the optimal solution (y_N, \bar{p}_N) satisfies

$$\left\{ \begin{array}{ll} -\partial_t^\alpha(y_N(t), \phi) - a(y_N(t), \phi; \mu) = \frac{1}{\gamma}(\bar{p}_N(T-t), \phi), & \phi \in Y_N, \\ -\partial_t^\alpha(\bar{p}_N(T-t), \varphi) - a(\bar{p}_N(T-t), \varphi; \mu) + (y_N(t), \varphi) = (y_d(t), \varphi), & \varphi \in Y_N, \\ y_N(\cdot, 0) = 0, \quad \bar{p}_N(\cdot, T-t_K) = 0, & \text{in } \Omega, \\ y_N(\cdot, t) = 0, \quad \bar{p}_N(\cdot, T-t) = 0, & \text{on } \partial\Omega. \end{array} \right.$$

The system for $t = t_k$ is as following:

$$\begin{cases} -\partial_t^\alpha(y_N(t_k), \phi) - a(y_N(t_k), \phi; \mu) = \frac{1}{\gamma}(\bar{p}_N(T - t_k), \phi), & \phi \in Y_N, k \in \frac{\mathbb{K}}{\{0\}}, \\ -\partial_t^\alpha(\bar{p}_N(T - t_k), \varphi) - a(\bar{p}_N(T - t_k), \varphi; \mu) + (y_N(t_k), \varphi) = (y_d(t_k), \varphi), & \varphi \in Y_N, k \in \frac{\mathbb{K}}{\{K\}}, \\ y_N(., 0) = 0, \quad \bar{p}_N(., T - t_K) = 0, & \text{on } \partial\Omega, \\ y_N(., t_k) = 0, \quad \bar{p}_N(., T - t_k) = 0, & \text{in } \Omega. \end{cases} \quad (14)$$

The reduced state and adjoint solution are defined as

$$y_N(t_k) = \sum_{i=1}^N y_{Ni}(t_k)\zeta_i, \quad \bar{p}_N(T - t_k) = \sum_{i=1}^N \bar{p}_{Ni}(T - t_k)\xi_i.$$

Denote the coefficient vectors as $y_N^k(t_k) = [y_{N1}(t_k), \dots, y_{NN}(t_k)]^T \in \mathbb{R}^N$ and $\bar{p}_N(T - t_k) = [\bar{p}_{N1}(T - t_k), \dots, \bar{p}_{NN}(T - t_k)]^T \in \mathbb{R}^N$, respectively. By using Lemma 5.1 the algebraic form of equation (14) is as following:

$$\begin{cases} -[D \otimes M_N + \mu I_K \otimes A_N]vec(y_N) - \frac{1}{\gamma}[I_b \otimes B_N]vec(\bar{p}_N) = 0, \\ [I_b^T \otimes B'_N]vec(y_N) - [D^T \otimes M'_N + \mu I_K \otimes A'_N]vec(\bar{p}_N) = vec(Y_{dN}). \end{cases} \quad (15)$$

Here, $M_N, M'_N, B_N, B'_N \in R^{N \times N}$ are matrices with entries

$$(M_N)_{ij} = (\zeta_i, \zeta_j), \quad (M'_N)_{ij} = (\xi_i, \xi_j),$$

$$(B_N)_{ij} = (\zeta_i, \xi_j), \quad (B'_N)_{ij} = (\xi_i, \zeta_j).$$

The parameter dependent matrices $A_N, A'_N \in R^{N \times N}$ are given by

$$(A_N)_{ij} = a(\zeta_i, \zeta_j), \quad (A'_N)_{i,j} = a(\xi_i, \xi_j)$$

and Y_{dN}^k is defined as :

$$Y_{dN}(t_k) = (y_d(t_k), \xi_i),$$

$$vec(y_N) = [y_1(t_1), \dots, y_N(t_1), y_1(t_2), \dots, y_N(t_2), \dots, y_1(t_K), \dots, y_N(t_K)],$$

$$vec(y_{dN}) = [y_{d1}(t_0), \dots, y_{dN}(t_0), y_{d1}(t_1), \dots, y_{dN}(t_1), \dots, y_{d1}(t_{K-1}), \dots, y_{dN}(t_{K-1})],$$

$$\text{vec}(\bar{p}_N) = [\bar{p}_1(T - t_0), \dots, \bar{p}_N(T - t_0), \bar{p}_1(T - t_1), \dots, \bar{p}_N(T - t_1), \dots, \bar{p}_1(T - t_{K-1}), \dots, \bar{p}_N(T - t_{K-1})].$$

6.2. Greedy Sampling Procedure

We generate the reduced basis space using the greedy sampling procedure summarized in Algorithm 1. $D_{\text{train}} \subset D$ is a finite but suitably large parameter train sample; μ_1 is the initial parameter value and $\epsilon > 0$ is a prescribed desired error tolerance. Since we can only guarantee the desired error tolerance for all $\mu \in D_{\text{train}}$, we note that we have to choose the train sample sufficiently fine. Now we present the greedy algorithm to generate the reduced basis spaces.

