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Abstract

We consider the inverse problem of finding the temperature distribution and the heat source whenever the temperatures at the initial time and the final time are given. The problem considered is one dimensional and the unknown heat source is supposed to be space dependent only. The existence and uniqueness results are proved.

Key words: Inverse problem, fractional derivative, heat equation, integral equations, biorthogonal system of functions, Fourier series
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1. Introduction

In this paper, we are concerned with the problem of finding $u(x, t)$ (the temperature distribution) and $f(x)$ (the source term) in the domain $Q_T = (0, 1) \times (0, T)$ for the following system

$$D_{0_+}^\alpha (u(x, t) - u(x, 0)) - u_{xx}(x, t) = f(x), \qquad (x, t) \in Q_T, \tag{1}$$

$$u(x, 0) = \varphi(x) \quad u(x, T) = \psi(x), \qquad x \in [0, 1], \tag{2}$$

$$u(1, t) = 0, \quad u_x(0, t) = u_x(1, t), \quad t \in [0, T], \tag{3}$$

where $D_{0_+}^\alpha$ stands for the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $\varphi(x)$ and $\psi(x)$ are the initial and final temperatures respectively. Our choice of the term $D_{0_+}^\alpha (u(x, .) - u(x, 0))(t)$ rather then the usual term $D_{0_+}^\alpha u(x, .)(t)$ is to avoid not only the singularity at zero but also to impose meaningful initial condition.

The problem of determination of temperature at interior points of a region when the initial and boundary conditions along with heat source term are specified are known as direct heat conduction problems. In many physical problems, determination of coefficients or right hand side (the source term, in case of the heat equation) in a differential equation from some available information is required; these problems are known as *inverse problems*. These kind of problems are ill posed in the sense of Hadamard. A number of articles address the solvability problem of the inverse problems (see [1], [2], [3], [4], [5], [6], [7], [8] and references therein).

From last few decades fractional calculus grabbed great attention of not only mathematicians and engineers but also of many scientists from all fields. Indeed fractional calculus tools have numerous applications in nanotechnology, control theory, viscoplasticity flow, biology, signal and image processing etc, see the latest monographs, [9], [10], [11], [12], [13] articles [14], [15] and reference therein. The mathematical analysis of initial and boundary value problems (linear or nonlinear) of fractional differential equations has been studied extensively by many authors, we refer to [16], [17], [18], [19], [20] and references therein.

In the next section we recall some definitions and notations from fractional calculus. Section 3 is devoted to our main results; we provide the expressions for the temperature distribution and the source term for the problem (1)-(3). Furthermore, we proved the existence and uniqueness of the inverse problem.

2. Preliminaries and notations

In this section, we recall basic definitions, notations from fractional calculus (see [11], [12], [13]).

For an integrable function $f : \mathbb{R}^+ \to \mathbb{R}$ the left sided Riemann-Liouville fractional integral of order $0 < \alpha < 1$ is defined by

$$J_{0_+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad t > 0, \tag{4}$$

where $\Gamma(\alpha)$ is the Euler Gamma function. The integral (4) can be written as a convolution

$$J_{0_+}^\alpha f(t) = (\phi_\alpha \star f)(t), \tag{5}$$

where

$$\phi_\alpha := \begin{cases} t^{\alpha-1}/\Gamma(\alpha), & t > 0, \\ 0 & t \leq 0. \end{cases} \tag{6}$$

The Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ is defined by

$$D_{0_+}^\alpha f(t) := \frac{d}{dt} J_{0_+}^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau. \tag{7}$$

In particular, when $\alpha = 0$ and $\alpha = 1$ then $D_{0_+}^0 f(t) = f(t)$ and $D_{0_+}^1 f(t) = f'(t)$ respectively; notice that the Riemann-Liouville fractional derivative of a constant is not equal to zero.

