

# On regular 2-path Hamiltonian graphs

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**Abstract:** Kronk introduced the  $l$ -path hamiltonianicity of graphs in 1969. A graph is  $l$ -path Hamiltonian if every path of length not exceeding  $l$  is contained in a Hamiltonian cycle. We have shown that if  $P = uvz$  is a 2-path of a 2-connected,  $k$ -regular graph on at most  $2k$  vertices and  $G - V(P)$  is connected, then there must exist a Hamiltonian cycle in  $G$  that contains the 2-path  $P$ . In this paper, we characterize a class of graphs that illustrate the sharpness of the bound  $2k$ . Additionally, we show that by excluding the class of graphs, both 2-connected,  $k$ -regular graphs on at most  $2k + 1$  vertices and 3-connected,  $k$ -regular graphs on at most  $3k - 6$  vertices satisfy that there is a Hamiltonian cycle containing the 2-path  $P$  if  $G \setminus V(P)$  is connected.

**Keywords:** Hamiltonian cycle;  $l$ -path Hamiltonian;  $k$ -regular graph

## 1 Introduction

A *Hamiltonian path (cycle)* in a graph  $G$  is a path (cycle) containing all the vertices of  $G$ , and a graph with a Hamiltonian cycle is called *Hamiltonian*. A graph is *Hamilton-connected* when every pair of distinct vertices is connected by a Hamiltonian path. The

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existence of Hamiltonian cycles in 2-connected,  $k$ -regular graphs has been the subject of research in several publications [4, 2, 6, 16, 3, 10]. The findings of these studies indicate that all 2-connected,  $k$ -regular graphs on at most  $3k + 4$  vertices, except for two kinds of graphs which is not 3-connected, are Hamiltonian. In 1976, Häggkvist proposed the following conjecture .

**Conjecture 1** ([5]). *For  $k \geq 4$ , every  $m$ -connected,  $k$ -regular graph on at most  $(m + 1)k$  vertices is Hamiltonian.*

The example constructed independently by Jackson and Jung (refer to [7]) provides evidence to refute Conjecture 1 for  $m \geq 4$ . However, there exists a remaining unresolved case when  $m = 3$ .

**Conjecture 2** (refer to [7]). *For  $k \geq 4$ , every 3-connected,  $k$ -regular graph on at most  $4k$  vertices is Hamiltonian.*

Considerable progress has been made in investigating the existence of Hamiltonian cycles within 3-connected,  $k$ -regular graphs. Conjecture 2 has been resolved for significantly large graphs according to [9], but for a very large (albeit finite) number of cases it remains open. In addition, the proof provided in [9] is extensive and intricate, making a simpler proof highly desirable.

Another noteworthy subarea within Hamiltonian graph theory focuses on Hamiltonian cycles that contain specified elements of a graph. Examples include  $k$ -ordered Hamiltonian graphs [14, 13], edge-Hamiltonian graphs [11], and others. One of these directions is the study of  $l$ -path Hamiltonicity. A graph  $G$  on  $n$  vertices is said to be  *$l$ -path Hamiltonian* if every path of length not exceeding  $l$ ,  $1 \leq l \leq n - 2$ , is contained in a Hamiltonian cycle. Kronk in [8] proved that for a graph  $G$  on  $n$  vertices, if the sum of the degrees of every pair of non-adjacent vertices of  $G$  is at least  $n + l$ , where  $l$  is a positive integer, then  $G$  is  $l$ -path Hamiltonian. The idea of combining  $l$ -path Hamiltonicity with regular graphs comes from the following result proved by Li in [11].

**Theorem 3** ([11]). *Let  $G$  be a 2-connected,  $k$ -regular graph on at most  $3k - 1$  vertices, and let  $e = uv$  be any edge of  $G$  such that  $\{u, v\}$  is not a cut-set. Then  $G$  has a Hamiltonian cycle containing  $e$ .*

Theorem 3 shows that a 2-connected,  $k$ -regular graph  $G$  on at most  $3k - 1$  vertices satisfying  $G - V(P)$  is connected for every path  $P$  of length 1 is 1-path Hamiltonian. Motivated by above result, Li and Yang in [12] proved the following result.

**Theorem 4** ([12]). *Let  $G$  be a 2-connected,  $k$ -regular graph on at most  $2k$  vertices, and let  $P = uvz$  be any path of  $G$  such that  $\{u, v, z\}$  is not a cut-set. Then  $G$  has a Hamiltonian cycle containing  $P$ .*

From Theorem 3 and Theorem 4, it can be deduced that if a 2-connected,  $k$ -regular graph  $G$  on at most  $2k$  vertices has the property that the graph  $G - V(P)$  is connected for every path  $P$  of length at most 2, then  $G$  is 2-path Hamiltonian.

For positive integer  $q \geq k$ , we define that a class  $\mathcal{H}$  of graphs of path  $P = uvz$  is a 2-connected,  $k$ -regular graph on  $2q + 1$  vertices, which contains two disjoint sets  $X$  and  $Y$  of vertices such that  $Y$  is independent,  $X$  contains  $\{u, v, z\}$ ,  $|Y| = q$ ,  $|X| = q + 1$ ,  $N(Y) \subseteq X$  and  $v$  is adjacent to  $u$  and  $z$ . Figure 1 below illustrates a 2-connected,  $k$ -regular graph on  $2k + 1$  vertices belonging to  $\mathcal{H}$  with  $q = k = 4$ .

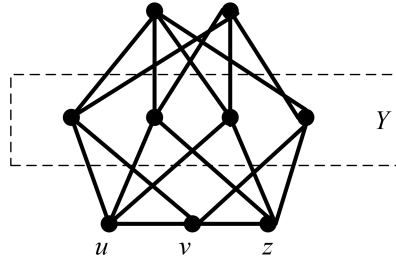


Figure 1

The existence of the graph class  $\mathcal{H}$  demonstrates that the upper bound  $2k$  in Theorem 4 is sharp. Clearly, when  $k$  is odd, the class of graphs  $\mathcal{H}$  does not exist. Therefore,

it is worthwhile to investigate whether the result in Theorem 4 can be strengthened by excluding the graph class  $\mathcal{H}$ . In this paper, we are going to prove the following.

**Theorem 5.** *Let  $G$  be a 2-connected,  $k$ -regular graph on  $n$  vertices, and let  $P = uvz$  be any path of  $G$  such that  $\{u, v, z\}$  is not a cut-set. If  $G \notin \mathcal{H}$  and  $n \leq 2k + 1$ , then  $G$  has a Hamiltonian cycle containing  $P$ .*

The following corollary follows from Theorem 3 and Theorem 5.