Algorithm 1

Greedy procedure

1	Choose $D_{\text{train}} \subset D$, $\mu_1 \in D_{\text{train}}$ (arbitrary), and $\epsilon > 0$.
2	$N=1$, $\mu^* \leftarrow \mu_1$, $S_N = \{\mu^*\}$, $\Delta_N(\mu^*) \leftarrow \Delta_{Ny}(\mu^*) + \Delta_{Np}(\mu^*)$,
3	Set $Y_N = \text{POD}(\{y^1(\mu^*), \dots, y^K(\mu^*)\})$ and $P_N = \text{POD}(\{p^0(\mu^*), \dots, p^{K-1}(\mu^*)\})$,
4	while $\Delta_N(\mu^*) > \epsilon$ do
5	$\forall \mu \in D_{\text{train}}$ compute the reduced solution (y_N, p_N) and $\Delta_{Ny}(\mu) \leftarrow \ y_h(\mu) - y_N(\mu)\ $, $\Delta_{Np}(\mu) \leftarrow \ p_h(\mu) - p_N(\mu)\ $, $\Delta_N(\mu) \leftarrow \ \Delta_{Ny}(\mu)\ + \ \Delta_{Np}(\mu)\ $,
6	$\mu^* \leftarrow \text{argmax}_{\mu \in D_{\text{train}}} \Delta_N(\mu)$, $S_N = S_N \cup \{\mu^*\}$, $Y_N = Y_N \cup \text{POD}(\{y^1(\mu^*), \dots, y^K(\mu^*)\})$, $P_N = P_N \cup \text{POD}(\{p^0(\mu^*), \dots, p^{K-1}(\mu^*)\})$,
7	$\Delta_N(\mu^*) \leftarrow \max_{\mu \in D_{\text{train}}} \Delta_N(\mu)$,
8	$N \leftarrow N + 1$,
9	end while.

We note that the reduced basis Y_N and P_N are enlarged in step 6 according to the evaluation of $\Delta_N(\mu)$ defined in step 5 as the real error between the FE solution and the RB one. This is computationally expensive and time-consuming. To avoid this, we obtain the error bound described in Section 6.

6.3. Online Stage

Let $\mu \in D$ is given. By using the information obtained in the offline stage, we solve the system (15) for the given parameter. For more detail about reduced basis method, see Karcher et al. (2013)

7. Error bound

To begin, we introduce the dual norm of the primal (state) residual

$$\varepsilon^{pr}(\mu, t_k) \equiv \sup_{v \in Y} \frac{R^{pr}(v; \mu, t_k)}{\|v\|_Y}, \quad (16)$$

and the dual norm of the dual (adjoint) residual

$$\varepsilon^{du}(\mu, T - t_k) \equiv \sup_{v \in Y} \frac{R^{du}(v; \mu, T - t_k)}{\|v\|_Y}, \quad (17)$$

where

$$R^{pr}(v; \mu, t_k) = (u^k, v) + a(y_N(\mu, t_k), v) - cb_{k-1}(y_N(t_0), v) \quad (18)$$

$$- c \sum_{m=1}^{k-1} (b_{k-m-1} - b - k - m)(y_N(\mu, t_m), v) + cb_0(y_N(\mu, t_k), v),$$

$$R^{du}(v; \mu, T - t_k) = (y_d(t_k), v) - (y(\mu, t^k), v) + a(\bar{p}_N(\mu, T - t_k), v) \quad (19)$$

$$- c \sum_{m=1}^{K-k-1} (b_{K-k-m-1} - b_{K-k-m})(\bar{p}_N(\mu, T - t_{K-m}), v) + cb_0(\bar{p}_N(\mu, T - t_k), v).$$

Lemma 7.1. *Let $e^{pr}(\mu, t_k) \equiv y(\mu, t_k) - y_N(\mu, t_k)$ be the error in the primal variable and define the norm*

$$\|v(\mu, t_k)\|^{pr} = ((v(\mu, t_k), v(\mu, t_k); \mu) + a(v(\mu, t_k), v(\mu, t_k); \mu))^{\frac{1}{2}}.$$