The Laplace transform of the Riemann-Liouville integral of order $0 < \alpha < 1$ of a function with at most exponentially growth is

$$\mathcal{L}\{J_{0_+}^\alpha f(t) : s\} = \mathcal{L}\{f(t) : s\}/s^\alpha.$$

Also, for $0 < \alpha < 1$ we have

$$J_{0_+}^\alpha D_{0_+}^\alpha \left(f(t) - f(0) \right) = f(t) - f(0). \tag{8}$$

The Mittag-Leffler function plays an important role in the theory of fractional differential equations, for any $z \in \mathbb{C}$ the Mittag-Leffler function with parameter ξ is

$$E_\xi(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + 1)} \qquad (Re(\xi) > 0). \tag{9}$$

In particular, $E_1(z) = e^z$

The Mittag-Leffler function of two parameters $E_{\xi,\beta}(z)$ which is a generalization of (9) is defined by

$$E_{\xi,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\xi k + \beta)} \qquad (z, \beta \in \mathbb{C}; \ Re(\xi) > 0). \tag{10}$$

3. Main Results

Let $e_\xi(t, \mu) := E_\xi(-\mu t^\xi)$ where $E_\xi(t)$ is the Mittag-Leffler function with one parameter ξ as defined in (9) and μ is a positive real number. Let us state and prove the

Lemma 3.1. *The solution of equation*

$$v(t) + \mu J_{0_+}^\alpha v(t) = g(t) \tag{11}$$

for $\mu \in \mathbb{R}$ satisfies the integral equation

$$v(t) = \int_0^t g'(t-\tau)e_\alpha(\tau, \mu)d\tau + g(0)e_\alpha(t, \mu), \tag{12}$$

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Proof. Taking the Laplace transform of both sides of the equation (11), it follows

$$\mathcal{L}\{v(t) : s\} = \frac{s^\alpha \mathcal{L}\{g(t) : s\}}{s^\alpha + \mu}. \tag{13}$$

There are different ways for getting the solution from the equation (13) by the inverse Laplace transform.

Writing the equation (13) as

$$\mathcal{L}\{v(t); s\} = \frac{s^{\alpha-1}}{s^\alpha + \mu}\left(s\mathcal{L}\{g(t) : s\} - g(0)\right) + \frac{s^{\alpha-1}}{s^\alpha + \mu}g(0);$$

and using the inverse Laplace transform, we obtain

$$v(t) = \int_0^t g'(t - \tau)e_\alpha(\tau, \mu)d\tau + g(0)e_\alpha(t, \mu),$$

which is (12). $\qquad\square$

3.1. Solution of the inverse problem

The key factor in determination of the temperature distribution and the unknown source term for the system (1)-(3) is the specially chosen basis for the space $L^2(0, 1)$ which is

$$\{2(1 - x), \quad \{4(1 - x)\cos 2\pi nx\}_{n=1}^\infty, \quad \{4\sin 2\pi nx\}_{n=1}^\infty\}. \tag{14}$$

In [21], it is proved that the set of functions in (14) forms a Riesz basis for the space $L^2(0, 1)$, hence the set is closed, minimal and spans the space $L^2(0, 1)$.

The set of functions (14) is not orthogonal and in [22] a biorthogonal set of functions given by

$$\{1, \quad \{\cos 2\pi nx\}_{n=1}^\infty, \quad \{x\sin 2\pi nx\}_{n=1}^\infty\}, \tag{15}$$

is constructed.

The solution of the inverse problem for the linear system (1)-(3) can be written in the form

$$u(x, t) = 2u_0(t)(1 - x) + \sum_{n=1}^\infty u_{1n}(t)4(1 - x)\cos 2\pi nx + \sum_{n=1}^\infty u_{2n}(t)4\sin 2\pi nx \tag{16}$$

$$f(x) = 2f_0(1 - x) + \sum_{n=1}^\infty f_{1n}4(1 - x)\cos 2\pi nx + \sum_{n=1}^\infty f_{2n}4\sin 2\pi nx \tag{17}$$

