Reduction of fuzzy automata by means of fuzzy quasi-orders[☆]

Aleksandar Stamenković[a], Miroslav Ćirić[*,a], Jelena Ignjatović[a]

[a]Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, P. O. Box 224, 18000 Niš, Serbia

Abstract

In our recent paper we have established close relationships between state reduction of a fuzzy recognizer and resolution of a particular system of fuzzy relation equations. In that paper we have also studied reductions by means of those solutions which are fuzzy equivalences. In this paper we will see that in some cases better reductions can be obtained using the solutions of this system that are fuzzy quasi-orders. Generally, fuzzy quasi-orders and fuzzy equivalences are equally good in the state reduction, but we show that right and left invariant fuzzy quasi-orders give better reductions than right and left invariant fuzzy equivalences. We also show that alternate reductions by means of fuzzy quasi-orders give better results than alternate reductions by means of fuzzy equivalences. Furthermore we study a more general type of fuzzy quasi-orders, weakly right and left invariant ones, and we show that they are closely related to determinization of fuzzy recognizers. We also demonstrate some applications of weakly left invariant fuzzy quasi-orders in conflict analysis of fuzzy discrete event systems.

Key words: Fuzzy automaton; non-deterministic automaton; fuzzy quasi-order; fuzzy equivalence; state reduction; afterset; alternate reduction; simulation; bisimulation; fuzzy relation equation; complete residuated lattice, fuzzy discrete event system

1. Introduction

Unlike deterministic finite automata (DFA), whose efficient minimization is possible, the state minimization problem for non-deterministic finite automata (NFA) is computationally hard (PSPACE-complete, [41, 77]) and known algorithms like in [16, 42, 55, 56, 76] cannot be used in practice. For that reason, many researchers aimed their attention to NFA state reduction methods which do not necessarily give a minimal one, but they give "reasonably" small NFAs that can be constructed efficiently. The basic idea of reducing the number of states of NFAs by computing and merging indistinguishable states resembles the minimization algorithm for DFAs, but is more complicated. That led to the concept of a right invariant equivalence on an NFA, studied by Ilie and Yu [36, 37], who showed that they can be used to construct small NFAs from regular expressions. In particular, both the partial derivative automaton and the follow automaton of a given regular expression are factor automata of the position automaton with respect to the right invariant equivalences (cf. [19, 20, 35, 37, 38]). Right invariant equivalences have been also studied in [10, 11, 18, 37, 39, 40]. Moreover, the same concept was studied under the name "bisimulation equivalence" in many areas of computer science and mathematics, such as modal logic, concurrency theory, set theory, formal verification, model checking, etc., and numerous algorithms have been proposed to compute the greatest bisimulation equivalence on a given labeled graph or a labeled transition system (cf. [52, 57, 58, 59, 62, 73, 75]). The faster algorithms are based on the crucial equivalence between the greatest bisimulation equivalence and the relational coarsest partition problem (see [28, 29, 43, 72, 61]).

Better results in the state reduction of NFAs can be achieved in two ways. The first one was also proposed by Ilie and Yu in [36, 37, 39, 40] who introduced the dual concept of a left invariant equivalence on an NFA and showed that even smaller NFAs can be obtained alternating reductions by means of right invariant and left invariant equivalences. On the other hand, Champarnaud and Coulon in [17, 18] proposed use of quasi-orders (preorders) instead of equivalences and showed that the method based on quasi-orders gives better reductions than the method based on equivalences. They gave an algorithm for computing the greatest right invariant and left invariant quasi-orders on an NFA working in a polynomial time, which was later improved in [39, 40].

Fuzzy finite automata are generalizations of NFAs, and the above mentioned problems concerning minimization and reduction of NFAs are also present in the work with fuzzy automata. Reduction of the number of states of fuzzy automata was studied in [2, 21, 46, 53, 60, 66], and the algorithms given there were also based on the idea of computing and merging indistinguishable states. They were called minimization algorithms, but the term minimization is not adequate because it does not involve the usual construction of the minimal fuzzy automaton in the set of all fuzzy automata recognizing a given fuzzy language, but just the procedure of computing and merging indistinguishable states. Therefore, these are essentially just state reduction algorithms.

In the deterministic case we can effectively detect and merge indistinguishable states, but in the non-deterministic case we have sets of states and it is seemingly very difficult to decide whether two states are distinguishable or not. What we shall do in this paper is find a superset such that one is certain not to merge state that should not be merged. There can always be states which could be merged but detecting those is too computationally expensive. In the case of fuzzy automata, this problem is even worse because we work with fuzzy sets of states. However, it turned out that in the non-deterministic case indistinguishability can be successfully modelled by equivalences and quasi-orders. In [24, 25] we have shown that in the fuzzy case the indistinguishability can be modelled by fuzzy equivalences, and here we show that this can be done by fuzzy quasi-orders. It is worth noting that in all previous papers dealing with reduction of fuzzy automata (cf. [2, 21, 46, 53, 60, 66]) only reductions by means of crisp equivalences have been investigated. In this paper, as well as [24, 25], show that better reductions can be achieved employing fuzzy relations, namely, fuzzy equivalences and fuzzy quasi-orders.

In contrast to [24, 25], where we have started from a fuzzy equivalence on a set of states A of a fuzzy automaton $\mathscr{A}$, here we start from an arbitrary fuzzy quasi-order R on A, we form the set A/R of all aftersets of R, and we turn the fuzzy transition function on A into a related fuzzy transition function on A/R. This results in the afterset fuzzy automaton $\mathscr{A}/R$. If, in addition, $\mathscr{A}$ is a fuzzy recognizer, then we also turn its fuzzy sets of initial and terminal states into related fuzzy sets of initial and terminal states of the afterset fuzzy recognizer $\mathscr{A}/R$. In a similar way, we construct the foreset fuzzy recognizer of $\mathscr{A}$ w.r.t. R, but we show that they are isomorphic, and hence, it is enough to consider only afterset fuzzy automata and recognizers. However, if we do not impose any restriction on R, then the afterset fuzzy recognizer $\mathscr{A}/R$ does not necessary recognize the same fuzzy language as $\mathscr{A}$. We show that $\mathscr{A}$ and $\mathscr{A}/R$ recognize the same fuzzy language, i.e., they are equivalent, if and only if R is a solution to a particular system of fuzzy relation equations including R, as an unknown fuzzy quasi-order, transition relations on $\mathscr{A}$ and fuzzy sets of initial and terminal states. This system, called the *general system*, has at least one solution in the set $\mathscr{Q}(A)$ of all fuzzy quasi-orders on A, the equality relation on A. Nevertheless, to obtain the best possible reduction of $\mathscr{A}$, we have to find the greatest solution to the general system in $\mathscr{Q}(A)$, if it exists, or to find as big a solution as possible. The general system does not necessary have the greatest solution (Example 3.2), and also, it may consist of infinitely many equations, and finding its nontrivial solutions may be a very difficult task. For that reason we aim our attention to some instances of the general system. These instances have to be as general as possible, but they have to be easier to solve. From a practical point of view, these instances have to consist of finitely many equations.

In Section 4 we study two instances of the general system whose solutions, called the right and left invariant fuzzy quasi-orders, are common generalization of right and left invariant quasi-orders and equivalences, studied in [17, 18, 39, 40], and right and left invariant fuzzy equivalences, studied in [24, 25]. Using a methodology similar to the one developed in [24, 25] for fuzzy equivalences, we give a characterization of right invariant fuzzy quasi-orders on a fuzzy automaton $\mathscr{A}$, and we prove that they form a complete lattice

whose greatest element gives the best reduction of $\mathscr{A}$ by means of fuzzy quasi-orders of this type. Then by Theorem 4.3 we give a procedure for computing the greatest right invariant fuzzy quasi-order contained in a given fuzzy quasi-order, which works if the underlying structure $\mathscr{L}$ of truth values is locally finite, but it does not necessary work if $\mathscr{L}$ is not locally finite. In particular, it works for classical fuzzy automata over the Gödel structure, but it does not necessary work for fuzzy automata over the Goguen (product) structure. We also characterize the greatest right invariant fuzzy quasi-order in the case when the structure $\mathscr{L}$ satisfies certain distributivity conditions for join and multiplication over infima. This characterization hold, for example, whenever multiplication is assumed to be a continuous t-norm on the real unit interval, and hence, they hold for Łukasiewicz, Goguen and Gödel structures. Although the results, as well as the methodology, are similar to those obtained in [24, 25] for fuzzy equivalences, there are some important differences which justify our study of state reductions by means of fuzzy quasi-orders. Generally, fuzzy quasi-orders and fuzzy equivalences are equally good in the reduction of fuzzy automata and recognizers, as we have shown by Theorem 3.4. However, Example 4.3 shows that the right invariant fuzzy quasi-orders give better reductions than right invariant fuzzy equivalences. Moreover, the iterative procedure for computing the greatest right invariant fuzzy quasi-order on a fuzzy automaton $\mathscr{A}$, given in Theorem 4.3, can terminate in a finite number of steps even if a similar iteration procedure for computing the greatest right invariant fuzzy equivalence on $\mathscr{A}$, developed in [24, 25], does not terminate in a finite number of steps (Example 4.2). It is worth noting that the greatest right and left invariant fuzzy quasi-orders are calculated using iterative procedures, but these calculations are not approximative. Whenever these procedures terminate in a finite number of steps, exact solutions to the considered systems of fuzzy relation equations are obtained.

As we have noted, the procedure for computing the greatest right invariant fuzzy quasi-orders on fuzzy automata does not necessary work if the underlying structure $\mathscr{L}$ of truth values is not locally finite. For that reason in Section 5 we consider some special types of right and left invariant fuzzy quasi-orders, and we show that the greatest fuzzy quasi-orders of these types can be effectively computed even if $\mathscr{L}$ is not necessary locally finite. By Theorem 5.1 we give an iterative procedure for computing the greatest right invariant crisp quasi-order contained in a given crisp or fuzzy quasi-order. This procedure works if $\mathscr{L}$ is any complete residuated lattice, and even if $\mathscr{L}$ is a lattice ordered monoid. On the other hand, as Example 5.1 shows, in cases when we are able to effectively compute the greatest right invariant fuzzy quasi-order, using it, better state reduction can be achieved than by using the greatest right invariant crisp quasi-order. We also study the strongly right invariant fuzzy quasi-orders, which can be effectively computed without any iteration procedure, by solving a simpler system of fuzzy relation equations. This procedure works if $\mathscr{L}$ is any complete residuated lattice, even though we also show that reductions by means of the greatest strongly right invariant fuzzy quasi-orders give worse results than reductions by means of the greatest right invariant ones (Example 5.2).

In addition to special types of right and left invariant fuzzy quasi-orders considered in Section 5, in Section 6 we study some more general types of these fuzzy quasi-orders – the weakly right and left invariant fuzzy quasi-orders. We show that the weakly right invariant fuzzy quasi-orders on a fuzzy recognizer $\mathscr{A}$ form a principal ideal of the lattice of quasi-orders on the set of states of $\mathscr{A}$. We give a procedure for computing the greatest element of this principal ideal (Theorem 6.1), and we show that weakly right invariant fuzzy quasi-orders give better reductions than right invariant ones. However, although the system of fuzzy relation equations that defines the weakly right invariant fuzzy quasi-orders consists of fuzzy relation equations whose greatest solutions can be easily computed, computing the greatest solution to the whole system is computationally hard. Namely, the number of equations may be exponential in the number of states of $\mathscr{A}$, or it may even be infinite. This is an immediate consequence of the fact that the procedure for computing the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}$ includes the procedure for determinization of the reverse fuzzy recognizer of $\mathscr{A}$ developed in [32], whereas the procedure for computing the greatest weakly left invariant one includes determinization of $\mathscr{A}$.

In Section 7 we show that even better results in the state reduction can be obtained by alternating reductions by means of the greatest right and left invariant fuzzy quasi-orders, or by means of the greatest weakly right and left invariant ones. First we show that if we reduce a fuzzy automaton using the greatest right invariant fuzzy quasi-order, repeated reduction using right invariant fuzzy quasi-orders can not decrease

the number of states. The number of states can be decreased if we apply reduction by means of the greatest left invariant fuzzy quasi-order. The same observation is true for left invariant fuzzy quasi-orders, as well as for weakly right and left invariant ones. We also show that alternate reductions starting with a (weakly) right invariant fuzzy quasi-order, and those starting with a (weakly) left invariant one, can have different lengths, and related alternate reducts can have different number of states (Example 7.1). Moreover, there is no a general procedure to decide which of these alternate reductions will give better results. Also, there is no a general procedure to decide whether we have reached the smallest number of states in alternate reductions.