The error in the primal variable is then bounded by

$$\|e^{pr}(\mu, t_k)\|^{pr} \leq \left[\frac{1}{\alpha(\mu)} (\varepsilon^{pr}(\mu, t_k))^2 + \sum_{k'=1}^{k-1} \Delta_{k'} \right]^{\frac{1}{2}},$$

where

$$\Delta_{k'} = \sum_{k''=1}^{k'} \frac{1}{\alpha(\mu)} (\varepsilon^{pr}(\mu, t_{k''}))^2.$$

Proof. We immediately derive from Equation (16) that $e^{pr}(\mu, t_k)$ satisfies

$$a(e^{pr}(\mu, t_k), v) + cb_0(e^{pr}(\mu, t_k), v) = c \sum_{m=1}^{k-1} (b_{k-m-1} - b_{k-m})(e^{pr}(\mu, t_m), v) + R^{pr}(v; \mu, t_k).$$

Set $v = e^{pr}(\mu, t_k)$, by equation (6), we obtain

$$\begin{aligned}
& a(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) + cb_0(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) \\
& \leq c \sum_{m=1}^{k-1} (b_{k-m-1} - b_{k-m})(e^{pr}(\mu, t_m), e^{pr}(\mu, t_k)) + \varepsilon^{pr}(\mu, t_k) \|e^{pr}(\mu, t_k)\| \\
& \leq c \sum_{m=1}^{k-1} (b_{k-m-1} - b_{k-m})(e^{pr}(\mu, t_m), e^{pr}(\mu, t_k)) + \varepsilon^{pr}(\mu, t_k) \|e^{pr}(\mu, t_k)\| \\
& \leq c \sum_{m=1}^{k-1} \max(b_{k-m-1} - b_{k-m})(e^{pr}(\mu, t_m), e^{pr}(\mu, t_k)) + \varepsilon^{pr}(\mu, t_k) \|e^{pr}(\mu, t_k)\|.
\end{aligned}$$

We know that

$$b_0 = 1, \quad \max(b_{k-m-1} - b_{k-m}) = b_0 - b_1 < 1, \quad (a, b) \leq \|a\| \|b\|,$$

so we have

$$\begin{aligned}
& (e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) + c(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) \leq \\
& \leq c \sum_{m=1}^{k-1} \|e^{pr}(\mu, t_m)\| \|e^{pr}(\mu, t_k)\| + \varepsilon^{pr}(\mu, t_k) \|e^{pr}(\mu, t_k)\|.
\end{aligned}$$

Since

$$\text{and (for } c, d \in \mathbb{R}, \rho \in \mathbb{R}_+), \quad 2|a||b| \leq \frac{1}{\rho^2}a^2 + \rho^2b^2,$$

we apply it twice,

$$a = \sqrt{c} \sum_{m=1}^{k-1} \|e^{pr}(\mu, t_m)\|, \quad b = \sqrt{c} \|e^{pr}(\mu, t_k)\|, \quad \rho = 1,$$

$$a = \varepsilon^{pr}(\mu, t_k), \quad b = \|e^{pr}(\mu, t_k)\|, \quad \rho = \alpha(\mu),$$

and by Equation (6), we obtain

$$a(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) + c(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) \leq c \sum_{m=1}^{k-1} (e^{pr}(\mu, t_m), e^{pr}(\mu, t_m)) + \frac{1}{\alpha(\mu)} (\varepsilon^{pr}(\mu, t_k))^2,$$

$$a(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) + c \|e^{pr}(\mu, t_k)\|^2 \leq c \sum_{m=1}^{k-1} \|e^{pr}(\mu, t_m)\|^2 + \frac{1}{\alpha(\mu)} (\varepsilon^{pr}(\mu, t_k))^2.$$

We now perform it from $k' = 1$ to k and $e^{pr}(\mu, t_0) = 0$, which completes the proof.

For $k' = 1$,

$$\begin{aligned} a(e^{pr}(\mu, t_1), e^{pr}(\mu, t_1)) + c\|e^{pr}(\mu, t_1)\|^2 &\leq \frac{1}{\alpha(\mu)}(\varepsilon^{pr}(\mu, t_k))^2 = \Delta_1 \Rightarrow \\ c\|e^{pr}(\mu, t_1)\|^2 &\leq \Delta_1, \end{aligned} \quad (20)$$

$k' = 2$,

$$a(e^{pr}(\mu, t_2), e^{pr}(\mu, t_2)) + c\|e^{pr}(\mu, t_2)\|^2 \leq c\|e^{pr}(\mu, t_1)\|^2 + \frac{1}{\alpha(\mu)}(\varepsilon^{pr}(\mu, t_2))^2.$$