where $u_0(t), u_{1n}(t), u_{2n}(t), f_0, f_{1n}$ and f_{2n} are to be determined. By plugging the expressions of $u(x, t)$ and $f(x)$ from (16) and (17) into the equation (1), the following system of fractional differential equations is obtained

$$D_{0_+}^\alpha(u_{2n}(t) - u_{2n}(0)) - 4\pi nu_{1n}(t) + 4\pi^2n^2u_{2n}(t) = f_{2n}, \tag{18}$$

$$D_{0_+}^\alpha(u_{1n}(t) - u_{1n}(0)) + 4\pi^2n^2u_{1n}(t) = f_{1n}, \tag{19}$$

$$D_{0_+}^\alpha(u_0(t) - u_0(0)) = f_0. \tag{20}$$

Since $J_{0_+}^\alpha 1 = t^\alpha/\Gamma(1 + \alpha)$ and for $0 < \alpha < 1$, $J_{0_+}^\alpha D_{0_+}^\alpha(u_0(t) - u_0(0)) = u_0(t) - u_0(0)$ then the solution of equation (20) is

$$u_0(t) = \frac{f_0t^\alpha}{\Gamma(1 + \alpha)} + u_0(0). \tag{21}$$

Setting $\lambda := 4\pi^2n^2$, $F(t) := f_{1n}t^\alpha/\Gamma(1 + \alpha) + u_{1n}(0)$ then equation (19) can be written as

$$u_{1n}(t) + \lambda J_{0_+}^\alpha u_{1n}(t) = F(t),$$

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and using lemma (3.1), the solution of the equation (19) is given by

$$u_{1n}(t) = \frac{f_{1n}\alpha}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} e_\alpha(\tau,\lambda) d\tau + u_{1n}(0) e_\alpha(t,\lambda). \tag{22}$$

Noticing that equation (18) can be written as

$$u_{2n}(t) + \lambda J_{0_+}^\alpha u_{2n}(t) = h(t) \tag{23}$$

where $h(t) := J_{0_+}^\alpha \left(f_{2n} + 4\pi n u_{1n}(t) \right) + u_{2n}(0)$, so its solution by virtue of lemma (3.1) is

$$u_{2n}(t) = \int_0^t h'(t-\tau) e_\alpha(\tau,\lambda) d\tau + h(0) e_\alpha(t,\lambda). \tag{24}$$

As $h(0) = u_{2n}(0)$ and

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left[J_{0_+}^\alpha(f_{2n} + 4\pi n u_{1n}(t)) - u_{2n}(0) \right] \\ &= D_{0_+}^\alpha \left(f_{2n} + 4\pi n u_{1n}(t) \right). \end{aligned}$$

From equation (22), we have

$$D_{0_+}^\alpha u_{1n}(t) = \frac{f_{1n}\alpha}{\Gamma(1+\alpha)} D_{0_+}^\alpha t^{\alpha-1} \star e_\alpha(t,\lambda) + u_{1n}(0) D_{0_+}^\alpha e_\alpha(t,\lambda),$$

since $D_{0_+}^\alpha t^{\alpha-1} = 0$ and $D_{0_+}^\alpha e_\alpha(t,\lambda) = e_{\alpha,1-\alpha}(t,\lambda)$, where $e_{\alpha,\beta}(t,\lambda) := t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)$; consequently the equation (24) becomes

$$u_{2n}(t) = \int_0^t s(t-\tau) e_\alpha(\tau,\lambda) d\tau + u_{2n}(0) e_\alpha(t,\lambda). \tag{25}$$

where we have set

$$s(t) := 4\pi n u_{1n}(0) e_{\alpha,1-\alpha}(t,\lambda) + f_{2n}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