Let us note that Champarnaud and Coulon [17, 18], Ilie, Navarro and Yu [39], and Ilie, Solis-Oba and Yu [40] studied the state reduction of non-deterministic recognizers by means of right and left invariant quasi-orders. However, they do not used the afterset or foreset recognizers w.r.t. a quasi-order R. Instead, they used the factor recognizer w.r.t. the natural equivalence E_R of R. Although these recognizers are equivalent and have the same number of states, there are some differences in their use if one works with alternate reductions. Indeed, by Example 7.1 we also show that in some cases alternate reductions by means of natural equivalences of right and left invariant quasi-orders and alternate reductions by means of right and left invariant equivalences do not decrease the number of states, while the alternate reductions by means of right and left invariant quasi-orders decrease this number.

Finally, in Section 8 we demonstrate some applications of weakly left invariant fuzzy quasi-orders in the fuzzy discrete event systems theory. We show that every fuzzy recognizer $\mathscr{A}$ is conflict-equivalent with the afterset fuzzy recognizer $\mathscr{A}/R$ w.r.t. any weakly left invariant fuzzy quasi-order R on $\mathscr{A}$. For the sake of conflict analysis, this means that in the parallel composition of fuzzy recognizers every component can be replaced by such afterset fuzzy recognizer, what results in a smaller fuzzy recognizer to be analysed, and do not affect conflicting properties of the components. It is also interesting to study applications of fuzzy quasi-orders for reducing automaton states in other branches of the theory of discrete event systems, for example in the fault diagnosis, and these applications will be a subject of our future research.

Note again that the meaning of state reductions by means of fuzzy quasi-orders and fuzzy equivalences is in their possible effectiveness, as opposed to the minimization which is not effective. However, by Theorem 3.5 we show that there exists a fuzzy recognizer such that no its state reduction by means of fuzzy quasi-orders or fuzzy equivalences provide a minimal fuzzy recognizer.

2. Preliminaries

2.1. Fuzzy sets and relations

In this paper we will use complete residuated lattices as structures of membership values. A *residuated lattice* is an algebra $\mathscr{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

(L1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1,
(L2) $(L, \otimes, 1)$ is a commutative monoid with the unit 1,
(L3) $\otimes$ and $\rightarrow$ form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all $x, y, z \in L$,

$$x \otimes y \leqslant z \ \Leftrightarrow \ x \leqslant y \rightarrow z. \tag{1}$$

If, in addition, $(L, \wedge, \vee, 0, 1)$ is a complete lattice, then $\mathscr{L}$ is called a *complete residuated lattice*.

The operations $\otimes$ (called *multiplication*) and $\rightarrow$ (called *residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of the existential and general quantifier, respectively. An operation $\leftrightarrow$ defined by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x), \tag{2}$$

called *biresiduum* (or *biimplication*), is used for modeling the equivalence of truth values. It can be easily verified that with respect to $\leqslant$, $\otimes$ is isotonic in both arguments, and $\rightarrow$ is isotonic in the second and antitonic in the first argument. Emphasizing their monoidal structure, in some sources residuated lattices are called

integral, commutative, residuated ℓ-monoids [30]. It can be easily verified that with respect to $\leqslant$, $\otimes$ is isotonic in both arguments, $\rightarrow$ is isotonic in the second and antitonic in the first argument, and for any $x, y, z \in L$ and any $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \subseteq L$, the following hold:

$$x \rightarrow y \leqslant x \otimes z \rightarrow y \otimes z, \tag{3}$$

$$\left(\bigvee_{i \in I} x_i \right) \otimes x = \bigvee_{i \in I} (x_i \otimes x), \tag{4}$$

$$\bigwedge_{i \in I} (x_i \rightarrow y_i) \leqslant \left(\bigwedge_{i \in I} x_i \right) \rightarrow \left(\bigwedge_{i \in I} y_i \right) \tag{5}$$

$$\bigwedge_{i \in I} (x_i \rightarrow y_i) \leqslant \left(\bigvee_{i \in I} x_i \right) \rightarrow \left(\bigvee_{i \in I} y_i \right). \tag{6}$$

For other properties of complete residuated lattices one can refer to [3, 4].

The most studied and applied structures of truth values, defined on the real unit interval $[0, 1]$ with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are the *Łukasiewicz structure* ($x \otimes y = \max(x + y - 1, 0)$, $x \rightarrow y = \min(1 - x + y, 1)$), the *Goguen (product) structure* ($x \otimes y = x \cdot y$, $x \rightarrow y = 1$ if $x \leqslant y$ and $= y/x$ otherwise) and the *Gödel structure* ($x \otimes y = \min(x, y)$, $x \rightarrow y = 1$ if $x \leqslant y$ and $= y$ otherwise). More generally, an algebra $([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and the residuum is defined by $x \rightarrow y = \bigvee \{u \in [0, 1] \,|\, u \otimes x \leqslant y\}$. Another important set of truth values is the set $\{a_0, a_1, \ldots, a_n\}$, $0 = a_0 < \cdots < a_n = 1$, with $a_k \otimes a_l = a_{\max(k+l-n, 0)}$ and $a_k \rightarrow a_l = a_{\min(n-k+l, n)}$. A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support $\{0, 1\}$. The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the *Boolean structure*. A residuated lattice $\mathscr{L}$ satisfying $x \otimes y = x \wedge y$ is called a *Heyting algebra*, whereas a Heyting algebra satisfying the prelinearity axiom $(x \rightarrow y) \vee (y \rightarrow x) = 1$ is called a *Gödel algebra*. If any finitelly generated subalgebra of residuated lattice $\mathscr{L}$ is finite, then $\mathscr{L}$ is called *locally finite*. For example, every Gödel algebra, and hence, the Gödel structure, is locally finite, whereas the product structure is not locally finite.

In the further text $\mathscr{L}$ will be a complete residuated lattice. A *fuzzy subset* of a set A over $\mathscr{L}$, or simply a *fuzzy subset* of A, is any mapping from A into L. Ordinary crisp subsets of A are considered as fuzzy subsets of A taking membership values in the set $\{0, 1\} \subseteq L$. Let f and g be two fuzzy subsets of A. The *equality* of f and g is defined as the usual equality of mappings, i.e., $f = g$ if and only if $f(x) = g(x)$, for every $x \in A$. The *inclusion* $f \leqslant g$ is also defined pointwise: $f \leqslant g$ if and only if $f(x) \leqslant g(x)$, for every $x \in A$. Endowed with this partial order the set L^A of all fuzzy subsets of A forms a complete residuated lattice, in which the meet (intersection) $\bigwedge_{i \in I} f_i$ and the join (union) $\bigvee_{i \in I} f_i$ of an arbitrary family $\{f_i\}_{i \in I}$ of fuzzy subsets of A are mappings from A into L defined by

$$\left(\bigwedge_{i \in I} f_i \right)(x) = \bigwedge_{i \in I} f_i(x), \qquad \left(\bigvee_{i \in I} f_i \right)(x) = \bigvee_{i \in I} f_i(x),$$

and the *product* $f \otimes g$ is a fuzzy subset defined by $f \otimes g(x) = f(x) \otimes g(x)$, for every $x \in A$. The *crisp part* of a fuzzy subset f of A is a crisp subset $\hat{f} = \{a \in A \,|\, f(a) = 1\}$ of A. We will also consider $\hat{f}$ as a mapping $\hat{f} : A \rightarrow L$ defined by $\hat{f}(a) = 1$, if $f(a) = 1$, and $\hat{f}(a) = 0$, if $f(a) < 1$.

A *fuzzy relation* on a set A is any mapping from $A \times A$ into L, that is to say, any fuzzy subset of $A \times A$, and the equality, inclusion, joins, meets and ordering of fuzzy relations are defined as for fuzzy sets. The set of all fuzzy relations on A will be denoted by $\mathscr{R}(A)$.

For fuzzy relations $P, Q \in \mathscr{R}(A)$, their *composition* $P \circ Q$ is a fuzzy relation on A defined by

$$(P \circ Q)(a, b) = \bigvee_{c \in A} P(a, c) \otimes Q(c, b), \tag{7}$$

for all $a, b \in A$, and for a fuzzy subset f of A and a fuzzy relation $P \in \mathcal{R}(A)$, the *compositions* $f \circ P$ and $P \circ f$ are fuzzy subsets of A defined by

$$(f \circ P)(a) = \bigvee_{b \in A} f(b) \otimes P(b, a), \quad (P \circ f)(a) = \bigvee_{b \in A} P(a, b) \otimes f(b), \tag{8}$$

for any $a \in A$. Finally, for fuzzy subsets f and g of A we write

$$f \circ g = \bigvee_{a \in A} f(a) \otimes g(a). \tag{9}$$

The value $f \circ g$ can be interpreted as the "degree of overlapping" of f and g.

For any $P, Q, R \in \mathcal{R}(A)$ and any $\{P_i\}_{i \in I}, \{Q_i\}_{i \in I} \subseteq \mathcal{R}(A)$, the following hold:

$$(P \circ Q) \circ R = P \circ (Q \circ R), \tag{10}$$

$$P \leqslant Q \text{ implies } P \circ R \leqslant Q \circ R \text{ and } R \circ P \leqslant R \circ Q, \tag{11}$$

$$P \circ \left(\bigvee_{i \in I} Q_i \right) = \bigvee_{i \in I} (P \circ Q_i), \quad \left(\bigvee_{i \in I} P_i \right) \circ Q = \bigvee_{i \in I} (P_i \circ Q) \tag{12}$$

$$P \circ \left(\bigwedge_{i \in I} Q_i \right) \leqslant \bigwedge_{i \in I} (P \circ Q_i), \quad \left(\bigwedge_{i \in I} P_i \right) \circ Q \leqslant \bigwedge_{i \in I} (P_i \circ Q). \tag{13}$$

We can also easily verify that

$$(f \circ P) \circ Q = f \circ (P \circ Q), \quad (f \circ P) \circ g = f \circ (P \circ g), \tag{14}$$

for arbitrary fuzzy subsets f and g of A, and fuzzy relations P and Q on A, and hence, the parentheses in (10) can be omitted. For $n \in \mathbb{N}$, an *n-th power* of a fuzzy relation R on A is a fuzzy relation R^n on A defined inductively by $R^1 = R$ and $R^{n+1} = R^n \circ R$. We also define R^0 to be the equality relation on A.

Note also that if A is a finite set with n elements, then P and Q can be treated as $n \times n$ fuzzy matrices over $\mathcal{L}$ and $P \circ Q$ is the matrix product, whereas $f \circ P$ can be treated as the product of a $1 \times n$ matrix f and an $n \times n$ matrix P, and $P \circ f$ as the product of an $n \times n$ matrix P and an $n \times 1$ matrix f^t (the transpose of f).

A fuzzy relation R on A is said to be

(R) *reflexive* (or *fuzzy reflexive*) if $R(a, a) = 1$, for every $a \in A$;

(S) *symmetric* (or *fuzzy symmetric*) if $R(a, b) = R(b, a)$, for all $a, b \in A$;

(T) *transitive* (or *fuzzy transitive*) if for all $a, b, c \in A$ we have

$$R(a, b) \otimes R(b, c) \leqslant R(a, c).$$

For a fuzzy relation R on a set A, a fuzzy relation R^∞ on A defined by

$$R^\infty = \bigvee_{n \in \mathbb{N}} R^n$$

is the least transitive fuzzy relation on A containing R, and it is called the *transitive closure* of R.

A fuzzy relation on A which is reflexive, symmetric and transitive is called a *fuzzy equivalence*. With respect to the ordering of fuzzy relations, the set $\mathcal{E}(A)$ of all fuzzy equivalences on A is a complete lattice, in which the meet coincide with the ordinary intersection of fuzzy relations, but in the general case, the join in $\mathcal{E}(A)$ does not coincide with the ordinary union of fuzzy relations.