By equation (20), we obtain

$$\begin{aligned} a(e^{pr}(\mu, t_2), e^{pr}(\mu, t_2)) + c\|e^{pr}(\mu, t_2)\|^2 &\leq \Delta_1 + \frac{1}{\alpha(\mu)}(\varepsilon^{pr}(\mu, t_2))^2 = \Delta_2, \Rightarrow \\ c\|e^{pr}(\mu, t_2)\|^2 &\leq \Delta_2, \end{aligned}$$

\vdots

$k' = k$,

$$\begin{aligned} a(e^{pr}(\mu, t_k), e^{pr}(\mu, t_k)) + c\|e^{pr}(\mu, t_k)\|^2 &\leq \left[\frac{1}{\alpha(\mu)}(\varepsilon^{pr}(\mu, t_k))^2 + \sum_{k'=1}^{k-1} \Delta_{k'}\right] \Rightarrow \\ \|e^{pr}(\mu, t_k)\| &\leq \left[\frac{1}{\alpha(\mu)}(\varepsilon^{pr}(\mu, t_k))^2 + \sum_{k'=1}^{k-1} \Delta_{k'}\right]^{\frac{1}{2}}. \end{aligned}$$

□

Lemma 7.2. Let $e^{du}(\mu, T - t_k) \equiv \bar{p}(\mu, T - t_k) - \bar{p}_N(\mu, T - t_k)$ be the error in the dual variable and define the norm

$$\|v(\mu, t_k)\|^{du} = ((v(\mu, t_k), v(\mu, t_k); \mu) + a(v(\mu, t_k), v(\mu, t_k); \mu))^{\frac{1}{2}}.$$

The error in the dual variable is then bounded by

$$\|e^{du}(\mu, T - t_k)\|^{du} \leq \left[\frac{1}{\alpha(\mu)}(\varepsilon^{du}(\mu, T - t_k))^2 + \sum_{k'=k+1}^K \Delta_{k'}\right]^{\frac{1}{2}}$$

where

$$\Delta_{k'} = \sum_{k''=k'}^K \frac{1}{\alpha(\mu)}(\varepsilon^{du}(\mu, T - t_{k''}))^2.$$

Proof. Follow proof of Lemma 7.1. For more detail, see Karcher et al. (2013) □

8. Numerical simulations

In this section, numerical experiments are conducted to validate the proposed method. We have solved the following three fractional PDE constrained optimization problems which are similar to test cases in (Nazemi & Kheyrinataj, 2015). The simulation is conducted on Matlab 7.

Example 8.1. In the problem (P), we take $T = 1$, $x \in [0, 1]$, $\alpha = 0.7$ and

$$y_d = \nu(2(t-1)^3x(x-1) + 12t(t-1)^2x(x-1) + 3t^2(2t-2)x(x-1)) + t^2(1-t)^3x(x-1).$$

The graphs of finite element solution and reduced solution of $y(x, t)$ and $p(x, t)$ for $t = 0.2, 0.4, 0.6, 0.8$ with $N = 7$, $\gamma = 10^{-6}$ and $\mu = 0.75$ are plotted in Fig. 1. As you can see in Fig. 1 the finite element solution and reduced solution are extremely close. In Fig. 2, the error functions $\|y_h - y_N\|_2$ and $\|p_h - p_N\|_2$ with $\nu = 10^{-6}$ are depicted.

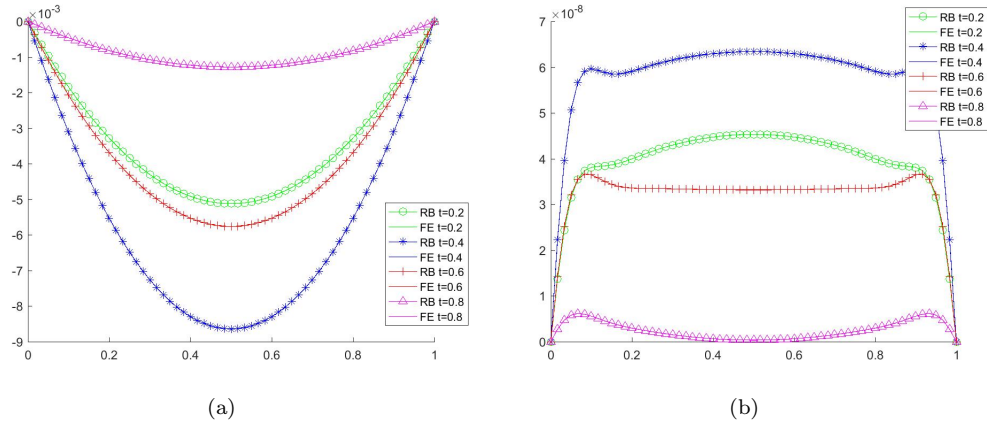


Figure 1. Comparisons between finite element and reduced solutions of $y(x, t)$ (1a) and $p(x, t)$ (1b) in $t = 0.2s$, $t = 0.4s$, $t = 0.6s$, $t = 0.8s$ with $\gamma = 10^{-6}$, $\mu = 0.75$ in Example 8.1.