The unknown constants are $u_{2n}(0), u_{1n}(0), u_0(0), f_{2n}, f_{1n}$ and f_0. In order to get these unknowns, we use the initial and final temperatures as given in (2)

$$\begin{aligned} 2u_0(0)(1-x) \quad + \quad & \sum_{n=1}^\infty u_{1n}(0)4(1-x)\cos 2\pi nx + \sum_{n=1}^\infty u_{2n}(0)4\sin 2\pi nx = 2\varphi_0(1-x) \\ + \quad & \sum_{n=1}^\infty \varphi_{1n}4(1-x)\cos 2\pi nx + \sum_{n=1}^\infty \varphi_{2n}4\sin 2\pi nx, \end{aligned}$$

$$\begin{aligned} 2u_0(T)(1-x) \quad + \quad & \sum_{n=1}^\infty u_{1n}(T)4(1-x)\cos 2\pi nx + \sum_{n=1}^\infty u_{2n}(T)4\sin 2\pi nx = 2\psi_0(1-x) \\ + \quad & \sum_{n=1}^\infty \psi_{1n}4(1-x)\cos 2\pi nx + \sum_{n=1}^\infty \psi_{2n}4\sin 2\pi nx. \end{aligned}$$

By identification we get

$$u_{2n}(0) = \varphi_{2n}, \quad u_{1n}(0) = \varphi_{1n}, \quad u_0(0) = \varphi_0, \tag{26}$$

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$$u_{2n}(T) = \psi_{2n}, \quad u_{1n}(T) = \psi_{1n}, \quad u_0(T) = \psi_0, \tag{27}$$

where $\{\varphi_0, \varphi_{1n}, \varphi_{2n}\}$ and $\{\psi_0, \psi_{1n}, \psi_{2n}\}$ are the coefficients of the series expansion in the basis (14) of the functions $\varphi(x)$ and $\psi(x)$, respectively. In terms of the biorthogonal basis (15) these are

$$\varphi_0 = \int_0^1 \varphi(x)dx, \quad \varphi_{1n} = \int_0^1 \varphi(x)\cos 2\pi n dx, \quad \varphi_{2n} = \int_0^1 \varphi(x)x\sin 2\pi n dx,$$

$$\psi_0 = \int_0^1 \psi(x)dx, \quad \varphi_{1n} = \int_0^1 \psi(x)\cos 2\pi n dx, \quad \psi_{2n} = \int_0^1 \psi(x)x\sin 2\pi n dx.$$

By virtue of (21), (22), (25), and conditions (27) we have the expressions of unknowns f_0, f_{1n} and f_{2n} as

$$f_0 = \Gamma(1+\alpha)\left[\frac{\psi_0 - \varphi_0}{T^\alpha}\right], \tag{28}$$

$$f_{1n} = \Gamma(1+\alpha)\left[\frac{\psi_{1n} - \varphi_{1n}e_\alpha(T,\lambda)}{\alpha\int_0^T (T-\tau)^{\alpha-1}e_\alpha(\tau,\lambda)d\tau}\right], \tag{29}$$

$$f_{2n} = \Gamma(1-\alpha)\left[\frac{\psi_{2n} - \varphi_{2n}e_\alpha(T,\lambda) - \int_0^T \varphi_{1n}e_{\alpha,1-\alpha}(T-\tau,\lambda)e_\alpha(\tau,\lambda)d\tau}{\int_0^T (T-\tau)^{-\alpha}e_\alpha(\tau,\lambda)d\tau}\right]. \tag{30}$$

The unknown source term and the temperature distribution for the problem (1)-(3) are given by the series (17) and (16), where the unknowns $u_0(0), u_{1n}(0), u_{2n}(0)$ are calculated from (26), while f_0, f_{1n}, f_{2n} are given by (28)-(30). In the next subsection we will show the uniqueness of the solution.