For a fuzzy equivalence E on A and $a \in A$ we define a fuzzy subset E_a of A by $E_a(x) = E(a, x)$, for every $x \in A$. We call E_a an *equivalence class* of E determined by a. The set $A/E = \{E_a \mid a \in A\}$ is called the *factor set* of A w.r.t. E (cf. [3, 22]). For an equivalence π on A, the related factor set will be denoted by A/π and the equivalence class of an element $a \in A$ by π_a. A fuzzy equivalence E on a set A is called a *fuzzy equality* if for

all $x, y \in A$, $E(x, y) = 1$ implies $x = y$. In other words, E is a fuzzy equality if and only if its crisp part $\widehat{E}$ is a crisp equality.

A fuzzy relation on a set A which is reflexive and transitive is called a *fuzzy quasi-order*, and a reflexive and transitive crisp relation on A is called a *quasi-order*. In some sources quasi-orders and fuzzy quasi-orders are called preorders and fuzzy preorders (for example, see [17, 18, 39, 40]). Note that a reflexive fuzzy relation R is a fuzzy quasi-order if and only if $R^2 = R$. With respect to the ordering of fuzzy relations, the set $\mathcal{Q}(A)$ of all fuzzy quasi-orders on A is a complete lattice, in which the meet coincide with the ordinary intersection of fuzzy relations. Nevertheless, in the general case, the join in $\mathcal{Q}(A)$ does not coincide with the ordinary union of fuzzy relations. Namely, if R is the join in $\mathcal{Q}(A)$ of a family $\{R_i\}_{i \in I}$ of fuzzy quasi-orders on A, then R can be represented by

$$R = \left(\bigvee_{i \in I} R_i \right)^{\infty} = \bigvee_{n \in \mathbb{N}} \left(\bigvee_{i \in I} R_i \right)^{n}. \tag{15}$$

If R is a fuzzy quasi-order on a set A, then a fuzzy relation E_R defined by $E_R = R \wedge R^{-1}$ is a fuzzy equivalence on A, and is called a *natural fuzzy equivalence* of R. A fuzzy quasi-order R on a set A is a *fuzzy order* if for all $a, b \in A$, $R(a, b) = R(b, a) = 1$ implies $a = b$, i.e., if the natural fuzzy equivalence E_R of R is a fuzzy equality. Clearly, a fuzzy quasi-order R is a fuzzy order if and only if its crisp part $\widehat{R}$ is a crisp order.

It is worth noting that different concepts of a fuzzy order have been discussed in literature concerning fuzzy relations (for example, see [5, 6, 7, 8] and other sources cited there). In particular, fuzzy orders defined here differ from fuzzy orderings defined in [5, 6, 8].

For more information about fuzzy sets and fuzzy logic we refer to the books [3, 4, 45], as well as to recent papers [78, 79], which review fuzzy logic and uncertainty in a much broader perspective.

2.2. Fuzzy automata and languages

By a *fuzzy automaton over $\mathcal{L}$*, or simply a *fuzzy automaton*, is defined as a triple $\mathcal{A} = (A, X, \delta^A)$, where A and X are the *set of states* and the *input alphabet*, and $\delta^A : A \times X \times A \to L$ is a fuzzy subset of $A \times X \times A$, called the *fuzzy transition function*. We can interpret $\delta^A(a, x, b)$ as the degree to which an input letter $x \in X$ causes a transition from a state $a \in A$ into a state $b \in A$. The input alphabet X will be always finite, but for methodological reasons we will allow the set of states A to be infinite. A fuzzy automaton whose set of states is finite is called a *fuzzy finite automaton*. Cardinality of a fuzzy automaton $\mathcal{A} = (A, X, \delta^A)$, denoted as $|\mathcal{A}|$, is defined as the cardinality of its set of states A.

Let X^* denote the free monoid over the alphabet X, and let $e \in X^*$ be the empty word. The mapping δ^A can be extended up to a mapping $\delta^A_* : A \times X^* \times A \to L$ as follows: If $a, b \in A$, then

$$\delta^A_*(a, e, b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}, \tag{16}$$

and if $a, b \in A$, $u \in X^*$ and $x \in X$, then

$$\delta^A_*(a, ux, b) = \bigvee_{c \in A} \delta^A_*(a, u, c) \otimes \delta^A(c, x, b). \tag{17}$$

By (4) and Theorem 3.1 [47] (see also [67, 68, 70]), we have that

$$\delta^A_*(a, uv, b) = \bigvee_{c \in A} \delta^A_*(a, u, c) \otimes \delta^A_*(c, v, b), \tag{18}$$

for all $a, b \in A$ and $u, v \in X^*$, i.e., if $w = x_1 \cdots x_n$, for $x_1, \ldots, x_n \in X$, then

$$\delta^A_*(a, w, b) = \bigvee_{(c_1, \ldots, c_{n-1}) \in A^{n-1}} \delta^A(a, x_1, c_1) \otimes \delta^A(c_1, x_2, c_2) \otimes \cdots \otimes \delta^A(c_{n-1}, x_n, b). \tag{19}$$

7

Intuitively, the product $\delta^A(a, x_1, c_1) \otimes \delta^A(c_1, x_2, c_2) \otimes \cdots \otimes \delta^A(c_{n-1}, x_n, b)$ represents the degree to which the input word w causes a transition from a state a into a state b through the sequence of intermediate states $c_1, \ldots, c_{n-1} \in A$, and $\delta^A_*(a, w, b)$ represents the supremum of degrees of all possible transitions from a into b caused by w.

For any $u \in X^*$, and any $a, b \in A$ define a fuzzy relation δ^A_u on A by

$$\delta^A_u(a, b) = \delta^A_*(a, u, b), \tag{20}$$

called the *fuzzy transition relation* determined by u. Then (18) can be written as

$$\delta^A_{uv} = \delta^A_u \circ \delta^A_v, \tag{21}$$

for all $u, v \in X^*$.

An *initial fuzzy automaton* is defined as a quadruple $\mathscr{A} = (A, X, \delta^A, \sigma^A)$, where (A, X, δ^A) is a fuzzy automaton and $\sigma^A \in L^A$ is the fuzzy set of *initial states*, and a *fuzzy recognizer* is defined as a five-tuple $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, where $(A, X, \delta^A, \sigma^A)$ is as above, and $\tau^A \in L^A$ is the fuzzy set of *terminal states*. We also say that $\mathscr{A}$ is a *fuzzy recognizer belonging to the fuzzy automaton* (A, X, δ^A).

A *fuzzy language* in X^* over $\mathscr{L}$, or briefly a *fuzzy language*, is any fuzzy subset of X^*, i.e., any mapping from X^* into L. A *fuzzy language recognized by a fuzzy recognizer* $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, denoted as $L(\mathscr{A})$, is a fuzzy language in L^{X^*} defined by

$$L(\mathscr{A})(u) = \bigvee_{a,b \in A} \sigma^A(a) \otimes \delta^A_*(a, u, b) \otimes \tau^A(b), \tag{22}$$

or equivalently,

$$\begin{aligned} L(\mathscr{A})(e) &= \sigma^A \circ \tau^A, \\ L(\mathscr{A})(u) &= \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A, \end{aligned} \tag{23}$$

for any $u = x_1 x_2 \ldots x_n \in X^+$, where $x_1, x_2, \ldots, x_n \in X$. In other words, the equality (22) means that the membership degree of the word u to the fuzzy language $L(\mathscr{A})$ is equal to the degree to which $\mathscr{A}$ recognizes or accepts the word u.

The *reverse fuzzy automaton* of a fuzzy automaton $\mathscr{A} = (A, X, \delta^A)$ denoted as $\bar{\mathscr{A}} = (A, X, \bar{\delta}^A)$, is a fuzzy automaton with the fuzzy transition function defined by $\bar{\delta}^A(a, x, b) = \delta^A(b, x, a)$, for all $a, b \in A$ and $x \in X$. A *reverse fuzzy recognizer* of a fuzzy recognizer $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is $\bar{\mathscr{A}} = (A, X, \bar{\delta}^A, \bar{\sigma}^A, \bar{\tau}^A)$, a fuzzy recognizer with the fuzzy transition function $\bar{\delta}^A$ defined as above, and fuzzy sets of initial and terminal states defined by $\bar{\sigma}^A = \tau^A$ and $\bar{\tau}^A = \sigma^A$.

Fuzzy automata $\mathscr{A} = (A, X, \delta^A)$ and $\mathscr{A}' = (A', X, \delta^{A'})$ are *isomorphic* if there is a bijective mapping $\phi : A \to A'$ such that $\delta^A(a, x, b) = \delta^{A'}(\phi(a), x, \phi(b))$, for all $a, b \in A$ and $x \in X$. It is easy to check that in this case we also have that $\delta^A_*(a, u, b) = \delta^{A'}_*(\phi(a), u, \phi(b))$, for all $a, b \in A$ and $u \in X^*$. Similarly, fuzzy recognizers $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and $\mathscr{A}' = (A', X, \delta^{A'}, \sigma^{A'}, \tau^{A'})$ are *isomorphic* if there exists a bijective mapping $\phi : A \to A'$ such that $\delta^A(a, x, b) = \delta^{A'}(\phi(a), x, \phi(b))$, for all $a, b \in A$ and $x \in X$, and also, $\sigma^A(a) = \sigma^{A'}(\phi(a))$ and $\tau^A(a) = \tau^{A'}(\phi(a))$, for every $a \in A$.

If $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy automaton such that δ^A is a crisp relation, then $\mathscr{A}$ is an ordinary crisp *non-deterministic automaton*, while if δ^A is a mapping of $A \times X$ into A, then $\mathscr{A}$ is an ordinary *deterministic automaton*. Evidently, in these two cases we have that δ^A_* is also a crisp subset of $A \times X^* \times A$, and a mapping of $A \times X^*$ into A, respectively. In other words, non-deterministic automata are fuzzy automata over the Boolean structure. If $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ such that δ^A is a crisp relation and σ^A and τ^A are crisp subsets of A, then $\mathscr{A}$ is called a *non-deterministic recognizer*.

For undefined notions and notation one can refer to [3, 4, 60].

3. Afterset and foreset fuzzy automata

Let R be a fuzzy quasi-order on a set A. For each $a \in A$, the *R-afterset* of a is the fuzzy set $R_a \in L^A$ defined by $R_a(b) = R(a,b)$, for any $b \in A$, while the *R-foreset* of a is the fuzzy set $R^a \in L^A$ defined by $R^a(b) = R(b,a)$, for any $b \in A$ (see [1, 26, 27]). The set of all R-aftersets will be denoted by A/R, and the set of all R-foresets will be denoted by $A\backslash R$. Clearly, if R is a fuzzy equivalence, then $A/R = A\backslash R$ is the set of all equivalence classes of R.

If f is an arbitrary fuzzy subset of A, then fuzzy relations R_f and R^f on A defined by

$$R_f(a,b) = f(a) \to f(b), \quad R^f(a,b) = f(b) \to f(a), \tag{24}$$

for all $a,b \in A$, are fuzzy quasi-orders on A. In particular, if f is a normalized fuzzy subset of A, then it is an afterset of R_f and a foreset of R^f.

Theorem 3.1. *Let R be a fuzzy quasi-order on a set A and E the natural fuzzy equivalence of R. Then*

(a) *For arbitrary $a,b \in A$ the following conditions are equivalent:*
   (i) $E(a,b) = 1$;
   (ii) $E_a = E_b$;
   (iii) $R_a = R_b$;
   (iv) $R^a = R^b$.
(b) *Functions $R_a \mapsto E_a$ of A/R to A/E, and $R_a \mapsto R^a$ of A/R to $A\backslash R$, are bijective functions.*

Proof. (a) Consider arbitrary $a,b \in A$.

(i)$\Rightarrow$(ii). Let $E(a,b) = 1$, that is $R(a,b) = R(b,a) = 1$. Then for every $c \in A$ we have that

$$R_b(c) = R(b,c) = R(a,b) \otimes R(b,c) \leqslant R(a,c) = R_a(c),$$

whence $R_b \leqslant R_a$. Analogously we prove that $R_a \leqslant R_b$, and therefore, $R_a = R_b$.