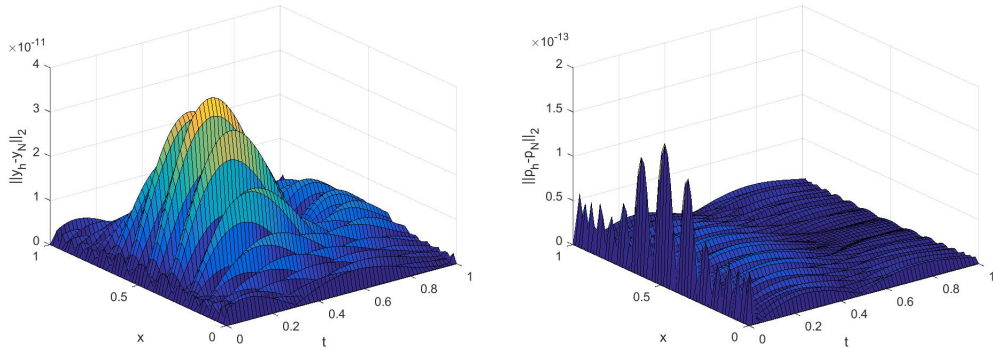


Figure 2. Plots of $\|y_h - y_N\|_2$ and $\|p_h - p_N\|_2$ with $\gamma = 10^{-6}$, $\mu = 0.75$ in Example 8.1.

See Fig. 8.1 for the error bound of the reduced basis method for the state variable

and adjoint variable. This method choose 7 parameters to construct the state space and adjoint space.

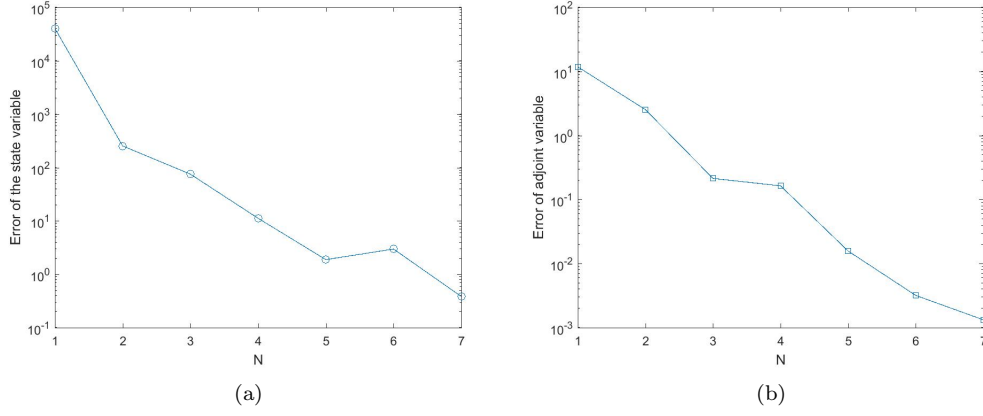


Figure 3. The error bound for the state variable (3a) and adjoint variable (3b).

Example 8.2. In problem (P) , we take $T = 1$, $x \in [0, 1]$, $\alpha = 0.99$ and

$$\begin{aligned}
 y_d = & \gamma(2\pi^2 t(t-1)^2(t-2)^2 + \pi^4 t^2(t-1)^2(t-2)^2 - 2t^2(t-1)^2 - 2t^2(t-2)^2 \\
 & - 2(t-1)^2(t-2)^2 - 2t^2(2t-2)(2t-4) - 4t(2t-2)(t-2)^2 - 4t(2t-4)(t-1)^2 - \\
 & 2\pi^2 t(t-1)^2(t-2)^2) \sin(\pi x) + t^2(1-t)^2(2-t)^2 \sin(\pi x).
 \end{aligned}$$

The graphs of finite element and reduced solutions of $y(x, t)$ and $p(x, t)$ for $t = 0.2, 0.4, 0.6, 0.8$ with $N = 1$ and $\gamma = 10^{-8}$ are plotted in Fig. 4. In Fig. 5, the error functions $\|y_h - y_N\|$ and $\|p_h - p_N\|$ with $\gamma = 10^{-8}$ are shown.