**Corollary 6.** *Let  $G$  be a 2-connected,  $k$ -regular graph on at most  $2k + 1$  vertices. If  $G - V(P)$  is connected for every path  $P$  of length at most 2 and  $G \notin \mathcal{H}$ , then  $G$  is 2-path Hamiltonian.*

We present an illustrative example that demonstrates the sharpness of the bound in Corollary 6. Let  $H_i$ ,  $i \in \{1, 2\}$ , be a graph which is obtained from  $K_{k+1}$  by deleting one edge  $e_i = a_i b_i$ . We can construct a 2-connected,  $k$ -regular graph  $G$  on  $2k + 2$  vertices from  $H_1$  and  $H_2$  by adding  $a_1 a_2$  and  $b_1 b_2$ . Notably, there is a 2-path in  $G$  that is not contained in any Hamiltonian cycle of  $G$ . Obviously, this example is not 3-connected graph. Is it possible to improve the result for 3-connected,  $k$ -regular graphs? In this paper, we also prove the following.

**Theorem 7.** *Let  $G$  be a 3-connected,  $k$ -regular graph on  $n$  vertices, and let  $P = uvz$  be any path of  $G$  such that  $\{u, v, z\}$  is not a cut-set. If  $G \notin \mathcal{H}$  and  $n \leq 3k - 6$ , then  $G$  has a Hamiltonian cycle containing  $P$ .*

The following corollary follows from Theorem 3 and Theorem 7.

**Corollary 8.** *Let  $G$  be a 3-connected,  $k$ -regular graph on at most  $3k - 6$  vertices. If  $G - V(P)$  is connected for every path  $P$  of length at most 2 and  $G \notin \mathcal{H}$ , then  $G$  is 2-path Hamiltonian.*

The  $L$ -graph (as discussed in [11]) as a counterexample of Theorem 3 with  $3k$  vertices is 3-connected for  $k \geq 6$ . Consequently, the bound provided in Corollary 8 is nearly optimal.

## 2 Notation and preliminaries

All graphs mentioned in this paper are finite simple graphs. For standard graph theory notation and terminology not explained in this paper, we refer the reader to [1]. Let  $G$  be a graph.  $|G|$  and  $\delta$  denote the number of vertices and the minimum degree of  $G$ , respectively. For  $x \in V(G)$  and  $S \subseteq V(G)$ ,  $N_G(x)$  denotes the neighbors of  $x$  in  $G$ ,  $N_G(S) = \bigcup_{x \in S} N_G(x)$  and  $d_G(x) = |N_G(x)|$ . For a cycle  $C$  in  $G$  with a fixed orientation, and any two vertices  $x, y$  on  $C$ , we denote by  $x^+$  and  $x^-$  the following vertex and the preceding vertex of  $x$  according to the orientation of  $C$ , respectively. We define the segment  $C[x, y]$  to be the set of vertices on  $C$  from  $x$  to  $y$  (including  $x$  and  $y$ ) according to the orientation and let  $C(x, y) = C[x, y] - \{x, y\}$ . Analogously,  $C[x, y)$  and  $C(x, y]$  are also defined. Let  $A$  be a set of vertices of  $G$ . An  $A$ -segment is a  $C[x, y]$  segment such that  $C[x, y] \cap A = \{x, y\}$ . We put  $x^{+2} = (x^+)^+$  ( $x^{-2} = (x^-)^-$ ) and  $x^{+i} = (x^{+(i-1)})^+$  ( $x^{-i} = (x^{-(i-1)})^-$ ).

For a cycle  $C$  and any  $A, B \subseteq V(G)$ , let  $E(A, B) = \{uv \in E(G) : u \in A, v \in B\}$  ( $E'(A, B) = \{uv \in E(G) - E(C) : u \in A, v \in B\}$ ) and  $E(A) = \{uv \in E(G) : u, v \in A\}$  ( $E'(A) = \{uv \in E(G) - E(C) : u, v \in A\}$ ). Put  $e(A, B) = |E(A, B)|$ ,  $e'(A, B) = |E'(A, B)|$ ,  $e(A) = |E(A)|$  and  $e'(A) = |E'(A)|$ . For convenience, we often use a subgraph  $H$  of a graph  $G$  to denote its vertex set  $V(H)$  if no confusion arises. We shall use the following result.

**Theorem 9** ([7]). *Let  $G$  be a connected graph such that for every longest path  $P$  in  $G$ , the sum of the degrees of the end-vertices of  $P$  is at least  $|P| + 1$ . Then  $G$  is Hamilton-connected.*

We first introduce an *operation* used during the proofs of Theorems 5 and Theorems 7. Let  $G$  be a connected graph on  $n$  vertices, and let  $P = uvz$  be a path of  $G$ . We define a new graph  $G_1$  by inserting two vertices  $w_1$  and  $w_2$  on the edges  $e_1 = uv$  and  $e_2 = vz$  of  $P$  respectively. Then we have  $G_1 = (G - \{e_1, e_2\}) \cup \{w_1, w_2\} \cup \{uw_1, w_1v, vw_2, w_2z\}$ ,  $P_1 = uw_1vw_2z$  and  $|G_1| = n_1 = n + 2$ . Clearly, if we want to prove that  $P$  is contained in

a Hamiltonian cycle in  $G$ , it is sufficient to prove that  $G_1$  is Hamiltonian.

### 3 Proof of Theorem 5

From Theorem 4, we have that Theorem 5 holds for  $n \leq 2k$ . So we only need to consider the case  $n = 2k + 1$ . Let  $G$  be a 2-connected,  $k$ -regular graph on  $n = 2k + 1$  vertices, and let  $P = uvz$  be a path of  $G$  such that  $\{u, v, z\}$  is not a cut-set. After the operation in section 2, we have  $|G_1| = n_1 = 2k + 3$ . Suppose  $G_1$  is not Hamiltonian. Let  $C$  be a longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , such that the number of components of  $R = G_1 - C$  is as small as possible. Let  $r = |R|$  and  $C = c_1c_2 \cdots c_{n_1-r}$ . The subscripts of  $c_i$  will be reduced modulo  $n_1 - r$  throughout. Obviously, we have  $|C| = n_1 - r \geq 6$ .

Suppose  $R$  is an independent set, then let  $v_0$  be an isolated vertex in  $R$ . Put  $Y_0 = \emptyset$ , and for any  $j \geq 1$ ,  $X_j = N(Y_{j-1} \cup \{v_0\})$ ,  $Y_j = \{c_i \in V(C) : c_{i-1}, c_{i+1} \in X_j\}$ ,  $X = \bigcup_{j=1}^{\infty} X_j$ ,  $Y = \bigcup_{j=0}^{\infty} Y_j$ ,  $x = |X| \geq k$  and  $y = |Y|$ . By the hopping lemma ([15]), we have  $X \subset V(C)$ ,  $X \cap Y = \emptyset$  and  $X$  does not contain two consecutive vertices of  $C$ . Let  $S_1, S_2, \dots, S_x$  be the sets of vertices contained in the open  $X$ -segment of  $C$  (the sets of vertices on  $C$  between  $X$  satisfying  $S_i \cap X = \emptyset$  for each  $i$  with  $1 \leq i \leq x$ ). Put  $\phi = \{S_i : |S_i| \geq 2, 1 \leq i \leq x\}$ . Then  $S_i = \{c_l, c_{l+1}, \dots, c_m\} \in \phi$  is said to be  $\psi$ -connected to  $S_j = \{c_q, c_{q+1}, \dots, c_z\} \in \phi$  if  $|S_i|$  is odd and  $c_q$  and  $c_z$  are both joined to  $c_{l+e}$  for all odd  $e$ ,  $1 \leq e \leq m - l - 1$ . Now,  $c_{l+1}, c_{l+3}, \dots, c_{m-1}$  are called  $P$ -vertices of  $S_i$ . Set  $P = \{c_i \in V(C) : c_i \text{ is a } P\text{-vertex of some } S_j \text{ which is } \psi\text{-connected to some } S_t \text{ of } \phi\}$ , and  $p = |V(P)|$ . By the same proof as the case 1 in [12], we have the following inequality which is the inequality (4) in [12],

$$p + 4 \leq (n_1 - 2x - k)(n_1 - 1 - 2x - p) + k - 2(r_1 - 1)(x - y - 1). \quad (1)$$