3.2. Uniqueness of the solution

Suppose $\{u_1(x,t), f_1(x)\}, \{u_2(x,t), f_2(x)\}$ are two solution sets of the inverse problem for the system (1)-(3). Define $\overline{u}(x,t) = u_1 - u_2$ and $\overline{f}(x) = f_1 - f_2$ then the function $\overline{u}(x,t)$ satisfies the following equation and boundary conditions

$$D_{0_+}^\alpha \overline{u}(x,t) - \overline{u}_{xx}(x,t) = \overline{f}(x), \qquad (x,t) \in Q_T, \tag{31}$$
$$\overline{u}(x,0) = 0 \quad \overline{u}(x,T) = 0, \qquad x \in [0,1], \tag{32}$$
$$\overline{u}(1,t) = 0, \quad \overline{u}_x(0,t) = \overline{u}_x(1,t), \qquad t \in [0,T]. \tag{33}$$

Using the biorthogonal basis (15), we have

$$\overline{u}_0(t) = \int_0^1 \overline{u}(x,t)dx, \tag{34}$$

$$\overline{u}_{1n}(t) = \int_0^1 \overline{u}(x,t)\cos 2\pi n x dx, \quad (n = 1,2,...), \tag{35}$$

and

$$\overline{u}_{2n}(t) = \int_0^1 \overline{u}(x,t)x\sin 2\pi n x dx \quad (n = 1,2,...). \tag{36}$$

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Alike,

$$\overline{f}_0 = \int_0^1 \overline{f}(x)dx, \tag{37}$$

$$\overline{f}_{1n} = \int_0^1 \overline{f}(x)\cos 2\pi nx dx, \quad (n = 1, 2, ...), \tag{38}$$

and

$$\overline{f}_{2n} = \int_0^1 \overline{f}(x)x \sin 2\pi nx dx \quad (n = 1, 2, ...). \tag{39}$$

Taking the fractional derivative $D_{0_+}^\alpha$ under the integral sign of equation (34), using (31), (37) and integration by parts, we get

$$D_{0_+}^\alpha \overline{u}_0(t) = \overline{f}_0, \tag{40}$$

with the boundary conditions

$$\overline{u}_0(0) = 0, \quad \overline{u}_0(T) = 0. \tag{41}$$

The solution of the problem (40) is

$$\overline{u}_0(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\overline{f}_0;$$

taking into account the boundary conditions (41), we get

$$\overline{f}_0 = 0, \quad \Rightarrow \quad \overline{u}_0(t) = 0. \tag{42}$$

Taking the fractional derivative $D_{0_+}^\alpha$ under the integral sign of equation (35) then by using (31), (32), (38) and integration by parts, we obtain

$$D_{0_+}^\alpha \overline{u}_{1n}(t) + \lambda \overline{u}_{1n}(t) = \overline{f}_{1n}; \tag{43}$$

the associated boundary conditions are

$$\overline{u}_{1n}(0) = 0, \quad \overline{u}_{1n}(T) = 0. \tag{44}$$

The solution of the problem (43) is

$$\overline{u}_{1n}(t) = e_{\alpha,\alpha}(t, \lambda) \star \overline{f}_{1n}.$$

Using (44) we have

$$\overline{f}_{1n} = 0, \quad \Rightarrow \quad \overline{u}_{1n}(t) = 0, \quad (n = 1, 2, ...).$$

In the same manner from (36), we obtain the equation

$$^C D_{0_+}^\alpha \overline{u}_{2n}(t) + 4\pi^2 n^2 \overline{u}_{2n}(t) = 4\pi n \overline{u}_{1n}(t) + \overline{f}_{2n}; \tag{45}$$

the associated boundary conditions are

$$\overline{u}_{2n}(0) = 0, \quad \overline{u}_{2n}(T) = 0. \tag{46}$$

Taking account of $\overline{u}_{1n}(t) = 0$ then the solution of the equation (45) satisfies

$$\overline{u}_{2n}(t) = e_{\alpha,\alpha}(t, \lambda) \star \overline{f}_{2n};$$

using (46) we have

$$\overline{f}_{2n} = 0, \quad \Rightarrow \quad \overline{u}_{2n}(t) = 0, \quad (n = 1, 2, ...).$$