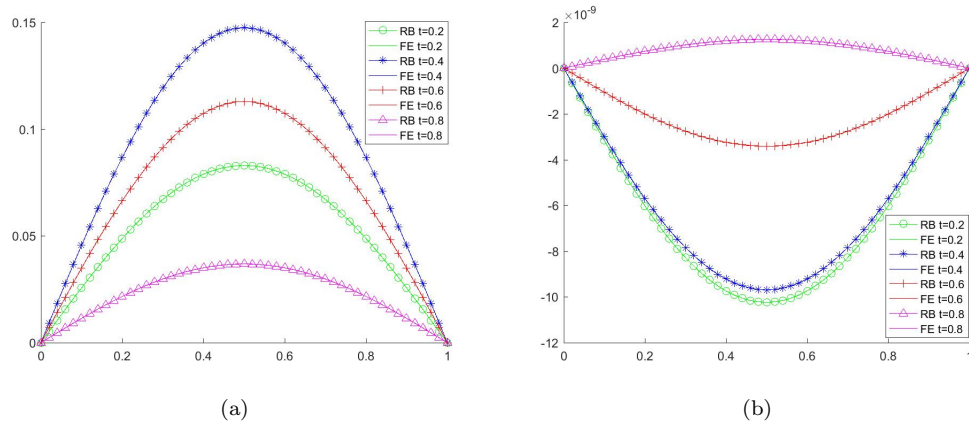


Figure 4. Comparisons between finite element and reduced solutions of $y(x, t)$ (4a) and $p(x, t)$ (4b) in $t = 0.2s$, $t = 0.4s$, $t = 0.6s$, $t = 0.8s$ with $\gamma = 10^{-8}$, $\mu = 0.6$ in Example 8.2.

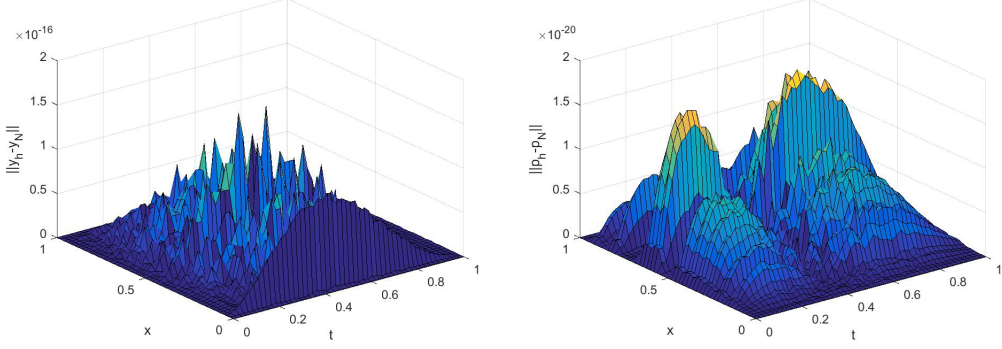


Figure 5. Plots of $\|y_h - y_N\|_2$ and $\|p_h - p_N\|_2$ with $\gamma = 10^{-8}, \mu = 0.6$ in Example 8.2.

In this example, the number of chosen parameter is one, so we did not plot the error bound.

Example 8.3. In the problem (P) , we take $T = 1, x \in [0, 1], \alpha = 0.7$ and

$$y_d = \gamma((16\pi^4 t^3(t-1)^3 - 3t^3(2t-2) - 18t^2(t-1)^2 - 6t(t-1)^3) \cos(2\pi x)) + \gamma(3t^3(2t-2) + 18t^2(t-1)^2 + 6t(t-1)^3) + t^3(1-t)^3(1 - \cos(2\pi x)).$$

The graphs of finite element and reduced solutions of $y(x, t)$ and $p(x, t)$ for $t = 0.2, 0.3, 0.5, 0.8$ with $N = 5$ and $\gamma = 10^{-7}$ are plotted in Fig. 6. In Fig. 7, the error functions $\|y_h - y_N\|$ and $\|p_h - p_N\|$ with $\gamma = 10^{-7}$ plotted. See Fig. 8 for the error