From the definition of  $P$ , we have  $p \leq \frac{n_1 - 1 - 2x}{2}$ , which implies  $n_1 - 1 - 2x - p \geq 2p - p \geq 0$ . And  $k \geq n_1 - 1 - 2x - p$  by  $n_1 = 2k + 3$  and  $x \geq k$ . So we have

$$p + 4 \leq (n_1 - 2x - k + 1)k - 2(r_1 - 1)(x - y - 1). \quad (2)$$

By the definitions of  $X$  and  $Y$ , we have  $x \geq y$ . Now, we claim  $x \geq y + 1$ . Otherwise, when  $x = y$ , since  $d_{G_1}(w_1) = d_{G_1}(w_2) = 2$ , we have  $u, v, z$  belong to  $X$  and  $w_1, w_2$  belong to  $Y$ . If  $R$  contains at least two isolated vertices, then  $|Y \cup R| \geq y + 2 = x + 2$ . Because  $Y \cup R$  is an independent set and  $N(Y \cup R) \subseteq X$ , we have that there are at least  $ky + 4 = kx + 4$  edges from  $Y \cup R$  to  $X$ , but  $X$  accepts at most  $kx$  edges, a contradiction. So, when  $x = y$ ,  $R$  contains only one isolated vertex, which implies  $n_1 = 2x + 1$  and  $n = 2x - 1 = 2q + 1$  for  $q = x - 1$ . By definition of  $\mathcal{H}$ , we have  $G \in \mathcal{H}$ , a contradiction. Therefore by (2), we have  $n_1 - 2x - k + 1 > 0$ , and then  $n_1 > 3k - 1 \geq 3k$ , which contradicts  $n_1 = 2k + 3$ .

Thus in the following proof, we assume that there exists a component  $H$  in  $R$  such that  $|H| \geq 2$ . For a path  $Q = q_1q_2 \cdots q_g, g \geq 2$ , in  $H$ , let  $t(Q)$  denote the number of  $C[c_i, c_j]$  such that  $c_i$  is joined to one of  $q_1$  and  $q_g$ ,  $c_j$  is joined to the other, and  $e(\{q_1, q_g\}, \{c_{i+1}, c_{i+2}, \dots, c_{j-1}\}) = 0$ . We say that  $Q$  satisfies the condition (\*) if  $t(Q) \geq 2$ ,  $N_C(\{q_1, q_g\}) \not\subseteq \{u, v, z\}$  and there is a  $C[c_i, c_j]$  such that  $u, v, z, w_1$  and  $w_2 \notin \{c_{i+1}, c_{i+2}, \dots, c_{j-1}\}$ . Now, let  $H$  be the largest component of  $R$  and  $h = |H|$ . The rest of the proof of Theorem 5 is divided into two cases. We first consider the case of  $k \geq 6$ , and we prove it in the following two cases.

**Case 1.**  $2 \leq h \leq k$ .

**Claim 1.** For  $k \geq 6$ , if  $2 \leq h \leq k$ , then  $H$  is Hamilton-connected.

*Proof.* By contradiction, suppose  $H$  is not Hamilton-connected. From Theorem 9, we can choose a longest path  $Q = q_1q_2 \cdots q_g, g \geq 2$ , in  $H$ , satisfying  $d_H(q_1) + d_H(q_g) \leq |Q| = g$ . For any  $v \in V(H)$ , since  $h \leq k$ , we have  $N_C(v) \geq 1$ . Let  $X = N_C(q_1) \cap N_C(q_g)$  and  $|X| = x$ .

First, we prove that  $N_C(q_1) = N_C(q_g)$ . Otherwise, without loss of generality assume that  $d_C(q_1) \leq d_C(q_g)$ . If  $d_C(q_1) = 1$ , we have  $d_C(q_g) \geq 2k - g - 1$  since  $d_C(q_1) + d_C(q_g) \geq 2k - g$ . Since  $C$  is the longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , we have that every  $N_C(q_g)$ -segment of  $C$  contains at least one interior vertex. Hence  $n_1 \geq |H| + |C| \geq g + 2d_C(q_g) \geq g + 2(2k - g - 1) = 4k - g - 2 \geq 3k - 2$ , a contradiction. Thus, we

have  $2 \leq d_C(q_1) \leq d_C(q_g)$ . It is easy to prove that  $t(Q) \geq 2$  and  $x + 1 \leq t(Q)$ . Since  $N_C(\{q_1, q_g\}) \cup (N_C(\{q_1, q_g\}))^+ \cup H \subseteq V(G_1)$ , we have

$$\begin{aligned}
n_1 &\geq |C| + |H| \geq |H| + 2|N_C(\{q_1, q_g\})| + (t(Q) - 2)(g - 1) \\
&\geq g + 2[d_C(q_1) + d_C(q_g)] - 2|N_C(q_1) \cap N_C(q_g)| + (t(Q) - 2)(g - 1) \\
&\geq g + 2(2k - g) - 2x + (t(Q) - 2)(g - 1) \\
&\geq 4k - g - 2(x + 1) + 2 + (t(Q) - 2)(g - 1) \\
&\geq 4k - g - 2t(Q) + 2 + (t(Q) - 2)(g - 1) \\
&\geq 3k - 2 + (t(Q) - 2)(g - 3).
\end{aligned}$$

Since  $H$  is not Hamilton-connected, we have  $g \geq 3$ . This implies  $(t(Q) - 2)(g - 3) \geq 0$ . Therefore, we have  $n_1 \geq 3k - 2$ , a contradiction.

From the above discussion, we have  $N_C(q_1) = N_C(q_g) = X$  and  $t(Q) = x = d_C(q_1) = d_C(q_g) \geq k - \frac{g}{2} \geq \frac{k}{2} \geq 3$ . Then

$$\begin{aligned}
n_1 &\geq |C| + |H| \geq (t(Q) - 2)g + 2 + t(Q) + h \geq (x - 2)g + 2 + x + g \\
&\geq (x - 1)(g + 1) + 3 \geq (k - \frac{g}{2} - 1)(g + 1) + 3.
\end{aligned}$$

Since  $f(g) = (k - \frac{g}{2} - 1)(g + 1) + 3$  is a concave function of  $g$ ,  $3 \leq g \leq k$ , we have  $f(3) = 4k - 7 > 2k + 3$  and  $f(k) = \frac{k^2}{2} - \frac{k}{2} + 2 > 2k + 3$  when  $k \geq 6$ . Hence  $f(g) > 2k + 3$ , a contradiction.