(ii)$\Rightarrow$(i). Let $R_a = R_b$. Then

$$R(a,b) = R_a(b) \geqslant R_b(b) = R(b,b) = 1,$$

which yields $R(a,b) = 1$. Analogously we prove that $R(b,a) = 1$, and hence, $E(a,b) = 1$.

Equivalence (i)$\Leftrightarrow$(iii) can be proved similarly as (i)$\Leftrightarrow$(ii).

The assertion (b) follows immediately by (a). $\square$

Let us consider the Gödel structure and a fuzzy quasi-order R on a set A given by

$$R = \begin{bmatrix} 1 & 0.3 & 0.3 \\ 0 & 1 & 0.2 \\ 0 & 1 & 1 \end{bmatrix}.$$

The natural fuzzy equivalence E_R of R is calculated by $E_R(a,b) = R(a,b) \wedge R^{-1}(a,b) = R(a,b) \wedge R(b,a)$, i.e.

$$E_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.2 \\ 0 & 0.2 & 1 \end{bmatrix}.$$

If A is a finite set with n elements and a fuzzy quasi-order R on A is treated as an $n \times n$ fuzzy matrix over $\mathscr{L}$, then R-aftersets are row vectors, whereas R-foresets are column vectors of this matrix. The previous theorem says that i-th and j-th row vectors of this matrix are equal if and only if its i-th and j-th column vectors are equal, and vice versa. Moreover, we have that R is a fuzzy order if and only if all its row vectors are different, or equivalently, if and only if all its column vectors are different.

9

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton and let R be a fuzzy quasi-order on A. We can define the fuzzy transition function $\delta^{A/R} : A/R \times X \times A/R \to L$ by

$$\delta^{A/R}(R_a, x, R_b) = \bigvee_{a', b' \in A} R(a, a') \otimes \delta^A(a', x, b') \otimes R(b', b), \tag{25}$$

or equivalently

$$\delta^{A/R}(R_a, x, R_b) = (R \circ \delta_x^A \circ R)(a, b) = R_a \circ \delta_x^A \circ R^b, \tag{26}$$

for all $a, b \in A$ and $x \in X$. According to the statement (a) of Theorem 3.1, $\delta^{A/R}$ is well-defined, and we have that $\mathscr{A}/R = (A/R, X, \delta^{A/R})$ is a fuzzy automaton, called the *afterset fuzzy automaton* of $\mathscr{A}$ w.r.t. R.

In addition, if $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy recognizer, then we define the fuzzy transition function $\delta^{A/R}$ as in (25), and we also define a fuzzy set $\sigma^{A/R} \in L^{A/R}$ of initial states and a fuzzy set $\tau^{A/R} \in L^{A/R}$ of terminal states by

$$\sigma^{A/R}(R_a) = \bigvee_{a' \in A} \sigma^A(a') \otimes R(a', a) = (\sigma^A \circ R)(a) = \sigma^A \circ R^a, \tag{27}$$

$$\tau^{A/R}(R_a) = \bigvee_{a' \in A} R(a, a') \otimes \tau^A(a') = (R \circ \tau^A)(a) = R_a \circ \tau^A, \tag{28}$$

for any $a \in A$. According to (a) of Theorem 3.1, $\sigma^{A/R}$ and $\tau^{A/R}$ are well-defined functions, and we have that $\mathscr{A}/R = (A/R, X, \delta^{A/R}, \sigma^{A/R}, \tau^{A/R})$ is a fuzzy recognizer, which is called the *afterset fuzzy recognizer* of $\mathscr{A}$ w.r.t. R.

Analogously, for a fuzzy automaton $\mathscr{A} = (A, X, \delta^A)$, the *foreset fuzzy automaton* of $\mathscr{A}$ w.r.t. R is a fuzzy automaton $\mathscr{A}\backslash R = (A\backslash R, X, \delta^{A\backslash R})$ with the fuzzy transition function $\delta^{A\backslash R}$ defined by

$$\delta^{A\backslash R}(R^a, x, R^b) = \bigvee_{a', b' \in A} R(a, a') \otimes \delta^A(a', x, b') \otimes R(b', b) = (R \circ \delta_x^A \circ R)(a, b) = R_a \circ \delta_x^A \circ R^b, \tag{29}$$

for all $a, b \in A$ and $x \in X$. In addition, for a fuzzy recognizer $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, the *foreset fuzzy recognizer* of $\mathscr{A}$ w.r.t. R is a fuzzy recognizer $\mathscr{A}\backslash R = (A\backslash R, X, \delta^{A\backslash R}, \sigma^{A\backslash R}, \tau^{A\backslash R})$ with a a fuzzy set $\sigma^{A\backslash R} \in L^{A\backslash R}$ of initial states and a fuzzy set $\tau^{A\backslash R} \in L^{A\backslash R}$ of terminal states by

$$\sigma^{A\backslash R}(R^a) = \bigvee_{a' \in A} \sigma^A(a') \otimes R(a', a) = (\sigma^A \circ R)(a) = \sigma^A \circ R^a, \tag{30}$$

$$\tau^{A\backslash R}(R^a) = \bigvee_{a' \in A} R(a, a') \otimes \tau^A(a') = (R \circ \tau^A)(a) = R_a \circ \tau^A, \tag{31}$$

for any $a \in A$.

We can easily prove the following:

Theorem 3.2. *For any fuzzy quasi-order R on a fuzzy recognizer (automaton) $\mathscr{A}$ the afterset fuzzy recognizer (automaton) $\mathscr{A}/R$ and the foreset fuzzy recognizer (automaton) $\mathscr{A}\backslash R$ are isomorphic.*

Proof. This follows immediately by (25), (29) and (b) of Theorem 3.1. $\square$

In view of Theorem 3.2, in the remainder of this paper we will consider only afterset fuzzy recognizers and automata. We will see in Example 4.3 that the factor fuzzy recognizer (automaton) $\mathscr{A}/E_R$ of $\mathscr{A}$, w.r.t. the natural fuzzy equivalence E_R of R, is not necessary isomorphic to fuzzy recognizers $\mathscr{A}/R$ and $\mathscr{A}\backslash R$, but by (b) of Theorem 3.1, it has the same cardinality as $\mathscr{A}/R$ and $\mathscr{A}\backslash R$, and if $L(\mathscr{A}) = L(\mathscr{A}/R) (= L(\mathscr{A}\backslash R))$, then we also have that $L(\mathscr{A}) = L(\mathscr{A}/E_R)$.

If $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy automaton and R is a fuzzy quasi-order on A, then we also define a new fuzzy transition function $\delta^{A|R} : A \times X \times A \to L$ by

$$\delta^{A|R}(a, x, b) = (R \circ \delta_x^A \circ R)(a, b), \quad \text{for all } a, b \in A \text{ and } x \in X,$$

i.e., $\delta_x^{A|R} = R \circ \delta_x^A \circ R$, for each $x \in X$, and we obtain a new fuzzy automaton $\mathscr{A}|R = (A, X, \delta^{A|R})$ with the same set of states and input alphabet as the original one. Furthermore, if $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy recognizer, then we also set $\sigma^{A|R} = \sigma^A$ and $\tau^{A|R} = \tau^A$, and we have that $\mathscr{A}|R = (A, X, \delta^{A|R}, \sigma^{A|R}, \tau^{A|R})$ is a fuzzy recognizer.

The following theorem can be conceived as a version of the well-known Second Isomorphism Theorem, concerning fuzzy automata and fuzzy quasi-orders on them. (cf. [9], §2.6).

Theorem 3.3. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer and let R and S be fuzzy quasi-orders on $\mathscr{A}$ such that $R \leqslant S$. Then a fuzzy relation S/R on A/R defined by*

$$S/R(R_a, R_b) = S(a, b), \qquad \text{for all } a, b \in A, \tag{32}$$

is a fuzzy quasi-order on A/R and fuzzy recognizers $\mathscr{A}/S$, $(\mathscr{A}/R)/(S/R)$ and $(\mathscr{A}|R)/S$ are isomorphic.

Proof. Let $a, a', b, b' \in A$ such that $R_a = R_{a'}$ and $R_b = R_{b'}$, i.e., $E_R(a, a') = E_R(b, b') = 1$. Since $R \leqslant S$, we also have that $R^{-1} \leqslant S^{-1}$, whence $E_R \leqslant E_S$, and by this it follows that $E_S(a, a') = E_S(b, b') = 1$, so $S(a, b) = S(a', b')$. Therefore, S/R is a well-defined fuzzy relation, and clearly, S/R is a fuzzy quasi-order.

For the sake of simplicity set $S/R = Q$. Define a mapping $\phi : A/S \to (A/R)/Q$ by

$$\phi(S_a) = Q_{R_a}, \qquad \text{for every } a \in A.$$

According to Theorem 3.1, for arbitrary $a, b \in A$ we have that

$$S_a = S_b \iff S(a, b) = S(b, a) = 1 \iff Q(R_a, R_b) = Q(R_b, R_a) = 1 \iff Q_{R_a} = Q_{R_b} \iff \phi(S_a) = \phi(S_b),$$

and hence, ϕ is a well-defined and injective function. It is clear that ϕ is also a surjective function. Thus, ϕ is a bijective function of A/S onto $(A/R)/Q$.

Since $R \leqslant S$ implies $R \circ S = S \circ R = S$, for arbitrary $a, b \in A$ and $x \in X$ we have that

$$
\begin{aligned}
\delta_x^{(A/R)/Q}(\phi(S_a), \phi(S_b)) &= \delta_x^{(A/R)/Q}(Q_{R_a}, Q_{R_b}) = (Q \circ \delta_x^{A/R} \circ Q)(R_a, R_b) \\
&= \bigvee_{c,d \in A} Q(R_a, R_c) \otimes \delta_x^{A/R}(R_c, R_d) \otimes Q(R_d, R_b) \\
&= \bigvee_{c,d \in A} S(a, c) \otimes (R \circ \delta_x^A \circ R)(c, d) \otimes S(d, b) \\
&= (S \circ R \circ \delta_x^A \circ R \circ S)(a, b) = (S \circ \delta_x^A \circ S)(a, b) = \delta_x^{A/S}(S_a, S_b).
\end{aligned}
$$

Moreover, for any $a \in A$ we have that

$$\sigma^{(A/R)/Q}(\phi(S_a)) = \sigma^{(A/R)/Q}(Q_{R_a}) = \sigma^{A/R}(R_a) = \sigma^A(a) = \sigma^{A/S}(S_a),$$

and similarly, $\tau^{(A/R)/Q}(\phi(S_a)) = \tau^{A/S}(S_a)$. Therefore, ϕ is an isomorphism of the fuzzy recognizer $\mathscr{A}/S$ onto the fuzzy recognizer $(\mathscr{A}/R)/(S/R)$.

Next, for all $a, b \in A$ and $x \in X$ we have that

$$
\begin{aligned}
\delta^{(A|R)/S}(S_a, x, S_b) &= (S \circ \delta_x^{A|R} \circ S)(a, b) = (S \circ R \circ \delta_x^A \circ R \circ S)(a, b) \\
&= (S \circ \delta_x^A \circ S)(a, b) = \delta^{A/S}(S_a, x, S_b),
\end{aligned}
$$

and $\sigma^{(A|R)/S} = \sigma^{A/S}$, $\tau^{(A|R)/S} = \tau^{A/S}$, so fuzzy recognizers $(\mathscr{A}|R)/S$ and $\mathscr{A}/S$ are isomorphic. $\quad \square$

If in the proof of the previous theorem we disregard fuzzy sets of initial and terminal states, we see that the theorem also hold for fuzzy automata.

Remark 3.1. For any given fuzzy quasi-order R on a fuzzy recognizer $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$, the rule $a \mapsto R_a$ defines a surjective function of A onto A/R. This means that the afterset fuzzy recognizer $\mathscr{A}/R$ has smaller or equal cardinality than the fuzzy recognizer $\mathscr{A}$.

Now, if R and S are fuzzy quasi-orders on $\mathscr{A}$ such that $R \leqslant S$, according to Theorem 3.3, the afterset fuzzy recognizer $\mathscr{A}/S$ has smaller or equal cardinality than $\mathscr{A}/R$. This fact will be frequently used in the rest of the paper.