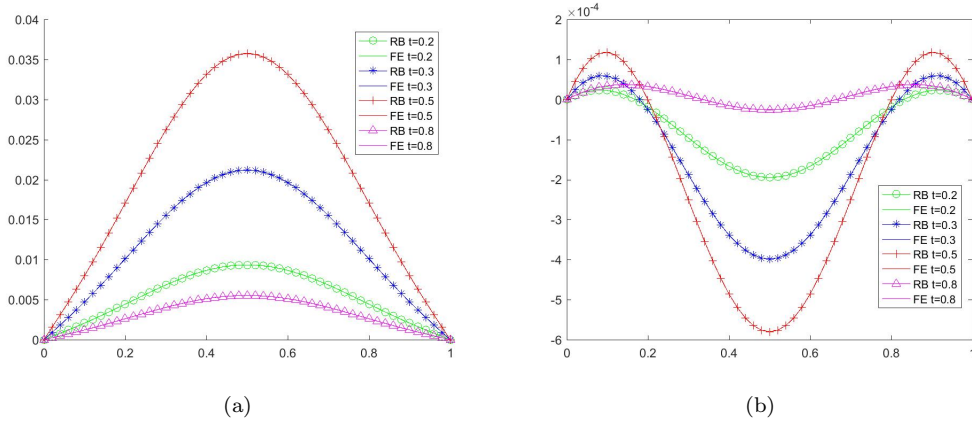


Figure 6. Comparisons between finite element and reduced solutions of $y(x, t)$ 4a and $p(x, t)$ 4b in $t = 0.2s, t = 0.3s, t = 0.5s, t = 0.8s$ with $\gamma = 10^{-3}, \mu = 0.45$ in Example 8.3.

bound of the reduced basis method for the state variable and adjoint variable. This method choose 8 parameters to construct the state space and adjoint space.

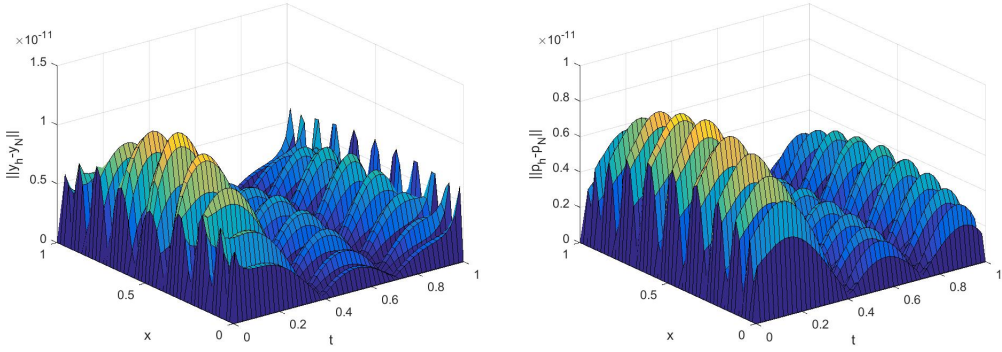


Figure 7. Plots of $\|y_h - y_N\|_2$ and $\|p_h - p_N\|_2$ with $\gamma = 10^{-3}$ in Example 8.3.

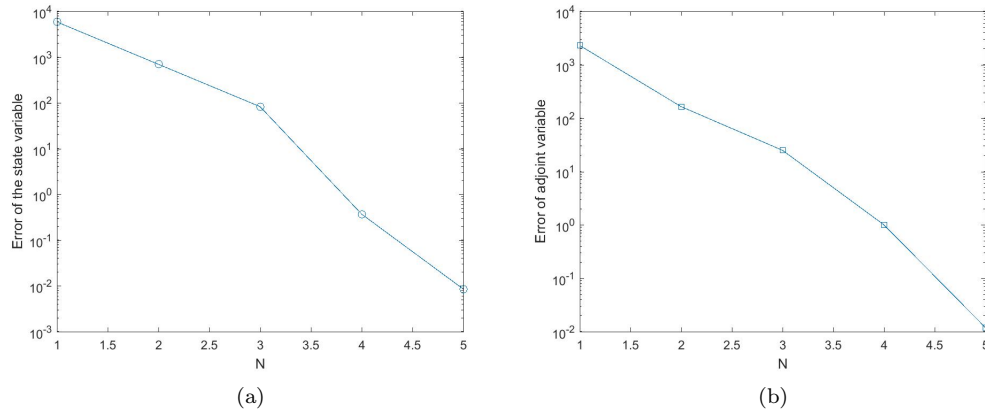


Figure 8. The error bound for the state variable (8a) and adjoint variable (8b).

9. Conclusion

In this paper, we use parametric model order reduction using reduced basis methods as an effective tool for obtaining a quick solution of fractional PDE constrained optimization problem. The used technique is applied to solve three test problems and the resulting solutions are in good agreement with the known exact solutions. For the sake of simplicity, we only considered the one-dimensional case with standard initial and boundary conditions, but the method can be extended to multi-dimensional cases with even non-classic boundary conditions which is the subject of the authors. The accuracy of numerical solution by this method is much higher than the classical numerical solutions. Numerical solutions are obtained efficiently and the stability is maintained for randomly perturbed data.