For every $t \in [0, T]$, the functions $\overline{u}(x, t)$ and $\overline{f}(x)$ are orthogonal to the system of functions given in (15), which form a basis of the space $L^2(0, 1)$, consequently,

$$\overline{u}(x, t) = 0, \quad \overline{f}(x) = 0.$$

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3.3. Existence of the solution

Suppose $\varphi(x) \in C^4[0,1]$ be such that $\varphi'(0) = \varphi'(1)$, $\varphi''(0) = \varphi''(1)$. As φ_{1n} is the coefficient of the cosine Fourier expansion of the function $\varphi(x)$ with respect to basis (15), from (26) the expression for $u_{1n}(0)$ can be written as

$$u_{1n}(0) = \int_0^1 \varphi(x) \cos 2\pi nx\, dx$$

which integrated by parts four times gives

$$
\begin{aligned}
u_{1n}(0) &= \frac{1}{16\pi^4 n^4} \int_0^1 \varphi^4(x) \cos 2\pi nx\, dx, \\
&= \frac{1}{16\pi^4 n^4} \varphi_{1n}^4,
\end{aligned}
$$

where φ_{1n}^4 is the coefficient of the cosine Fourier expansion of the function $\varphi^4(x)$ with respect to the basis (15).

Alike, we obtain

$$u_{2n}(0) = \frac{1}{16\pi^4 n^4} \varphi_{2n}^4, \tag{47}$$

where φ_{2n}^4 is the coefficient of sine Fourier transform of the function φ^4 with respect to the basis (15).

The set of functions $\{\varphi_{1n}^4, \varphi_{2n}^4\ n = 1, 2, ...\}$ is bounded as by supposition we have $\varphi(x) \in C^4[0,1]$, and due to the inequality

$$\sum_{n=1}^{\infty} (\varphi_{in}^4)^2 \leq C \parallel \varphi^4(x) \parallel_{L^2(0,1)}^2, \quad i = 1, 2$$

which is true by the Bessel inequality of the trigonometric series. Similarly, we have for $\psi(x) \in C^4[0,1]$ the set of functions $\{\psi_{1n}^4, \psi_{2n}^4\ n = 1, 2, ...\}$ is bounded and

$$\sum_{n=1}^{\infty} (\psi_{in}^4)^2 \leq C \parallel \psi^4(x) \parallel_{L^2(0,1)}^2, \quad i = 1, 2.$$

The Mittag-Leffler functions $e_\alpha(t; \mu)$ and $e_{\alpha,\beta}(t; \mu)$ for $\mu > 0$ and $0 < \alpha \leq 1$, $0 < \alpha \leq \beta \leq 1$ respectively, are completely monotone functions (see [23] page 268). Furthermore, we have

$$E_{\alpha,\beta}(\mu t^\alpha) \leq M, \quad t \in [a, b], \tag{48}$$

where $[a, b]$ is a finite interval with $a \geq 0$, and

$$\int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\beta}(\lambda t^\alpha)\, d\tau < \infty,$$

(see [24] page 9).

The solution $u(x, t)$ becomes

$$
\begin{aligned}
u(x, t) &= 2(1 - x)\left(f_0 t^\alpha / \Gamma(1 + \alpha) + \varphi_0 \right) \\
&+ \sum_{n=1}^{\infty} \left(\frac{f_{1n}\alpha}{\Gamma(1 + \alpha)} t^{\alpha-1} \star e_\alpha(t, \lambda) + \frac{\varphi_{1n}^4}{16\pi^4 n^4} e_\alpha(t, \lambda) \right) 4(1 - x) \cos 2\pi nx \\
&+ \sum_{n=1}^{\infty} \left(\frac{\varphi_{1n}^4}{4\pi^3 n^3} e_{\alpha,1-\alpha}(t, \lambda) + f_{2n} \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \star e_\alpha(t, \lambda) + \frac{\varphi_{2n}^4}{16\pi^4 n^4} e_\alpha(t, \lambda) \right) 4 \sin 2\pi nx \tag{49}
\end{aligned}
$$