Let us note that if $\mathscr{A}$ is a fuzzy recognizer or a fuzzy automaton, A is its set of states, and R, S and T are fuzzy quasi-orders on A such that $R \leqslant S$ and $R \leqslant T$, then

$$S \leqslant T \quad \Leftrightarrow \quad S/R \leqslant T/R, \tag{33}$$

and hence, a mapping $\Phi : \mathscr{Q}_R(A) = \{S \in \mathscr{Q}(A) \mid R \leqslant S\} \to \mathscr{Q}(A/R)$, given by $\Phi : S \mapsto S/R$, is injective (in fact, it is an order isomorphism of $\mathscr{Q}_R(A)$ onto a subset of $\mathscr{Q}(A/R)$). In particular, for a fuzzy quasi-order R on A, the fuzzy relation R/R on A/R will be denoted by $\widetilde{R}$. It can be easily verified that $\widetilde{R}$ is a fuzzy order on A/R, and if E is a fuzzy equivalence on A, then $\widetilde{E}$ is a fuzzy equality on A/E.

For a fuzzy recognizer $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and a fuzzy quasi-order R on A we have that the fuzzy language $L(\mathscr{A}/R)$ recognized by the afterset fuzzy recognizer $\mathscr{A}/R$ is given by

$$
\begin{aligned}
L(\mathscr{A}/R)(e) &= \sigma^A \circ R \circ \tau^A, \\
L(\mathscr{A}/R)(u) &= \sigma^A \circ R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ R \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \tau^A,
\end{aligned}
\tag{34}
$$

whereas the fuzzy language $L(\mathscr{A})$ recognized by $\mathscr{A}$ is given by

$$
\begin{aligned}
L(\mathscr{A})(e) &= \sigma^A \circ \tau^A, \\
L(\mathscr{A})(u) &= \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A,
\end{aligned}
$$

for any $u = x_1 x_2 \ldots x_n \in X^+$, where $x_1, x_2, \ldots, x_n \in X$. Let us note that the equation (34) follows immediately by definition of the afterset fuzzy recognizer $\mathscr{A}/R$ (the equations (26), (27) and (28)), by the equations (10) and (14), and the fact that $R \circ R = R$, for every fuzzy quasi-order R. Hence, the fuzzy recognizer $\mathscr{A}$ and the afterset fuzzy recognizer $\mathscr{A}/R$ are *equivalent*, i.e., they recognize the same fuzzy language, if and only if the fuzzy quasi-order R is a solution to a system of fuzzy relation equations

$$
\begin{aligned}
\sigma^A \circ \tau^A &= \sigma^A \circ R \circ \tau^A, \\
\sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A &= \sigma^A \circ R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ R \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \tau^A,
\end{aligned}
\tag{35}
$$

for all $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. We will call (35) the *general system*.

The general system has at least one solution in $\mathscr{Q}(A)$, the equality relation on A. It will be called the *trivial solution*. To attain the best possible reduction of $\mathscr{A}$, we have to find the greatest solution to the general system in $\mathscr{Q}(A)$, if it exists, or to find as big a solution as possible. However, the general system does not necessary have the greatest solution (see Example 3.2), and also, it may consist of infinitely many equations, and finding its nontrivial solutions may be a very difficult task. For that reason we will aim our attention to some instances of the general system. These instances have to be as general as possible, but they have to be easier to solve. From a practical point of view, these instances have to consist of finitely many equations.

The following theorem describes some properties of the set of all solutions to the general system.

Theorem 3.4. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer.*

The set of all solutions to the general system in $\mathscr{Q}(A)$ is an order ideal of the lattice $\mathscr{Q}(A)$.

Consequently, if a fuzzy quasi-order R on A is a solution to the general system, then its natural fuzzy equivalence E_R is also a solution to the general system.

Proof. Consider arbitrary $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in X$, and fuzzy quasi-orders R and S on A such that S is a solution to the general system and $R \leqslant S$. By the facts that S is a solution to the general system and $R \leqslant S$, by reflexivity of R, and by (11) we obtain that

$$\sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A \leqslant \sigma^A \circ R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ R \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \tau^A$$
$$\leqslant \sigma^A \circ S \circ \delta^A_{x_1} \circ S \circ \delta^A_{x_2} \circ S \circ \cdots \circ S \circ \delta^A_{x_n} \circ S \circ \tau^A$$
$$= \sigma^A \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \tau^A,$$

and hence, R is a solution to the general system. By this it follows that solutions to the general system in $\mathcal{Q}(A)$ form an order ideal of the lattice $\mathcal{Q}(A)$.

The second part of the theorem follows immediately by the fact that $E_R = R \wedge R^{-1} \leqslant R$. $\quad\square$

The following example shows that there are fuzzy quasi-orders which are not solutions to the general system, but their natural fuzzy equivalences are solutions to this system.

Example 3.1. Let $\mathscr{L}$ be the Boolean structure, let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x, y\}$, and $\delta^A_x, \delta^A_y, \sigma^A$ and τ^A are given by

$$\delta^A_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta^A_y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

and consider a fuzzy quasi-order R on A given by

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we have that

$$\sigma^A \circ R \circ \delta^A_x \circ R \circ \delta^A_y \circ R \circ \tau^A = 1 \neq 0 = \sigma^A \circ \delta^A_x \circ \delta^A_y \circ \tau^A,$$

so R is not a solution to the general system, but its natural fuzzy equivalence E_R is the equality relation on A, and hence, it is a solution to the general system.

The next example shows that the general system does not necessary have the greatest solution.

Example 3.2. Let $\mathscr{L}$ be the Boolean structure, let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x\}$, and δ^A_x, σ^A and τ^A are given by

$$\delta^A_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and consider fuzzy quasi-orders (in fact, fuzzy equivalences) E and F on A given by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

We have that both E and F are solutions to the general system (since E is right invariant and F is left invariant, see the next section for details). On the other hand, the join of E and F in the lattice $\mathcal{Q}(A)$ is a fuzzy quasi-order U given by

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and it is not a solution to the general system, since

$$\sigma^A \circ U \circ \delta_x^A \circ U \circ \delta_x^A \circ U \circ \tau^A = 1 \neq 0 = \sigma^A \circ \delta_x^A \circ \delta_x^A \circ \tau^A.$$

If the general system would have the greatest solution R in $\mathcal{Q}(A)$, then $E \leqslant R$ and $F \leqslant R$ would imply $U \leqslant R$, and by Theorem 3.4 we would obtain that U is a solution to the general system. Hence, we conclude that the general system does not have the greatest solution in $\mathcal{Q}(A)$.

The next theorem demonstrates one shortcoming of state reductions by means of fuzzy quasi-orders and fuzzy equivalences. Namely, we show that for some fuzzy recognizers no reduction will result in its minimal automaton.

Theorem 3.5. *There exists a fuzzy automaton $\mathscr{A}$ such that no reduction of $\mathscr{A}$ by means of fuzzy quasi-orders provide a minimal fuzzy recognizer.*

Proof. Let $\mathscr{L}$ be the Boolean structure and $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ a fuzzy recognizer over $\mathscr{L}$, where $|A| = 4$, $X = \{x\}$, and δ^A, σ^A, and τ^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \sigma^A = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \tau^A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

It is easy to check that for each $u \in X^*$ the following is true:

$$L(\mathscr{A})(u) = \begin{cases} 0 & \text{if } u = e \text{ or } u = x^n, \text{for } n \geqslant 2, \\ 1 & \text{if } u = x, \end{cases}$$

(in fact, $\mathscr{A}$ is a nondeterministic recognizer and $L(\mathscr{A})$ is an ordinary crisp language consisting only of the letter x). If $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$ is a fuzzy recognizer over $\mathscr{L}$ with $|B| = 2$, and

$$\delta_x^B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \sigma^B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \tau^B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

then $\mathscr{B}$ recognizes $L(\mathscr{A})$, and it is a minimal fuzzy recognizer of $L(\mathscr{A})$, since $L(\mathscr{A})$ can not be recognized by a fuzzy recognizer with only one state.

Consider now an arbitrary fuzzy equivalence

$$E = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ a_{12} & 1 & a_{23} & a_{24} \\ a_{13} & a_{23} & 1 & a_{34} \\ a_{14} & a_{24} & a_{34} & 1 \end{bmatrix}$$

on A, and suppose that E is a solution to the general system corresponding to the fuzzy automaton $\mathscr{A}$. We will show that E can not reduce $\mathscr{A}$ to a fuzzy recognizer with two states.

First, by $\sigma^A \circ E \circ \tau^A = a_{23} \vee a_{24}$ and $\sigma^A \circ E \circ \tau^A = \sigma^A \circ \tau^A = L(\mathscr{A})(e) = 0$ it follows $a_{23} = a_{24} = 0$. Next, reflexivity and transitivity of E yield $E \circ E = E$, what implies

$$\begin{array}{llr}
a_{12} \wedge a_{13} = 0, & a_{12} = 0 \text{ or } a_{13} = 0 & (36) \\
a_{12} \wedge a_{14} = 0, & a_{12} = 0 \text{ or } a_{14} = 0 & (37) \\
a_{13} \vee (a_{14} \wedge a_{34}) = a_{13}, \quad \text{i.e.,} & a_{13} = 0 \text{ implies } a_{14} = 0 \text{ or } a_{34} = 0, & (38) \\
a_{14} \vee (a_{13} \wedge a_{34}) = a_{14}, & a_{14} = 0 \text{ implies } a_{13} = 0 \text{ or } a_{34} = 0, & (39) \\
a_{34} \vee (a_{13} \wedge a_{14}) = a_{34}, & a_{34} = 0 \text{ implies } a_{13} = 0 \text{ or } a_{14} = 0. & (40)
\end{array}$$

14

If $a_{12} = 1$, then by (36) and (37) we obtain $a_{13} = a_{14} = 0$, and hence

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

However, none of these two matrices is a solution to the general system. Therefore, we conclude that $a_{12} = 0$. According to (38), (39) and (40), we distinguish the following five cases

$$a_{13} = a_{14} = a_{34} = 0,$$
$$a_{13} = a_{14} = 0, \quad a_{34} = 1,$$
$$a_{13} = a_{34} = 0, \quad a_{14} = 1,$$
$$a_{14} = a_{34} = 0, \quad a_{13} = 1,$$
$$a_{13} = a_{14} = a_{34} = 1,$$

and we obtain that E has one of the following forms

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}. \quad (41)$$

In the first case, E is the equality relation, and it does not provide any reduction of $\mathscr{A}$, and in the second and fourth case, it can be easily verified that E is a solution to the general system, but it reduces $\mathscr{A}$ to a fuzzy recognizer with three states. Finally, in the third and fifth case, E is not a solution to the general system, since

$$\sigma^A \circ E \circ \delta_x^A \circ E \circ \delta_x^A \circ E \circ \tau_x^A = 1 \neq 0 = \sigma^A \circ \delta_x^A \circ \delta_x^A \circ \tau_x^A.$$

Therefore, any state reduction of $\mathscr{A}$ by means of fuzzy equivalences does not provide fuzzy recognizer with less than three states. According to (b) of Theorem 3.1, the same conclusion also holds for fuzzy quasi-orders. This completes the proof of the theorem. $\square$

4. Right and left invariant fuzzy quasi-orders

As in [24, 25], where similar questions concerning fuzzy equivalences have been considered, here we study the following two instances of the general system. Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton. If a fuzzy quasi-order R on A is a solution to a system

$$R \circ \delta_x^A \circ R = \delta_x^A \circ R, \ \text{ for every } x \in X, \quad (42)$$

then it will be called a *right invariant fuzzy quasi-order* on $\mathscr{A}$, and if it is a solution to a system

$$R \circ \delta_x^A \circ R = R \circ \delta_x^A, \ \text{ for every } x \in X, \quad (43)$$

then it will be called a *left invariant fuzzy quasi-order* on $\mathscr{A}$. A crisp quasi-order on A which is a solution to (42) is called a *right invariant quasi-order* on $\mathscr{A}$, and a crisp quasi-order which is a solution to (43) is called a *left invariant quasi-order* on $\mathscr{A}$. Let us note that a fuzzy quasi-order on A is both right and left invariant if and only if it is a solution to system

$$R \circ \delta_x^A = \delta_x^A \circ R, \ \text{ for every } x \in X, \quad (44)$$

and then it is called an *invariant fuzzy quasi-order*.