□

**Subcase 1.1.**  $h = k$ .

If there is a  $N_C(H)$ -segment  $C[c_i, c_j]$  such that  $u, v, z, w_1$  and  $w_2 \notin \{c_{i+1}, c_{i+2}, \dots, c_{j-1}\}$ , we have  $|C(c_i, c_j)| \geq h$  and  $n_1 \geq |H| + |C| \geq |H| + |C(c_i, c_j)| + |\{v, w_1, w_2, c_i, c_j\}| \geq h + h + 5 = 2k + 5$ , a contradiction. Thus,  $|N_C(H)| = 2$  and  $v \in N_C(H)$ . Let  $X = N_C(H)$ . Since  $h = k$ , for any  $v_i \in H$ ,  $i \in \{1, \dots, h\}$ , we have  $|N_C(v_i)| \geq 1$ . So  $e(H, X) \geq k$ . Because  $|G_1 - H - X| = n_1 - k - 2 = 2k + 3 - k - 2 = k + 1$ , we have

$$e(G_1 - H - X) = e(G_1 - H - X - \{w_1, w_2\}) + e(G_1 - H - X, \{w_1, w_2\}) \leq \frac{(k - 2)(k - 1)}{2} + 2.$$



Since  $G$  is a  $k$ -regular graph, we have

$$e(G_1 - H - X, X) = k|G_1 - H - X - \{w_1, w_2\}| + 4 - 2e(G_1 - H - X) - e(G_1 - H - X, H).$$

Since  $e(G_1 - H - X, H) = 0$ , we have

$$e(G_1 - H - X, X) \geq k(k-1) + 4 - 2\left(\frac{(k-2)(k-1)}{2} + 2\right) = 2k - 2 \geq k + 1.$$

On the other hand  $e(G_1 - H - X, X) \leq e(G_1, X) - e(H, X) \leq 2k - k = k$ , a contradiction.

**Subcase 1.2.**  $\frac{k+1}{2} \leq h \leq k-1$ .

Since  $H$  is Hamilton-connected, it is easy to deduce that there exists a Hamiltonian path  $Q$  in  $H$  such that  $Q$  satisfies (\*). By a similar proof to Lemma 6 in [12], we have the following claim.

**Claim 2.** *There exists a Hamiltonian path  $Q$  in  $H$  such that  $t(Q) \geq 3$ .*

Since  $t(Q) \geq 3$ , there is a  $C[c_i, c_j]$  such that  $u, v, z, w_1$  and  $w_2 \notin \{c_{i+1}, c_{i+2}, \dots, c_{j-1}\}$ . Then there exists either  $c_i^- \notin \{w_1, w_2\}$  or  $c_j^+ \notin \{w_1, w_2\}$ . Without loss of generality, let  $c_i^- \notin \{w_1, w_2\}$ . Since  $C$  is a longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , we have  $N_{G_1}(c_i^-) \cap [H \cup c_{j-1}, c_{j-2}, \dots, c_{j-h} \cup \{c_i^-\}] = \emptyset$ . And there are at least two of  $\{v, w_1, w_2\}$  which can not be adjacent to  $c_i^-$ . This implies  $d_{G_1}(c_i^-) \leq 2k + 3 - (h + h + 3) \leq k - 1$ , a contradiction to  $d_{G_1}(c_i^-) = k$ .

**Subcase 1.3.**  $2 \leq h \leq \frac{k}{2}$ .

For any  $v \in V(H)$ , since  $2 \leq h \leq \frac{k}{2}$ , we have  $N_C(v) \geq k - h + 1 \geq k - \frac{k}{2} + 1 \geq 4$ . So  $N_C(H) \geq k - h + 1$ . Let  $a \in V(H)$  and  $N_C(a) = A$ .

**Claim 3.** *For any  $A$ -segment  $C[c_i, c_j]$  satisfying  $C[c_i, c_j] \cap \{w_1, w_2\} = \emptyset$ , we have  $C[c_i, c_j] \cap N_C((V(H) - a)) \neq \emptyset$ .*

*Proof.* Let  $N_C((V(H) - a)) = S$ . Suppose that there is an  $A$ -segment  $C[c_i, c_j]$  satisfying  $C[c_i, c_j] \cap [\{w_1, w_2\} \cup S] = \emptyset$ . Let  $S = \{c_{r_1}, c_{r_2}, \dots, c_{r_s}\}$ , where  $c_{r_1}$  and  $c_{r_s}$  are the closest vertices to  $c_j$  and  $c_i$  in  $S$ , respectively. Clearly,  $|S| = s \geq k - h + 1 \geq 4$ .

Therefore, there is at least one segment of  $C[c_j, c_{r_1}]$  and  $C[c_{r_s}, c_i]$  which does not contain  $w_1$  and  $w_2$ . Without loss of generality, let  $w_1, w_2 \notin C[c_{r_s}, c_i]$ . Obviously, we have  $N_C(c_i^+) \cap [H \cup \{c_{r_s+1}, c_{r_s+2}, \dots, c_{r_s+h}, w_1, w_2, c_i^+\}] = \emptyset$ . And for any  $C[c_{r_i}, c_{r_{i+1}}]$  satisfying  $C[c_{r_i}, c_{r_{i+1}}] \cap \{w_1, w_2\} = \emptyset$ ,  $i \in \{1, 2, \dots, s-1\}$ , we have  $N_C(c_i^+) \cap \{c_{r_{i+1}}, c_{r_{i+2}}\} = \emptyset$ . This implies

$$\begin{aligned} d_{G_1}(c_i^+) &\leq 2k + 3 - [h + 2(s - 2 - 1) + h + 3] \\ &\leq 2k + 3 - [h + 2(k - h + 1 - 2 - 1) + h + 3] \leq 4, \end{aligned}$$

a contradiction.  $\square$

By Claim 3, we have

$$|C| \geq |A| + (|A| - 2)h + 2 \geq (k - h + 1) + (k - h + 1 - 2)h + 2 \quad (3)$$

and  $n_1 \geq |C| + |H| \geq (k - h + 1) + (k - h + 1 - 2)h + 2 + h = k + 3 + (k - h - 1)h$ .

Put  $g(h) = k + 3 + (k - h - 1)h$ . For  $k \geq 7$ , since  $g(h)$  is a concave function of  $h$  with  $g(2) = 3k - 3 > 2k + 3$  and  $g(\frac{k}{2}) = \frac{k^2}{4} + \frac{k}{2} + 3 > 2k + 3$ . Hence  $g(h) > 2k + 3$ , a contradiction. When  $k = 6$  and  $n_1 = 15$ , there are two cases where  $h = 2$  and  $h = 3$  to consider as follows.