References

- Antil, H., & Otarola, E. (2015). A FEM for an optimal control problem of fractional powers of elliptic operators. *SIAM Journal on Control and Optimization*, 53(6), 3432-3456.
- Aquino, W., Kouri, D. P., & Zou, Z. (2018). A Locally Adapted Reduced Basis Method

- for Solving Risk-Averse PDE-Constrained Optimization Problems. In 2018 AIAA Non-Deterministic Approaches Conference (p. 2174).
- Bai, Y., Baleanu, D., & Wu, G. C. (2018). Existence and discrete approximation for optimization problems governed by fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation*, 59, 338-348.
- Balmes, E. (1996). Parametric families of reduced finite element models. Theory and applications. *Mechanical Systems and Signal Processing*, 10(4), 381-394.
- Bhrawy, A. H. (2016). A new spectral algorithm for a time-space fractional partial differential equations with subdiffusion and superdiffusion. In *Proc. Rom. Acad., Ser. A*, 17(1), 39-47.
- Bhrawy, A. H., Zaky, M. A., & Van Gorder, R. A. (2016). A space-time Legendre spectral tau method for the two-sided space-time Caputo fractional diffusion-wave equation. *Numerical Algorithms*, 71(1), 151-180.
- Darehmiraki, M., Farahi, M. H., & Effati, S. (2016). Solution for fractional distributed optimal control problem by hybrid meshless method. *Journal of Vibration and Control*, 1077546316678527.
- Dihlmann, M. A., & Haasdonk, B. (2015). Certified PDE-constrained parameter optimization using reduced basis surrogate models for evolution problems. *Computational Optimization and Applications*, 60(3), 753-787.
- Grepl, M. A., & Patera, A. T. (2005). A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(1), 157-181.
- Krcher, M., Grepl, M. (2016). A Posteriori Error Estimation for Reduced Order Solutions of Parametrized Parabolic Optimal Control Problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(6), 1615-1638.
- Lapichino, L., Ulbrich, S., & Volkwein, S. (2017). Multiobjective PDE-constrained optimization using the reduced-basis method. *Advances in Computational Mathematics*, 43(5), 945-972.
- Mang, A., Gholami, A., Davatzikos, C., & Biros, G. (2018). PDE-constrained optimization in medical image analysis. arXiv preprint arXiv:1803.00058.
- Mohebbi, A., Abbaszadeh, M., & Dehghan, M. (2013). The use of a meshless technique based on collocation and radial basis functions for solving the time fractional nonlinear Schrödinger equation arising in quantum mechanics. *Engineering Analysis with Boundary Elements*, 37(2), 475-485.
- Nazemi, A., & Kheyrinataj, F. (2015). Parabolic optimal control problems with a quintic B-spline dynamic model. *Nonlinear Dynamics*, 80(1-2), 653-667.
- Otarola, E. (2017). A piecewise linear FEM for an optimal control problem of fractional operators: error analysis on curved domains. *ESAIM: Mathematical Modelling and Numerical Analysis*, 51(4), 1473-1500.
- Pearson, J. W. (2013). Fast iterative solvers for PDE-constrained optimization problems (Doctoral dissertation, University of Oxford).
- Qian, E., Grepl, M., Veroy, K., & Willcox, K. (2017). A certified trust region reduced basis approach to PDE-constrained optimization. *SIAM Journal on Scientific Computing*, 39(5), S434-S460.
- Quiroga, A. A. I., Torres, G. A., Fernandez, D., & Turner, C. V. (2018). Nonlinear optimization for a tumor invasion PDE model. *Computational and Applied Mathematics*, 37(1), 485-499.
- Quiroga, A. A. I., Fernandez, D., Torres, G. A., & Turner, C. V. (2015). Adjoint method for a tumor invasion PDE-constrained optimization problem in 2D using adaptive finite element method. *Applied Mathematics and Computation*, 270, 358-368.
- Salati, A. B., Shamsi, M., & Torres, D. F. (2018). Direct transcription methods based on fractional integral approximation formulas for solving nonlinear fractional optimal control problems. *Communications in Nonlinear Science and Numerical Simulation*.
- Sun, H., Zhang, Y., Baleanu, D., Chen, W., & Chen, Y. (2018). A new collection of real world applications of fractional calculus in science and engineering. *Communications in Nonlinear Science and Numerical Simulation*.
- Ye, X., & Xu, C. (2014). A spectral method for optimal control problems governed by the time

fractional diffusion equation with control constraints. In *Spectral and High Order Methods for Partial Differential Equations-ICOSAHOM 2012* (pp. 403-414). Springer, Cham.

Zaky, M. A., & Machado, J. T. (2017). On the formulation and numerical simulation of distributed-order fractional optimal control problems. *Communications in Nonlinear Science and Numerical Simulation*, 52, 177-189.