15

If $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy recognizer, then by a *right invariant fuzzy quasi-order* on $\mathscr{A}$ we mean a fuzzy quasi-order R on A which is a solution to (42) and

$$R \circ \tau^A = \tau^A, \tag{45}$$

and a *left invariant fuzzy quasi-order* on $\mathscr{A}$ is a fuzzy quasi-order R on A which is a solution to (43) and

$$\sigma^A \circ R = \sigma^A. \tag{46}$$

It is clear that all right and left invariant fuzzy quasi-orders on a fuzzy recognizer $\mathscr{A}$ are solutions of the general system (35), and hence, the corresponding afterset fuzzy automata are equivalent to $\mathscr{A}$.

In other words, right (resp. left) invariant fuzzy quasi-orders on the fuzzy recognizer $\mathscr{A}$ are exactly those right (resp. left) invariant fuzzy quasi-orders on the fuzzy automaton (A, X, δ^A) which are solutions to the fuzzy relation equation (45) (resp. (46)). It is well-known (see [23, 63, 64, 65, 74]) that solutions to (45) (resp. (46)) in $\mathscr{Q}(A)$ form a principal ideal of $\mathscr{Q}(A)$ whose greatest element is a fuzzy quasi-order R^τ (resp. R_σ) defined by (24) (here we write $\tau^A = \tau$ and $\sigma^A = \sigma$). This means that right (resp. left) invariant fuzzy quasi-orders on the fuzzy recognizer $\mathscr{A}$ are those right (resp. left) invariant fuzzy quasi-orders on the fuzzy automaton (A, X, δ^A) which are contained in R^τ (resp. R_σ).

Let us note that fuzzy equivalences satisfying (42) and (43) have been studied in [24, 25]. They are respectively called right and left invariant fuzzy equivalences. Right and left invariant quasi-orders have been used for the state reduction of non-deterministic automata by Champarnaud and Coulon [17, 18], Ilie, Navarro and Yu [39], and Ilie, Solis-Oba and Yu [40] (see also [36, 37]).

By the following theorem we give a characterization of right invariant fuzzy quasi-orders:

Theorem 4.1. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton and R a fuzzy quasi-order on A. Then the following conditions are equivalent:*

  (i) *R is a right invariant fuzzy quasi-order;*
 (ii) *$R \circ \delta_x^A \leqslant \delta_x^A \circ R$, for every $x \in X$;*
(iii) *for all $a, b \in A$ we have*

$$R(a, b) \leqslant \bigwedge_{x \in X} \bigwedge_{c \in A} (\delta_x^A \circ R)(b, c) \to (\delta_x^A \circ R)(a, c). \tag{47}$$

Proof. (i)$\Leftrightarrow$(ii). Consider an arbitrary $x \in X$. If $R \circ \delta_x^A \circ R = \delta_x^A \circ R$, then by reflexivity of R it follows

$$R \circ \delta_x^A \leqslant R \circ \delta_x^A \circ R = \delta_x^A \circ R.$$

Conversely, if $R \circ \delta_x^A \leqslant \delta_x^A \circ R$ then $R \circ \delta_x^A \circ R \leqslant \delta_x^A \circ R \circ R = \delta_x^A \circ R$, and since the opposite inequality follows by reflexivity of R, we conclude that $R \circ \delta_x^A \circ R = \delta_x^A \circ R$.

(i)$\Rightarrow$(iii). Let R be a right invariant fuzzy equivalence. Then for all $x \in X$ and $a, b, c \in A$ we have that

$$R(a, b) \otimes (\delta_x^A \circ R)(b, c) \leqslant (R \circ \delta_x^A \circ R)(a, c) = (\delta_x^A \circ R)(a, c),$$

and by the adjunction property we obtain that $R(a, b) \leqslant (\delta_x^A \circ R)(b, c) \to (\delta_x^A \circ R)(a, c)$. Hence,

$$R(a, b) \leqslant (\delta_x^A \circ R)(b, c) \to (\delta_x^A \circ R)(a, c). \tag{48}$$

Since (48) is satisfied for all $c \in A$ and $x \in X$, we conclude that (47) holds.

(iii)$\Rightarrow$(i). If (iii) holds, then for arbitrary $x \in X$ and $a, b, c \in A$ we have that

$$R(a, b) \leqslant (\delta_x^A \circ R)(b, c) \to (\delta_x^A \circ R)(a, c),$$

16

and by the adjunction property we obtain that $R(a,b) \otimes (\delta_x^A \circ R)(b,c) \leqslant (\delta_x^A \circ R)(a,c)$. Now,

$$(R \circ \delta_x^A \circ R)(a,c) = \bigvee_{b \in A} R(a,b) \otimes (\delta_x^A \circ R)(b,c) \leqslant (\delta_x^A \circ R)(a,c),$$

whence $R \circ \delta_x^A \circ R \leqslant \delta_x^A \circ R$, and since the opposite inequality follows immediately by reflexivity of R, we conclude that $R \circ \delta_x^A \circ R = \delta_x^A \circ R$, for every $x \in X$, i.e., R is a right invariant fuzzy quasi-order. $\qquad\square$

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton and R a fuzzy quasi-order on A. Let us define a fuzzy relation R^r on A by

$$R^r(a,b) = \bigwedge_{x \in X} \bigwedge_{c \in A} (\delta_x^A \circ R)(b,c) \rightarrow (\delta_x^A \circ R)(a,c), \tag{49}$$

for all $a, b \in A$. Since R^r is an intersection of a family of fuzzy quasi-orders defined as in (24), we have that R^r is also a fuzzy quasi-order. According to Theorem 4.1, R is a right invariant fuzzy quasi-order if and only if $R \leqslant R^r$.

Moreover, we have the following:

Lemma 4.1. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton, and let R and S be fuzzy quasi-orders on A.*
If $R \leqslant S$, then $R^r \leqslant S^r$.

Proof. Consider arbitrary $a, b \in A$ and $x \in X$. By $R \leqslant S$ it follows $R \circ S = S$, and by (3), for arbitrary $c, d \in A$ we have that

$$(\delta_x^A \circ R)(b,c) \rightarrow (\delta_x^A \circ R)(a,c) \leqslant (\delta_x \circ R)(b,c) \otimes S(c,d) \rightarrow (\delta_x^A \circ R)(a,c) \otimes S(c,d).$$

Now, by (6) we obtain that

$$
\begin{aligned}
R^r(a,b) &\leqslant \bigwedge_{c \in A} (\delta_x^A \circ R)(b,c) \rightarrow (\delta_x^A \circ R)(a,c) \\
&\leqslant \bigwedge_{c \in A} \Big[(\delta_x^A \circ R)(b,c) \otimes S(c,d) \rightarrow (\delta_x^A \circ R)(b,c) \otimes S(c,d) \Big] \\
&\leqslant \Big[\bigvee_{c \in A} (\delta_x^A \circ R)(b,c) \otimes S(c,d) \Big] \rightarrow \Big[\bigvee_{c \in A} (\delta_x^A \circ R)(a,c) \otimes S(c,d) \Big] \\
&= (\delta_x^A \circ R \circ S)(b,d) \rightarrow (\delta_x^A \circ R \circ S)(a,d) = (\delta_x^A \circ S)(b,d) \rightarrow (\delta_x^A \circ S)(a,d).
\end{aligned}
$$

Since this holds for all $x \in X$ and $d \in A$, we conclude that

$$R^r(a,b) \leqslant \bigwedge_{x \in X} \bigwedge_{d \in A} (\delta_x^A \circ S)(b,d) \rightarrow (\delta_x^A \circ S)(a,d) = S^r(a,b),$$

and hence, $R^r \leqslant S^r$. $\qquad\square$

Now we prove the following:

Theorem 4.2. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton and let $\mathscr{A}' = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer belonging to $\mathscr{A}$. Then*

(a) *The set $\mathscr{Q}^{ri}(\mathscr{A})$ of all right invariant fuzzy quasi-orders on $\mathscr{A}$ forms a complete lattice, which is a complete join-subsemilattice of the lattice $\mathscr{Q}(A)$ of all fuzzy quasi-orders on A.*

(b) *The set $\mathscr{Q}^{cri}(\mathscr{A})$ of all right invariant crisp quasi-orders on $\mathscr{A}$ forms a complete lattice, which is a complete join-subsemilattice of the lattice $\mathscr{Q}^{ri}(A)$.*

(c) *The set $\mathscr{Q}^{ri}(\mathscr{A}')$ of all right invariant fuzzy quasi-orders on $\mathscr{A}'$ is a principal ideal of the lattice $\mathscr{Q}^{ri}(\mathscr{A})$.*

17

Proof. (a) Let $\{R_i\}_{i \in I} \subseteq \mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$, and let R be the join of this family in $\mathscr{Q}(A)$. Then for each $i \in I$, by $R_i \leqslant R$ and Lemma 4.1 we obtain that $R_i \leqslant R_i^r \leqslant R^r$, whence $R \leqslant R^r$. Now, by Theorem 4.1 it follows that $R \in \mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$, and hence, $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$ is a complete join-subsemilattice of $\mathscr{Q}(\mathscr{A})$. Since $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$ contains the least element of $\mathscr{Q}(A)$, the equality relation on A, we conclude that $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$ is a complete lattice.

(b) This follows immediately by (a) and (15), since union and composition of fuzzy relations, applied to crisp relations, as results give crisp relations.

(c) By definition, $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A}')$ consists of all $R \in \mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$ which satisfy $R \circ \tau = \tau$. It is well-known that $R \circ \tau = \tau$ is equivalent to $R \leqslant R_\tau$, what implies that $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A}')$ is an ideal of $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$. Next, let $\{R_i\}_{i \in I}$ be an arbitrary family of elements of $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A}')$ and let R be the join of this family in $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$. According to (a) of this theorem, R is also the join of the family $\{R_i\}_{i \in I}$ in $\mathscr{Q}(\mathscr{A})$, and since $R_i \leqslant R_\tau$, for every $i \in I$, we conclude that $R \leqslant R_\tau$. By this it follows that, $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A}')$ is a complete join-subsemilattice of $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$, and hence, $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A}')$ is an ideal of $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$ having the greatest element, what means that $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A}')$ is a principal ideal of $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$. $\square$

As we have noted before, the problem of computing the greatest right invariant fuzzy quasi-order on a fuzzy recognizer $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ one reduces to the problem of computing the greatest right invariant fuzzy quasi-order on a fuzzy automaton (A, X, δ^A) contained in the fuzzy quasi-order R^τ ($\tau = \tau^A$). For that reason, in the sequel we consider the problem how to construct the greatest right invariant fuzzy quasi-order R^{ri} contained in a given fuzzy quasi-order R on a fuzzy automaton.

Theorem 4.3. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton, let R be a fuzzy quasi-order on $\mathscr{A}$ and let R^{ri} be the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R.*

Define inductively a sequence $\{R_k\}_{k \in \mathbb{N}}$ of fuzzy quasi-orders on $\mathscr{A}$ as follows:

$$R_1 = R, \quad R_{k+1} = R_k \wedge R_k^r, \quad \text{for each } k \in \mathbb{N}. \tag{50}$$

Then

(a) $R^{\mathrm{ri}} \leqslant \cdots \leqslant R_{k+1} \leqslant R_k \leqslant \cdots \leqslant R_1 = R$;

(b) *If $R_k = R_{k+m}$, for some $k, m \in \mathbb{N}$, then $R_k = R_{k+1} = R^{\mathrm{ri}}$;*

(c) *If $\mathscr{A}$ is finite and $\mathscr{L}$ is locally finite, then $R_k = R^{\mathrm{ri}}$ for some $k \in \mathbb{N}$.*

Proof. (a) Clearly, $R_{k+1} \leqslant R_k$, for each $k \in \mathbb{N}$, and $R^{\mathrm{ri}} \leqslant R_1$. Suppose that $R^{\mathrm{ri}} \leqslant R_k$, for some $k \in \mathbb{N}$. Then $R^{\mathrm{ri}} \leqslant (R^{\mathrm{ri}})^r \leqslant R_k^r$, so $R^{\mathrm{ri}} \leqslant R_k \wedge R_k^r = R_{k+1}$. Therefore, by induction we obtain that $R^{\mathrm{ri}} \leqslant R_k$, for every $k \in \mathbb{N}$.