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Setting $M = \max\{M_1, M_2\}$ where

$$|e_\alpha| \leq M_1, \quad |e_{\alpha,1-\alpha}| \leq M_2 \quad t \in [0,T], T \text{ is finite.}$$

Taking into account the values of f_0, f_{1n}, f_{2n} from (28)-(30) and the properties of Mittag-Leffler function, the series (49) is bounded above by

$$2(1-x)\left(|\psi_0| + (1 - \frac{t^\alpha}{T^\alpha})|\varphi_0|\right)$$

$$+ \sum_{n=1}^{\infty}\left(\left\{\frac{\psi_{1n}^4 - \varphi_{1n}^4 M}{16\pi^4 n^4}\right\}\frac{t^\alpha}{T^\alpha} + \frac{\varphi_{1n}^4}{16\pi^4 n^4}M\right)4(1-x)\cos 2\pi nx$$

$$+ \sum_{n=1}^{\infty}\left(\frac{\varphi_{1n}^4 M}{4\pi^3 n^3} + \frac{t^{1-\alpha}}{T^{1-\alpha}}\left\{\frac{\psi_{2n}^4 - \varphi_{2n}^4 M}{16\pi^4 n^4}\right\} - \frac{\varphi_{1n}^4 M}{16\pi^4 n^4} + \frac{\varphi_{2n}^4 M}{16\pi^4 n^4}\right)4\sin 2\pi nx. \tag{50}$$

For every $(x,t) \in Q_T$ the series (50) is bounded above by the uniformly convergent series

$$\sum_{n=1}^{\infty}\left(\frac{|\psi_{1n}^4| + |\varphi_{1n}^4|2M}{16\pi^4 n^4}\right) + \sum_{n=1}^{\infty}\left(\frac{(4\pi n - 1)|\varphi_{1n}^4|M}{16\pi^4 n^4} + \frac{|\psi_{2n}^4| - |\varphi_{2n}^4|2M}{16\pi^4 n^4}\right). \tag{51}$$

By the Weierstrass M-test, the series (49) is uniformly convergent in the domain Q_T. Hence, the solution $u(x,t)$ is continuous in the domain Q_T.

The uniformly convergent series doesn't provide any information about the convergence of the series obtained from its term by term differentiation. Now take the $D_{0_+}^\alpha$ derivative from the series expression of $u(x,t)$ given by (49)

$$\sum_{n=1}^{\infty} D_{0_+}^\alpha \mathcal{U}_n(x,t) = 2(1-x)\left(f_0 + \varphi_0\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right)$$

$$+ \sum_{n=1}^{\infty}\left(\frac{\varphi_{1n}^4}{16\pi^4 n^4}e_{\alpha,1-\alpha}(t,\lambda)\right)4(1-x)\cos 2\pi nx$$

$$+ \sum_{n=1}^{\infty}\left(\frac{\varphi_{1n}^4}{4\pi^3 n^3}e_{\alpha,1-2\alpha}(t,\lambda) + f_{2n}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\star e_{\alpha,1-\alpha}(t,\lambda) + \frac{\varphi_{2n}^4}{16\pi^4 n^4}e_{\alpha,1-\alpha}(t,\lambda)\right)4\sin 2\pi nx, \tag{52}$$

we have used $D_{0_+}^\alpha t^{\alpha-1} = 0$, $D_{0_+}^\alpha e_\alpha(t,\lambda) = e_{\alpha,1-\alpha}(t,\lambda)$, $D_{0_+}^\alpha e_{\alpha,1-\alpha}(t,\lambda) = e_{\alpha,1-2\alpha}(t,\lambda)$ and $\mathcal{U}_n(x,t)$ is defined by the right hand side of the series (16).

The series (52) is bounded above by the uniformly convergent series

$$\sum_{n=1}^{\infty}\left(\frac{|\psi_{2n}^4| + (4\pi n + 1)|\varphi_{1n}^4|M}{16\pi^4 n^4}\right),$$

hence it converges uniformly, so $D_{0_+}^\alpha u_n(x,t) = \sum_{n=1}^{\infty} D_{0_+}^\alpha \mathcal{U}_n(x,t)$.