Case (a):  $h = 2$ . By (3), we have  $13 = n_1 - h \geq |C| \geq (k - h + 1) + (k - h + 1 - 2)h + 2 = 13$ . So  $R$  has only one component  $H$  and  $|C| = |A| + (|A| - 2)h + 2 = 13$ . Since  $C$  is a longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , we have  $N_C(v_i) = N_C(v_j), i \neq j$ , for any  $v_i \in H, i \in \{1, 2\}$ . Let  $N_C(H) = X$  and  $Z = X^+ \cup X^-$ . Clearly, we have  $u, v, z \in X$  and  $e'(Z) = 0$ , otherwise, there exists a longer cycle containing  $w_1$  and  $w_2$ . Since  $e(Z, H) = 0$ ,  $|X| = 5$  and  $|Z| = 8$ , we have  $e(Z, X) = (6 - 1) \times 6 + 4 = 34$ . However,  $e(Z, X) \leq k|X| - e(H, X) \leq 20$ , a contradiction.

Case (b):  $h = 3$ . It is similar to case (a) above.

The following figure 2 shows the edges between  $H$  and  $C$  in the above two cases.

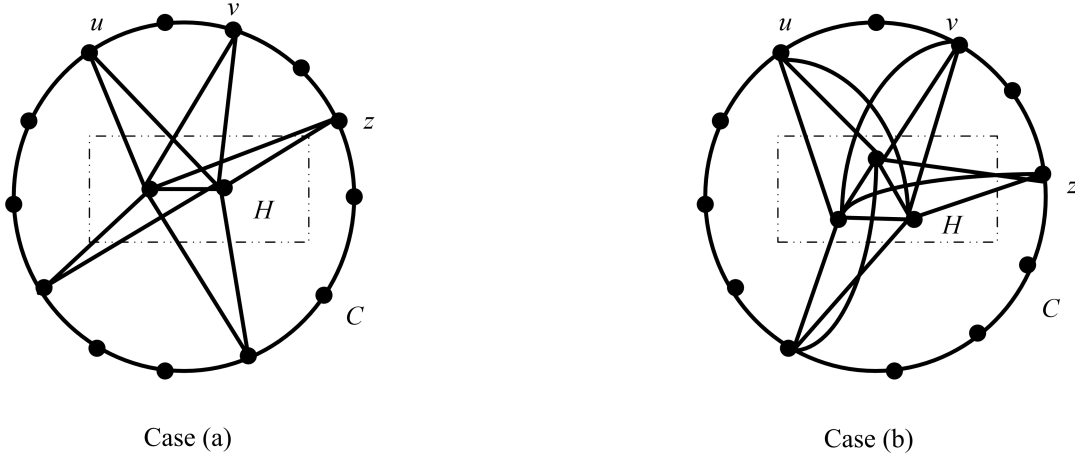


Figure 2

**Case 2.**  $h \geq k + 1$ .

By the assumption of connectivity and as  $\{u, v, z\}$  is not a cut-set, there exists  $x'' \in N_C(H)$ , such that  $x'' \notin \{w_1, w_2\}$ . It is clear that  $N_{G_1}(x'') \cap H = \emptyset$ , and at least two of  $\{v, w_1, w_2\}$  cannot be adjacent to  $x''$ . It follows that  $d_{G_1}(x'') \leq 2k + 3 - (k + 1) - 2 - 1 = k - 1$ , a contradiction.

These contradictions complete our proof for  $k \geq 6$ . We next discuss the case of  $3 \leq k \leq 5$ . Since  $2k + 1$  is odd, we only need to discuss the cases of  $k = 4$ . When  $k = 4$ , we have  $n_1 = 11$ . Since  $|C| \geq 6$ , we consider the following four cases. It is worth noting that we have already proved the case when  $R$  is an independent set in the preceding paragraph. Now, we first prove a simple claim.

**Claim 4.** *When  $k = 4$ , if  $R$  contains an edge  $v_0v_1$  and  $|C| \leq 9$ , then  $N_C(v_0) = N_C(v_1)$ .*

*Proof.* By contradiction. Let  $Q = v_0v_1$ . Suppose  $N_C(v_0) \neq N_C(v_1)$ , and without loss of generality,  $|N_C(v_0) - N_C(v_1)| = 1$ . Since  $G$  is a 4-regular graph, we have  $|N_C(v_0) \cup N_C(v_1)| = 4$  and  $t(Q) = 4$ . Since  $C$  is a longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , we have  $|C| \geq |N_C(v_0) \cup N_C(v_1)| + 2(t(Q) - 2) + 2 \geq 10$ , a contradiction.  $\square$

Case (a):  $|V(C)| = 9$ . Let  $C = uw_1vw_2zx_1x_2x_3x_4$ . Then  $R$  is an edge  $v_0v_1$ . By Claim 4

, we have  $N_C(v_0) = N_C(v_1)$ . By assumption and symmetry, we have  $N_C(R) = \{v, z, x_3\}$ ,  $N_C(R) = \{v, z, x_4\}$ , or  $N_C(R) = \{x_1, x_4, v\}$ . If  $N_C(R) = \{x_1, x_4, v\}$ , we have  $x_2u \notin E(G_1)$ , otherwise, there is a Hamiltonian cycle  $C' = v_0x_1zw_2vw_1ux_2x_3x_4v_1$ . So we have  $d_{G_1}(x_2) \leq 3$ , a contradiction. The proofs for  $N_C(R) = \{v, z, x_3\}$  and  $N_C(R) = \{v, z, x_4\}$  are similar.

Case (b):  $|V(C)| = 8$ . Let  $C = uw_1vw_2zx_1x_2x_3$ .

Subcase (b1):  $R$  consists of an edge  $v_0v_1$  and an isolated vertex  $v_2$ . By Claim 4, we have  $N_C(v_0) = N_C(v_1)$ . By assumption and symmetry, we have  $N_C(v_0) = N_C(v_1) = \{v, z, x_3\}$ . Since  $v \in N_C(v_2)$ , we have  $d_{G_1}(v) \geq 5$ , a contradiction.

Subcase (b2):  $R$  contains only one connected component  $H$  which has three vertices. Since  $G$  is a 4-regular graph, we have  $d_C(H) \geq 6$  and  $|N_C(H)| \geq 3$ . By assumption and symmetry, we have  $\{v, z, x_2\} \subseteq N_C(H)$ ,  $\{v, z, x_3\} \subseteq N_C(H)$ ,  $\{u, z, x_2\} \subseteq N_C(H)$  or  $\{v, x_1, x_3\} \subseteq N_C(H)$ . If  $\{v, z, x_2\} \subseteq N_C(H)$ , we have  $x_1, x_3 \notin N_C(H)$ . We claim  $x_1x_3 \notin E(G_1)$ , otherwise, there is a longer cycle  $C' = Hzw_2vw_1ux_3x_1x_2$  containing  $w_1$  and  $w_2$ . Since  $G$  is a 4-regular graph, we have  $v \in N_{G_1}(x_1)$  and  $v \in N_{G_1}(x_3)$ ,  $d_{G_1}(v) \geq 5$ , a contradiction. The proofs for the other three cases are similar.