(b) Let $R_k = R_{k+m}$, for some $k, m \in \mathbb{N}$. Then $R_k = R_{k+m} \leqslant R_{k+1} = R_k \wedge R_k^r \leqslant R_k^r$, what means that R_k is a right invariant fuzzy quasi-order. Since R^{ri} is the greatest right invariant fuzzy quasi-order contained in R, we conclude that $R_k = R_{k+1} = R^{\mathrm{ri}}$.

(c) Let $\mathscr{A}$ be a finite fuzzy automaton and $\mathscr{L}$ be a locally finite algebra. Let the carrier of a subalgebra of $\mathscr{L}$ generated by the set $\delta^A(A \times X \times A) \cup R(A \times A)$ be denoted by $L_{\mathscr{A}}$. This generating set is finite, so $L_{\mathscr{A}}$ is also finite, and hence, the set $L_{\mathscr{A}}^{A \times A}$ of all fuzzy relations on A with values in $L_{\mathscr{A}}$ is finite. By definitions of fuzzy relations R_k and R_k^r we have that $R_k \in L_{\mathscr{A}}^{A \times A}$, what implies that there exist $k, n \in \mathbb{N}$ such that $R_k = R_{k+m}$, and by (b) we conclude that $R_k = R^{\mathrm{ri}}$. $\square$

According to (c) of Theorem 4.3, if the structure $\mathscr{L}$ is locally finite, then for every fuzzy automaton $\mathscr{A}$ over $\mathscr{L}$ we have that every sequence of fuzzy quasi-orders defined by (50) is finite. However, this does not necessary hold if $\mathscr{L}$ is not locally finite, as the following example shows:

Example 4.1. Let $\mathscr{L}$ be the Goguen (product) structure and $\mathscr{A} = (A, X, \delta^A)$ a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2\}$, $X = \{x\}$, and δ_x^A is given by

$$\delta_x^A = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix},$$

and let R be the universal relation on A. Applying to R the procedure from Theorem 4.3, we obtain a sequence $\{R_k\}_{k \in \mathbb{N}}$ of fuzzy quasi-orders given by

$$R_k = \begin{bmatrix} 1 & 1 \\ \frac{1}{2^{k-1}} & 1 \end{bmatrix}, \quad k \in \mathbb{N},$$

whose all members are different, i.e., this sequence is infinite. We also have that the greatest right invariant fuzzy quasi-order contained in R is given by

$$R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

For a fuzzy automaton $\mathscr{A} = (A, X, \delta^A)$ over a complete residuated lattice $\mathscr{L}$, the greatest left invariant fuzzy quasi-order R^{li} contained in a given fuzzy quasi-order R on A can be computed in a similar way as R^{ri}. Indeed, inductively we define a sequence $\{R_k\}_{k \in \mathbb{N}}$ of fuzzy quasi-orders on A by

$$R_1 = R, \quad R_{k+1} = R_k \wedge R_k^l, \quad \text{for each } k \in \mathbb{N}, \tag{51}$$

where R_k^l is a fuzzy quasi-order on A defined by

$$R_k^l(a, b) = \bigwedge_{x \in X} \bigwedge_{c \in A} (R_k \circ \delta_x^A)(c, a) \rightarrow (R_k \circ \delta_x^A)(c, b), \qquad \text{for all } a, b \in A.$$

If $\mathscr{L}$ is locally finite, then this sequence is necessary finite and R^{li} equals the least element of this sequence.

It is worth noting that the greatest right and left invariant fuzzy quasi-orders are calculated using iterative procedures, but these calculations are not approximative. Whenever these procedures terminate in a finite number of steps, exact solutions to the considered systems of fuzzy relation equations are obtained.

Note also that for a fuzzy automaton $\mathscr{A} = (A, X, \delta^A)$ over a complete residuated lattice $\mathscr{L}$, in [24, 25] we gave a procedure for computing the greatest right invariant fuzzy equivalence E^{rie} contained in a given fuzzy equivalence E on A. This procedure is similar to the procedure given in Theorem 4.3 for fuzzy quasi-orders, and it also works for all fuzzy finite automata over a locally finite complete residuated lattice. Namely, inductively we define a sequence $\{E_k\}_{k \in \mathbb{N}}$ of fuzzy equivalences on A by

$$E_1 = E, \quad E_{k+1} = E_k \wedge E_k^{req}, \quad \text{for each } k \in \mathbb{N}, \tag{52}$$

where E_k^{req} is a fuzzy equivalence defined by

$$E_k^{req}(a, b) = \bigwedge_{x \in X} \bigwedge_{c \in A} (\delta_x^A \circ E_k)(a, c) \leftrightarrow (\delta_x^A \circ E_k)(b, c), \qquad \text{for all } a, b \in A.$$

It was proved in [24, 25] that if $\mathscr{L}$ is locally finite, then this sequence is necessary finite and E^{rie} equals the least element of this sequence.

By the next example we show that it is possible that the sequence of fuzzy equivalences defined by (52) is infinite, but the sequence of fuzzy quasi-orders defined by (50) is finite.

Example 4.2. Let $\mathscr{L}$ be the Goguen (product) structure and $\mathscr{A} = (A, X, \delta^A)$ a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x\}$, and δ_x^A is given by

$$\delta_x^A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

If we start from the universal relation on A, applying the rule (52) we obtain an infinite sequence $\{E_k\}_{k \in \mathbb{N}}$ of fuzzy equivalences on A, where

$$E_k = \begin{bmatrix} 1 & 1 & \frac{1}{2^{k-1}} \\ 1 & 1 & \frac{1}{2^{k-1}} \\ \frac{1}{2^{k-1}} & \frac{1}{2^{k-1}} & 1 \end{bmatrix}, \quad k \in \mathbb{N}.$$

19

On the other hand, if we also start from the universal relation, the rule (50) gives a finite sequence $\{R_k\}_{k \in \mathbb{N}}$ of fuzzy quasi-orders on A, where

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \quad R_k = R_2, \text{ for each } k \in \mathbb{N}, k \geqslant 3.$$

Reduction of fuzzy automata by means of right and left invariant fuzzy equivalences has been studied in [24, 25]. Since the set of all right invariant fuzzy equivalences is a subset of the set of all right invariant fuzzy quasi-orders, the greatest element of this subset (the greatest right invariant fuzzy equivalence) is less or equal than the greatest element of the whole set (the greatest right invariant fuzzy quasi-order). The next example shows that this inequality can be strict. Thus, reduction of a fuzzy automaton by using the greatest right invariant fuzzy quasi-order gives better results than its reduction by using the greatest right invariant fuzzy equivalence, according to Remark 3.1.

Furthermore, as we have shown by Theorem 3.4, if a fuzzy quasi-order R on a fuzzy automaton $\mathscr{A}$ is a solution to the general system, then its natural fuzzy equivalence E_R is also a solution to the general system. But, the next example also shows that if R is a right invariant fuzzy quasi-order, then E_R is not necessary a right invariant fuzzy equivalence.

Example 4.3. Let $\mathscr{L}$ be the Boolean structure, and let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x, y\}$, and fuzzy transition relations δ_x^A and δ_y^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The greatest right invariant fuzzy quasi order R^{ri} on $\mathscr{A}$, its natural fuzzy equivalence $E_{R^{\mathrm{ri}}}$, and the greatest right invariant fuzzy equivalence E^{ri} on $\mathscr{A}$ are given by

$$R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E_{R^{\mathrm{ri}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad E^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, E^{ri} do not reduce the number of states of $\mathscr{A}$, but R^{ri} reduces $\mathscr{A}$ to a fuzzy automaton with two states.

Moreover, R^{ri} is a right invariant fuzzy quasi-order, but its natural fuzzy equivalence $E_{R^{\mathrm{ri}}}$ is not a right invariant fuzzy equivalence, because $E^{\mathrm{ri}} < E_{R^{\mathrm{ri}}}$. We also have that the afterset fuzzy automaton $\mathscr{A}/R^{\mathrm{ri}}$ is not isomorphic to the factor fuzzy automaton $\mathscr{A}/E_{R^{\mathrm{ri}}}$, since

$$R^{\mathrm{ri}} \circ \delta_y^A \circ R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad E_{R^{\mathrm{ri}}} \circ \delta_y^A \circ E_{R^{\mathrm{ri}}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Next we consider the case when $\mathscr{L}$ is a complete residuated lattice satisfying the following conditions:

$$x \vee \left(\bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \vee y_i), \tag{53}$$

$$x \otimes \left(\bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \otimes y_i), \tag{54}$$

for all $x \in L$ and $\{y_i\}_{i \in I} \subseteq L$. Let us note that if $\mathscr{L} = ([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$, where $[0, 1]$ is the real unit interval and $\otimes$ is a left-continuous t-norm on $[0, 1]$, then (53) follows immediately by linearity of $\mathscr{L}$, and $\mathscr{L}$ satisfies (54) if and only if $\otimes$ is a continuous t-norm, i.e., if and only if $\mathscr{L}$ is a *BL*-algebra (cf. [3, 4]). Therefore, conditions (53) and (54) hold for every *BL*-algebra on the real unit interval. In particular, the Łukasiewicz, Goguen (product) and Gödel structures fulfill (53) and (54).

We have that the following is true:

20

Theorem 4.4. *Let $\mathscr{L}$ be a complete residuated lattice satisfying (53) and (54), let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy finite automaton over $\mathscr{L}$, let R be a fuzzy quasi-order on A, let R^{ri} be the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ contained in R, and let $\{R_k\}_{k\in\mathbb{N}}$ be the sequence of fuzzy quasi-orders on A defined by (50). Then*

$$R^{\mathrm{ri}} = \bigwedge_{k\in\mathbb{N}} R_k. \tag{55}$$

Proof. It was proved in [25] that if (53) holds, then for all non-increasing sequences $\{x_k\}_{k\in\mathbb{N}}, \{y_k\}_{k\in\mathbb{N}} \subseteq L$ we have

$$\bigwedge_{k\in\mathbb{N}}(x_k \vee y_k) = \left(\bigwedge_{k\in\mathbb{N}} x_k\right) \vee \left(\bigwedge_{k\in\mathbb{N}} y_k\right). \tag{56}$$

For the sake of simplicity set

$$S = \bigwedge_{k\in\mathbb{N}} R_k.$$

Clearly, S is a fuzzy quasi-order. To prove (55) it is enough to prove that S is a right invariant fuzzy quasi-order on $\mathscr{A}$. First, we have that

$$S(a,b) \leqslant R_{k+1}(a,b) \leqslant R_k^r(a,b) \leqslant \delta_x^A \circ R_k(b,c) \to \delta_x^A \circ R_k(a,c), \tag{57}$$

holds for all $a, b, c \in A$, $x \in X$ and $k \in \mathbb{N}$. Now, by (57) and (5) we obtain that

$$S(a,b) \leqslant \bigwedge_{k\in\mathbb{N}}\left(\delta_x^A \circ R_k(b,c) \to \delta_x^A \circ R_k(a,c)\right) \leqslant \bigwedge_{k\in\mathbb{N}}\left(\delta_x^A \circ R_k(b,c)\right) \to \bigwedge_{k\in\mathbb{N}}\left(\delta_x^A \circ R_k(a,c)\right), \tag{58}$$

for all $a, b, c \in A$ and $x \in X$. Next,

$$
\begin{aligned}
\bigwedge_{k\in\mathbb{N}}\left(\delta_x^A \circ R_k(b,c)\right) &= \bigwedge_{k\in\mathbb{N}}\left(\bigvee_{d\in A}\left(\delta_x^A(b,d) \otimes R_k(d,c)\right)\right) \\
&= \bigvee_{d\in A}\left(\bigwedge_{k\in\mathbb{N}}\left(\delta_x^A(b,d) \otimes R_k(d,c)\right)\right) \qquad \text{(by (56))} \\
&= \bigvee_{d\in A}\left(\delta_x^A(b,d) \otimes \left(\bigwedge_{k\in\mathbb{N}} R_k(d,c)\right)\right) \qquad \text{(by (54))} \\
&= \bigvee_{d\in A}\left(\delta_x^A(b,d) \otimes S(d,c)\right) = (\delta_x^A \circ S)(b,c).
\end{aligned}
\tag{59}
$$