Take the x-derivative of the series expression of $u(x,t)$ from (49), we obtain

$$\sum_{n=1}^{\infty} \mathcal{U}_{n,x}(x,t) := -2\left(f_0 t^\alpha/\Gamma(1+\alpha) + \varphi_0\right) - \sum_{n=1}^{\infty}\left(\frac{f_{1n}\alpha}{\Gamma(1+\alpha)}t^{\alpha-1}\star e_\alpha(t,\lambda) + \frac{\varphi_{1n}^4}{16\pi^4 n^4}e_\alpha(t,\lambda)\right)$$

$$\left(8\pi n\sin 2\pi nx + 4\cos 2\pi nx\right) - \sum_{n=1}^{\infty}\left(\frac{\varphi_{1n}^4}{4\pi^3 n^3}e_{\alpha,1-\alpha}(t,\lambda)\right.$$

$$+ \left. f_{2n}\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\star e_\alpha(t,\lambda) + \frac{\varphi_{2n}^4}{16\pi^4 n^4}e_\alpha(t,\lambda)\right)8\pi n\sin 2\pi nx. \tag{53}$$

8

$$\sum_{n=1}^{\infty} \mathcal{U}_{n,xx}(x,t) \ := \ \sum_{n=1}^{\infty} \left(\frac{f_{1n}\alpha}{\Gamma(1+\alpha)} t^{\alpha-1} \star e_\alpha(t,\lambda) + \frac{\varphi_{1n}^4}{16\pi^4 n^4} e_\alpha(t,\lambda) \right)$$

$$\left(16\pi^2 n^2 \sin 2\pi nx - 16\pi^2 n^2 (1-x)\cos 2\pi nx \right) - \sum_{n=1}^{\infty} \left(\frac{\varphi_{1n}^4}{4\pi^3 n^3} e_{\alpha,1-\alpha}(t,\lambda) \right.$$

$$+ \ f_{2n} \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \star e_\alpha(t,\lambda) + \frac{\varphi_{2n}^4}{16\pi^4 n^4} e_\alpha(t,\lambda) \left. \right) 16\pi^2 n^2 \sin 2\pi nx. \tag{54}$$

The series (53) and (54) are bounded above by the uniformly convergent series

$$\sum_{n=1}^{\infty} \left(\frac{(1+2\pi n)|\psi_{1n}^4| + 2\pi n|\psi_{2n}^4| + (4\pi^2 n^2 + 6\pi n + 2)M|\varphi_{1n}^4| + (1+M)2\pi n|\varphi_{2n}^4|}{4\pi^4 n^4} \right),$$

$$\sum_{n=1}^{\infty} \left(\frac{|\psi_{1n}^4| + 2|\psi_{2n}^4| + (1+TM)M|\varphi_{1n}^4| + M|\varphi_{2n}^4|}{\pi^2 n^2} \right),$$

respectively, then $\sum_{n=1}^{\infty} \mathcal{U}_{n,x}(x,t)$ and $\sum_{n=1}^{\infty} \mathcal{U}_{n,xx}(x,t)$ converges uniformly; consequently

$$u_{xx}(t,x) = \sum_{n=1}^{\infty} \mathcal{U}_{n,xx}(x,t),$$

is uniformly convergent.

Hence the term by term differentiation of the series (49) with respect to time t and x is valid. Similar we can show that $f(x)$ obtained by series (17) is continuous.

Remark 3.2. *The above analysis remains valid if the boundary conditions given by (3) are replaced by*

$$u(0,t) = u(1,t), \quad u_x(0,t) \ = \ u_x(1,t), \quad t \in [0,T],$$
$$or$$
$$u(1,t) = 0, \quad u_x(0,t) \ = \ 0, \quad t \in [0,T].$$

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