Case (c):  $|V(C)| = 7$ . Let  $C = uw_1vw_2zx_1x_2$ . Clearly, there cannot have isolated vertex in  $R$ .

Subcase (c1):  $R$  consists of two edges  $v_0v_1$  and  $v_2v_3$  that are not in the same component. By Claim 4, we have  $v \in N_C(v_i), i = 0, 1, 2, 3$ , implying  $d_{G_1}(v) \geq 6$ , a contradiction.

Subcase (c2):  $R$  contains only one connected component  $H$  which has four vertices. Since  $G$  is a 4-regular graph, we have  $d_C(H) \geq 4$  and  $|N_C(H)| \geq 2$ . By assumption and symmetry, we have  $\{v, x_1\} \subseteq N_C(H)$  or  $\{z, x_2\} \subseteq N_C(H)$ . If  $\{v, x_1\} \subseteq N_C(H)$ , we have  $z, x_2 \notin N_C(H)$ . Since  $G$  is a 4-regular graph, we have  $vx_2, zx_2 \in E(G_1)$ . So  $v$  can accept at most 1 edge from  $H$ . Then  $N_C(H) \geq 3$  and  $u \in N_C(H)$ . There is a longer cycle  $C' = Hw_1vw_2zx_2x_1$  containing  $w_1$  and  $w_2$ , a contradiction. The proof for  $\{z, x_2\} \subseteq N_C(H)$  is similar.

Case (d):  $|V(C)| = 6$ . Let  $C = uw_1vw_2zx_1$ . Clearly, there is no component  $H$  in  $R$  such that  $|H| = 1$  or  $|H| = 2$ . So  $R$  contains only one connected component  $H$  which has five vertices. Obviously, there exist two consecutive vertices of  $\{u, z, x_1\}$  which are adjacent to  $H$ , a contradiction.

Thus, we complete the proof of Theorem 5.

## 4 Proof of Theorem 7

Let  $G$  be a 3-connected,  $k$ -regular graph on  $n \leq 3k - 6$  vertices, and let  $P = uvz$  be a path of  $G$  such that  $\{u, v, z\}$  is not a cut-set. After the operation in section 2, we have  $|G_1| = n_1 \leq 3k - 4$ . From Theorem 4, we have  $3k - 6 \geq 2k + 1$ , so,  $k \geq 7$ . Suppose  $G_1$  is not Hamiltonian. Let  $C$  be a longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , such that the number of components of  $R = G_1 - C$  is as small as possible. Let  $r = |R|$  and  $C = c_1c_2 \cdots c_{n_1-r}$ . The subscripts of  $c_i$  will be reduced modulo  $n_1 - r$  throughout. Obviously, we have  $|C| = n_1 - r \geq 7$ .

When  $R$  is an independent set, by the same proof as in section 3, we get  $n_1 \geq 3k$ , which contradicts  $n_1 \leq 3k - 4$ . Thus in the following proof, we assume that there exists a component  $H$  in  $R$  such that  $|H| \geq 2$ . For a path  $Q = q_1q_2 \cdots q_g, g \geq 2$ , in  $H$ , let  $t(Q)$  denote the number of  $C[c_i, c_j]$  such that  $c_i$  is joined to one of  $q_1$  and  $q_g$ ,  $c_j$  is joined to the other, and  $e(\{q_1, q_g\}, \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}) = 0$ . We say that  $Q$  satisfies the condition (\*) if  $t(Q) \geq 2$ ,  $N_C(\{q_1, q_g\}) \not\subseteq \{u, v, z\}$  and there is a  $C[c_i, c_j]$  such that  $u, v, z, w_1$  and  $w_2 \notin \{c_{i+1}, c_{i+2}, \cdots, c_{j-1}\}$ . Now, let  $H$  be the largest component of  $R$  and  $h = |H|$ . Consider the following three cases.

**Case 1.**  $2 \leq h \leq k$ .

By the same proof as in claim 1, if  $H$  is not Hamilton-connected, we have  $n_1 \geq (k - \frac{g}{2} - 1)(g + 1) + 3$ . Since  $f(g) = (k - \frac{g}{2} - 1)(g + 1) + 3$  is a concave function of  $g$ ,  $3 \leq g \leq k$ , we have  $f(3) = 4k - 7 > 3k - 4$  and  $f(k) = \frac{k^2}{2} - \frac{k}{2} + 2 > 3k - 4$  when  $k \geq 7$ .

Hence  $f(g) > 3k - 4$ , a contradiction. Then  $H$  is Hamilton-connected.

Let  $M = \{r_i x_i \in E(G_1) : i \in \{1, 2, \dots, m\}, r_i \in V(H), x_i \in V(C)\}$  be a maximum matching in  $E_{G_1}(H, C)$  and  $m = |M|$ , then  $m \geq 3$  since  $G$  is 3-connected. In order to prove case 1, we next consider the following two subcases.

**Subcase 1.1.**  $k - 2 \leq h \leq k$ .

We claim  $m = 3$ . Otherwise, suppose  $m \geq 4$ , we must have  $n_1 \geq |H| + |C| \geq h + m + (m - 2)h + 2 \geq h + 2 + 2h + 4 = 3h + 6 \geq 3(k - 2) + 6 = 3k$ , a contradiction.

Let  $x_1, x_2, x_3$  be in this order on  $C$ . Since  $\{u, v, z\}$  is not a cut-set, without loss of generality, let  $x_3 \notin \{u, w_1, v, w_2, z\}$  and  $x_3^- \notin \{w_1, w_2\}, x_3^+ \notin \{w_1, w_2\}$ . Then we have either  $C[x_3, x_1]$  or  $C[x_2, x_3]$  which does not contain  $w_1$  and  $w_2$ . When  $w_1, w_2 \notin C[x_3, x_1]$ , since  $C$  is a longest cycle of  $G_1$  containing  $w_1$  and  $w_2$ , we have

$$N_C(x_3^-) \cap [H \cup \{x_1^-, x_1^{-2} \cdots, x_1^{-h}\} \cup x_3^-] = \emptyset.$$

And there are at least two of  $\{v, w_1, w_2\}$  which can not be adjacent to  $x_3^-$ . It follows that

$$d_{G_1}(x_3^-) \leq 3k - 4 - 2h - 3 \leq 3k - 4 - 2(k - 2) - 3 = k - 3,$$

this contradicts  $d_{G_1}(x_3^-) = k$ . When  $w_1, w_2 \notin C[x_2, x_3]$ , we have

$$N_C(x_3^+) \cap [H \cup \{x_2^+, x_2^{+2} \cdots, x_2^{+h}\} \cup x_3^+] = \emptyset.$$

And there are at least two of  $\{v, w_1, w_2\}$  which can not be adjacent to  $x_3^+$ . It follows that

$$d_{G_1}(x_3^+) \leq 3k - 4 - 2h - 3 \leq 3k - 4 - 2(k - 2) - 3 = k - 3,$$

this contradicts  $d_{G_1}(x_3^+) = k$ .

**Subcase 1.2.**  $2 \leq h \leq k - 3$ .

For any  $v \in V(H)$ , since  $2 \leq h \leq k - 3$ , we have  $N_C(v) \geq k - h + 1 \geq 4$ . By a similar proof as in Case 1.3, we have  $n_1 \geq |C| + |H| \geq k + 3 + (k - h - 1)h$ . Put  $g(h) =$

$k + 3 + (k - h - 1)h$ . Since  $g(h)$  is a concave function of  $h$  with  $g(2) = 3k - 3 = g(k - 3)$ . Hence  $g(h) > 3k - 4$ , a contradiction.

**Case 2.**  $k + 1 \leq h \leq 2k - 7$ .

Let  $Q = q_1q_2 \cdots q_g$  be a path in  $H$ , which is chosen as long as possible such that  $Q$  satisfies the condition (\*) (note that we are assuming 3-connectedness and that  $\{u, v, z\}$  is not a cut-set). Put  $A = N_C(q_1)$  and  $B = N_C(q_g)$ .

**Claim 5.**  $2 \leq g \leq k - 8$ .

*Proof.* Suppose that  $g \geq k - 7$ . By the definition of the condition (\*), there is a  $C[c_i, c_j]$  such that  $u, v, z, w_1$  and  $w_2 \notin \{c_{i+1}, c_{i+2}, \dots, c_{j-1}\}$ . By the assumption of 3-connectedness and as  $\{u, v, z\}$  is not a cut-set, we have either  $c_i^- \notin \{w_1, w_2\}$  or  $c_j^+ \notin \{w_1, w_2\}$ . Without loss of generality, let  $c_i^- \notin \{w_1, w_2\}$ . We have  $N_C(c_i^-) \cap \{H \cup \{c_i^-, c_{j-1}, c_{j-2}, \dots, c_{j-g}\}\} = \emptyset$ , and at least two of  $\{v, w_1, w_2\}$  cannot be adjacent to  $c_i^-$ . Thus

$$d_{G_1}(c_i^-) \leq 3k - 4 - [h + g + 2 + 1] \leq 3k - 4 - (k + 1) - (k - 7) - 2 - 1 \leq k - 1.$$

This contradicts that  $d_{G_1}(c_i^-) = k$ . □

**Claim 6.**  $Q$  is a maximal path in  $H$  satisfying (\*).

*Proof.* Suppose that  $Q$  is not a maximal path in  $H$ .

Let  $Q' = b_1b_2 \cdots b_sq_1q_2 \cdots q_gq_{g+1} \cdots q_e$  be a maximal path in  $H$  containing  $Q$ . Without loss of generality, we assume  $s \geq 1$ . From the definition of  $Q$ , we have  $N_C(b_1) \leq 3$ , which implies  $N_H(b_1) \geq k - 3$ .

When  $N_H(b_1) \cap \{q_2, q_3, \dots, q_e\} \neq \emptyset$ , one of the following three cases occurs.

- (1).  $N_H(b_1) \cap \{q_{g+1}, q_{g+2}, \dots, q_e\} = \emptyset$ .
- (2).  $N_H(b_1) \cap \{q_2, q_3, \dots, q_{g-1}\} = \emptyset$ .
- (3).  $N_H(b_1) \cap \{q_{g+1}, q_{g+2}, \dots, q_e\} \neq \emptyset$  and  $N_H(b_1) \cap \{q_2, q_3, \dots, q_{g-1}\} \neq \emptyset$ .

In (1), set  $i = \min\{j \geq 2 : b_1q_j \in E(G_1)\}$ . Let  $Q'' = q_1b_sb_{s-1} \cdots b_1q_iq_{i+1} \cdots q_g$ . In (2), set  $j = \max\{d \geq g : b_1q_d \in E(G_1)\}$ . Let  $Q'' = q_1b_sb_{s-1} \cdots b_1q_jq_{j-1} \cdots q_g$ . In both (1) and (2), since  $N_H(b_1) \cup \{b_1\} \subseteq Q''$ , then  $g \geq |V(Q'')| \geq k - 2$ , a contradiction to Claim 5.

In (3), set

$$l_1 = \min\{j : 2 \leq j \leq g - 1, b_1q_j \in E(G_1)\},$$

$$l_2 = \max\{j : 2 \leq j \leq g - 1, b_1q_j \in E(G_1)\},$$

and

$$l_3 = \max\{j : b_1q_j \in E(G_1)\}.$$

Let  $Q^* = q_1b_sb_{s-1} \cdots b_1q_{l_1}q_{l_1+1} \cdots q_g$  and  $Q^{**} = q_1q_2 \cdots q_{l_2}b_1q_{l_3}q_{l_3-1} \cdots q_g$ . Because  $g \geq \max\{|V(Q^*)|, |V(Q^{**})|\}$ , we have  $l_1 - 2 \geq s$  and  $g - 1 - l_2 \geq l_3 - g + 1$ , which implies  $g \geq (l_2 - l_1 + 1) + s + l_3 - g + 1 + 2$ . So

$$\begin{aligned} g &\geq 1 + |\{b_1, \dots, b_s\} \cup \{q_1, q_g\} \cup \{q_{l_1}, q_{l_1+1}, \dots, q_{l_2}\} \cup \{q_{g+1}, \dots, q_{l_3}\}| \\ &\geq 1 + |N_H(b_1) \cup \{b_1\}| \geq k - 1, \end{aligned}$$

a contradiction to Claim 5.

A similar argument holds if  $N_H(q_e) \cap \{b_1, b_2, \dots, b_s, q_1, q_2, \dots, q_{g-1}\} \neq \emptyset$ . Thus it is enough for completing our proof to discuss the following two cases.

(a).  $e > g$ ,  $N_H(b_1) \subseteq \{b_2, b_3, \dots, b_s, q_1\}$  and  $N_H(q_e) \subseteq \{q_g, q_{g+1}, \dots, q_{e-1}\}$

(b).  $e = g$ ,  $N_H(b_1) \subseteq \{b_2, b_3, \dots, b_s, q_1\}$ .

In case (a), since  $|N_C(b_1)| \leq 3$ , we have  $|\{b_1, \dots, b_s\}, q_1| \geq |N_H(b_1) \cup \{b_1\}| \geq k - 2$ . Similarly  $|\{q_g, \dots, q_{e-1}\}, q_e| \geq |N_H(q_e) \cup \{q_e\}| \geq k - 2$ . Then we have  $|H| \geq |N_H(b_1) \cup \{b_1\}| + |N_H(q_e) \cup \{q_e\}| \geq 2k - 4$ . This contradicts  $k + 1 \leq h \leq 2k - 7$ .

In case (b), denote  $q_1 = b_{s+1}$ . Let  $i = \max\{j : b_1b_j \in E(G_1)\}$ . We claim that  $N_C(b_l) = \{v\}$  and  $N_H(b_l) \cap \{q_2, q_3, \dots, q_g\} = \emptyset$ , for any  $1 \leq l \leq i - 1$ . Otherwise, let  $d = \min\{j : j > l \text{ and } b_1b_j \in E(G_1)\}$ . When  $|N_C(b_l)| \geq 2$ , either  $b_1b_{l-1} \cdots b_1b_db_{d+1} \cdots b_sq_1$



or  $b_l b_{l-1} \cdots b_1 b_d b_{d+1} \cdots b_s q_1 q_2 \cdots q_g$  is a path that satisfies (\*) and is longer than  $Q$ , a contradiction. When  $q_f \in N_H(b_l)$  for some  $2 \leq f \leq g$ , then

$$Q''' = q_1 b_s b_{s-1} \cdots b_d b_1 b_2 \cdots b_l q_f q_{f+1} \cdots q_g$$

is the path that satisfies (\*) and is longer than  $Q$ , a contradiction.

Since  $|Q'| \geq |N_H(b_1)| + |N_H(q_g)| + 2 - 1$ , we have

$$\begin{aligned} n_1 &\geq |Q'| + |C| \\ &\geq |N_H(b_1)| + k - |B| + 1 + |B| + (|B| - 1 + g) \\ &\geq |N_H(b_1)| + k + |B| + g \\ &\geq 2k - 1 + |B| + g. \end{aligned}$$

So  $g \leq 3k - 4 - 2k + 1 - |B| < k - |B| = |N_H(q_g)|$ . It follows that  $N_H(q_g) \cap \{b_1, b_2, \dots, b_s\} \neq \emptyset$ . Then  $|N_C(q_2)| \leq 3$  as there is a path in  $H$  with at least  $g + 1$  vertices connecting  $q_1$  and  $q_2$ , a contradiction. Thus we have  $|H| \geq |N_H(b_1)| + |N_H(q_2)| + 1 \geq k - 1 + k - 3 + 1 \geq 2k - 3$ , a contradiction.

□

**Claim 7.**  $t(Q) \geq 3$ .

*Proof.* From Claim 5 and Claim 6, we have  $|A| \geq 9$  and  $|B| \geq 9$ . Suppose  $t(Q) = 2$ . There is only one case:  $A = \{c_{i_1}, c_{i_2}, \dots, c_{i_s}\}$  and  $B = \{c_{j_1}, c_{j_2}, \dots, c_{j_l}\}$  such that  $s \geq 9, l \geq 9$  and  $\{c_{i_1}, c_{i_2}, \dots, c_{i_s}\} \cap \{c_{j_1}, c_{j_2}, \dots, c_{j_l}\} = \emptyset$ .

There is at least one segment of  $C[c_{j_l}, c_{i_1}]$  and  $C[c_{i_s}, c_{j_1}]$  which does not contain  $w_1, w_2$ . Without loss of generality, let  $w_1, w_2 \notin C[c_{j_l}, c_{i_1}]$ , then there exists some  $c_z \in A^+$  satisfying  $N_C(c_z) \cap [H \cup \{w_1, w_2, c_z, c_{j_{l+1}}, c_{j_{l+2}}, \dots, c_{j_{l+g}}\}] = \emptyset$ . And for any  $C[c_{j_f}, c_{j_{f+1}}]$  such that  $C[c_{j_f}, c_{j_{f+1}}] \cap \{w_1, w_2\} = \emptyset, f = 1, 2, \dots, l - 1$ , we have  $N_C(c_z) \cap \{c_{j_{f+1}}, c_{j_{f+2}}\} = \emptyset$ . This implies

$$d_{G_1}(c_z) \leq 3k - 4 - [h + g + 2(l - 1 - 2) + 3] \leq 3k - 1 - 2(g + l) \leq k - 3,$$

a contradiction (note that  $g + l \geq k + 1$ ).

□

In fact, we have  $t(Q) \geq 4$ . Otherwise, suppose  $t(Q) = 3$ , We only have to consider one case:  $A \cap B = \{c_m\}$ ,  $A \setminus \{c_m\} = \{c_{i_1}, c_{i_2}, \dots, c_{i_s}\}$  and  $B \setminus \{c_m\} = \{c_{j_1}, c_{j_2}, \dots, c_{j_l}\}$  such that  $s \geq 8, l \geq 8$  and  $\{c_{i_1}, c_{i_2}, \dots, c_{i_s}\} \cap \{c_{j_1}, c_{j_2}, \dots, c_{j_l}\} = \emptyset$ . It is worth noting that in this case,  $g + l \geq k$ . By a similar proof as in Claim 7, we have that there exists some  $c_z \in A^+$  or  $c_z \in B^+$  such that  $d_{G_1}(c_z) \leq 3k - 4 - [h + g + 2(l - 1 - 2) + 3] \leq 3k - 1 - 2(g + l) \leq k - 1$ , a contradiction. Since  $N_C(\{q_1, q_g\}) \cup (N_C(\{q_1, q_g\}))^+ \cup H \subseteq V(G_1)$ , we have

$$\begin{aligned} n_1 &\geq |C| + |H| \\ &\geq |H| + 2|N_C(\{q_1, q_g\})| + (t(Q) - 2)(g - 1) \\ &\geq h + 2(k - g + 1) + (t(Q) - 2)(g - 1) \\ &\geq 3k + 1 + (t(Q) - 4)(g - 1). \end{aligned}$$

Because of  $n_1 \leq 3k - 4$ , we have  $-5 \geq (t(Q) - 4)(g - 1)$ , a contradiction to  $t(Q) \geq 4$  and  $g \geq 2$ .

**Case 3.**  $h \geq 2k - 6$ .

Since  $G_1$  is a 3-connected graph and  $\{u, v, z\}$  is not a cut-set, there exists a vertex  $x' \in N_C(H)$ , such that  $x'^- \notin \{w_1, w_2\}$ . It is clear that  $N_{G_1}(x'^-) \cap H = \emptyset$ , and at least two of  $\{v, w_1, w_2\}$  cannot be adjacent to  $x'^-$ . It follows that

$$d_{G_1}(x'^-) \leq 3k - 4 - (2k - 6) - 2 - 1 = k - 1,$$

a contradiction.

Thus, we complete the proof of Theorem 7.

## 5 Conclusion

In this paper, we characterize a class of graphs that illustrate the sharpness of the bound  $2k$  in Theorem 4. By excluding these particular graphs, we are able to enhance the result and

establish that the bound is in fact  $2k + 1$  for 2-connected,  $k$ -regular graphs and  $3k - 6$  for 3-connected,  $k$ -regular graphs. The problem of regular 3-connected 2-path Hamiltonian graphs with  $n$  vertices remains intriguing whenever  $3k - 5 \leq n \leq 3k - 1$ . Naturally, we may inquire whether any interesting properties can be observed in regular  $m$ -connected graphs for  $m \geq 4$ ? The resolution of the aforementioned question is necessarily relevant to the study of the Hamiltonicity and edge-Hamiltonicity of regular graphs. However, it should be noted that the existence of an  $L$ -graph (as illustrated in [11]) presents a counterexample, showcasing higher connectivity that prevents the realization of edge-Hamiltonicity in regular graphs. As a result, we establish  $3k - 1$  as an upper bound for achieving the optimum outcome.

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