Use of condition (56) is justified by the facts that A is finite, and that $\{R_k(d,c)\}_{k\in\mathbb{N}}$ is a non-increasing sequence, so $\{\delta_x^A(b,d) \otimes R_k(d,c)\}_{k\in\mathbb{N}}$ is also a non-increasing sequence. In the same way we prove that

$$\bigwedge_{k\in\mathbb{N}}\left(\delta_x^A \circ R_k(a,c)\right) = (\delta_x^A \circ S)(a,c). \tag{60}$$

Therefore, by (58), (59) and (60) we obtain that

$$S(a,b) \leqslant (\delta_x^A \circ S)(b,c) \to (\delta_x^A \circ S)(a,c).$$

Since this inequality holds for all $x \in X$ and $c \in A$, we have that

$$S(a,b) \leqslant \bigwedge_{x\in X}\bigwedge_{c\in A}(\delta_x^A \circ S)(b,c) \to (\delta_x^A \circ S)(a,c),$$

and by (iii) of Theorem 4.1 we obtain that S is a right invariant fuzzy quasi-order on $\mathscr{A}$. $\square$

21

5. Some special types of right and left invariant fuzzy quasi-orders

For a given fuzzy quasi-order R on a fuzzy automaton $\mathscr{A}$, Theorem 4.3 gives a procedure for computing R^{ri} in case when the complete residuated lattice $\mathscr{L}$ is locally finite, and Theorem 4.4 characterizes R^{ri} in case when $\mathscr{L}$ satisfies some additional distributivity conditions. But, what to do if $\mathscr{L}$ do not satisfy any of these conditions? In this case we could consider some subset of $\mathscr{Q}^{\mathrm{ri}}(\mathscr{A})$ whose greatest element can be effectively computed when $\mathscr{L}$ is any complete residuated lattice. Here we consider two such subsets. The first one is the set $\mathscr{Q}^{\mathrm{cri}}(\mathscr{A})$ of all right invariant crisp quasi-orders on $\mathscr{A}$, and the second one is the set $\mathscr{Q}^{\mathrm{sri}}(\mathscr{A})$ of strongly right invariant fuzzy quasi-orders, which will be defined latter.

Note that for a crisp relation ϱ and a fuzzy relation R on a set A we have that $\varrho \leqslant R$ if and only if $\varrho \subseteq \widehat{R}$, where $\widehat{R}$ denotes the crisp part of R. Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton and R a fuzzy quasi-order on A. It is easy to verify that the crisp part of the fuzzy quasi-order R^r can be represented as follows: for all $a, b \in A$ we have

$$(a, b) \in \widehat{R^r} \quad \Leftrightarrow \quad (\forall x \in X)(\forall c \in A)\, (\delta_x^A \circ R)(b, c) \leqslant (\delta_x^A \circ R)(a, c). \tag{61}$$

We have that $\widehat{R^r}$ is a quasi-order, since the crisp part of any fuzzy quasi-order is a quasi-order.

The following theorem gives a procedure for computing the greatest right invariant crisp quasi-order on a fuzzy automaton contained in a given quasi-order.

Theorem 5.1. *Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton, let ϱ be a quasi-order on $\mathscr{A}$ and let ϱ^{ri} be the greatest right invariant quasi-order on $\mathscr{A}$ contained in ϱ.*

Define inductively a sequence $\{\varrho_k\}_{k \in \mathbb{N}}$ of quasi-orders on $\mathscr{A}$ as follows:

$$\varrho_1 = \varrho, \quad \varrho_{k+1} = \varrho_k \cap \widehat{\varrho_k^r}, \ \text{for each } k \in \mathbb{N}.$$

Then

(a) $\varrho^{\mathrm{ri}} \subseteq \cdots \subseteq \varrho_{k+1} \subseteq \varrho_k \subseteq \cdots \subseteq \varrho_1 = \varrho$;

(b) *If $\varrho_k = \varrho_{k+m}$, for some $k, m \in \mathbb{N}$, then $\varrho_k = \varrho_{k+1} = \varrho^{\mathrm{ri}}$;*

(c) *If $\mathscr{A}$ is finite, then $\varrho_k = \varrho^{\mathrm{ri}}$ for some $k \in \mathbb{N}$.*

Proof. (a) Clearly, $\varrho_{k+1} \subseteq \varrho_k$, for every $k \in \mathbb{N}$, and $\varrho^{\mathrm{ri}} \subseteq \varrho_1$. If $\varrho^{\mathrm{ri}} \subseteq \varrho_k$, for some $k \in \mathbb{N}$, then $(\varrho^{\mathrm{ri}})^r \leqslant \varrho_k^r$, and also, $\varrho^{\mathrm{ri}} \leqslant (\varrho^{\mathrm{ri}})^r$, so we have that

$$\varrho^{\mathrm{ri}} \subseteq \widehat{(\varrho^{\mathrm{ri}})^r} \subseteq \widehat{\varrho_k^r},$$

and by this it follows that $\varrho^{\mathrm{ri}} \subseteq \varrho_{k+1}$. Hence, by induction we obtain that $\varrho^{\mathrm{ri}} \subseteq \varrho_k$, for every $k \in \mathbb{N}$.

(b) If $\varrho_k = \varrho_{k+m}$, for some $k, m \in \mathbb{N}$, then

$$\varrho_k = \varrho_{k+m} \subseteq \varrho_{k+1} \subseteq \widehat{\varrho_k^r} \leqslant \varrho_k^r,$$

so we have that ϱ_k is a right invariant quasi-order on $\mathscr{A}$. Therefore, $\varrho_k = \varrho_{k+1} = \varrho^{\mathrm{ri}}$.

(c) If the set A is finite, then the set of all crisp relations on A is also finite, so there exist $k, m \in \mathbb{N}$ such that $\varrho_k = \varrho_{k+m}$, and then $\varrho_k = \varrho^{\mathrm{ri}}$. $\quad\square$

The previous theorem shows that the greatest right invariant crisp quasi-order can be effectively computed for any fuzzy finite automaton over an arbitrary complete residuated lattice, not necessary locally finite, and even for a fuzzy finite automaton over an arbitrary lattice-ordered monoid. However, in cases when we are able to effectively compute the greatest right invariant fuzzy quasi-order, using it we can attain better reduction than using the greatest right invariant crisp quasi-order, as the next example shows. Namely, the greatest right invariant crisp quasi-order ϱ^{ri} is less or equal than the greatest right invariant fuzzy quasi-order R^{ri} and according to Remark 3.1 there holds $|\mathscr{A}/R^{\mathrm{ri}}| \leqslant |\mathscr{A}/\varrho^{\mathrm{ri}}|$.

Example 5.1. Let $\mathscr{L}$ be the Gödel structure, and let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x\}$, and δ_x^A is given by

$$\delta_x^A = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix}.$$

Then the greatest right invariant fuzzy quasi-order R^{ri} and the greatest right invariant crisp quasi-order ϱ^{ri} on $\mathscr{A}$ are given by

$$R^{\mathrm{ri}} = \begin{bmatrix} 1 & 0.1 & 1 \\ 1 & 1 & 1 \\ 1 & 0.1 & 1 \end{bmatrix}, \quad \varrho^{\mathrm{ri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, ϱ^{ri} do not reduce the number of states of $\mathscr{A}$, but R^{ri} reduces $\mathscr{A}$ to a fuzzy automaton with two states.

Let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton. If a fuzzy quasi-order R on A is a solution to system

$$R \circ \delta_x^A = \delta_x^A, \quad \text{for every } x \in X, \tag{62}$$

then it is called a *strongly right invariant fuzzy quasi-order* on $\mathscr{A}$, and if it is a solution to system

$$\delta_x^A \circ R = \delta_x^A, \quad \text{for every } x \in X, \tag{63}$$

then it is a *strongly left invariant fuzzy quasi-order* on $\mathscr{A}$. Clearly, every strongly right (resp. left) invariant fuzzy quasi-order is right (resp. left) invariant. Let us note that a fuzzy quasi-order on A is both strongly right and left invariant if and only if it is a solution to system

$$R \circ \delta_x^A \circ R = \delta_x^A, \quad \text{for every } x \in X, \tag{64}$$

and then it is called a *strongly invariant fuzzy quasi-order*.

If $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ is a fuzzy recognizer, then by a *stongly right invariant fuzzy quasi-order* on $\mathscr{A}$ we mean a fuzzy quasi-order on A which is a solution to (62) and

$$R \circ \tau^A = \tau^A, \tag{65}$$

and a *strongly left invariant fuzzy quasi-order* on $\mathscr{A}$ is a fuzzy quasi-order which is a solution to (63) and

$$\sigma^A \circ R = \sigma^A. \tag{66}$$

In the further text we study strongly right invariant fuzzy quasi-orders.

Theorem 5.2. *Let* $\mathscr{A} = (A, X, \delta^A)$ *be a fuzzy automaton and let* $\mathscr{A}' = (A, X, \delta^A, \sigma^A, \tau^A)$ *be a fuzzy recognizer belonging to* $\mathscr{A}$. *Then*

(a) *The set* $\mathscr{Q}^{\mathrm{sri}}(\mathscr{A})$ *of all strongly right invariant fuzzy quasi-orders on* $\mathscr{A}$ *is a principal ideal of the lattice* $\mathscr{Q}(A)$. *The greatest element of this principal ideal is a fuzzy quasi-order* R^{sri} *defined by*

$$R^{\mathrm{sri}}(a, b) = \bigwedge_{x \in X} \bigwedge_{c \in A} \delta_x^A(b, c) \to \delta_x^A(a, c), \quad \text{for all } a, b \in A. \tag{67}$$

(b) *The set* $\mathscr{Q}^{\mathrm{sri}}(\mathscr{A}')$ *of all strongly right invariant fuzzy quasi-orders on* $\mathscr{A}'$ *is the principal ideal of the lattice* $\mathscr{Q}(A)$. *The greatest element of this principal ideal is a fuzzy quasi-order* $R_\tau \wedge R^{\mathrm{sri}}$.

Proof. (a) We have that R^{sri} is a fuzzy quasi-order, as an intersection of a family of fuzzy quasi-orders defined as in (24). Let R be an arbitrary fuzzy quasi-order on A. Then we have that

$$\begin{aligned}
R \leqslant R^{\mathrm{sri}} \quad &\Leftrightarrow \quad (\forall x \in X)(\forall a, b, c \in A)\; R(a,b) \leqslant \delta_x^A(b,c) \to \delta_x^A(a,c) \\
&\Leftrightarrow \quad (\forall x \in X)(\forall a, b, c \in A)\; R(a,b) \otimes \delta_x^A(b,c) \leqslant \delta_x^A(a,c) \\
&\Leftrightarrow \quad (\forall x \in X)(\forall a, c \in A)\; \bigvee_{b \in A} R(a,b) \otimes \delta_x^A(b,c) \leqslant \delta_x^A(a,c) \\
&\Leftrightarrow \quad (\forall x \in X)(\forall a, c \in A)\; R \circ \delta_x^A(a,c) \leqslant \delta_x^A(a,c) \\
&\Leftrightarrow \quad (\forall x \in X)\; R \circ \delta_x^A \leqslant \delta_x^A \\
&\Leftrightarrow \quad (\forall x \in X)\; R \circ \delta_x^A = \delta_x^A ,
\end{aligned}$$

so R is the strongly right invariant if and only if it belongs to the principal ideal of $\mathscr{Q}(A)$ generated by R^{sri}.

(b) This follows immediately by (a). □

According to (67), the greatest strongly right invariant crisp quasi-order can be effectively computed for any fuzzy finite automaton over an arbitrary complete residuated lattice, not necessary locally finite. However, in cases when we are able to effectively compute the greatest right invariant fuzzy quasi-order, using it we can attain better reduction than using the greatest strongly right invariant quasi-order. Indeed, the following example presents a fuzzy automaton whose number of states can be reduced by means of right invariant fuzzy quasi-orders, but it can not be reduced using strongly right invariant ones.

Example 5.2. Consider again the fuzzy automaton $\mathscr{A}$ from Example 4.3. In this example we showed that the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$ reduces $\mathscr{A}$ to a fuzzy automaton with two states. On the other hand, the greatest strongly right invariant fuzzy quasi-order R^{sri} on $\mathscr{A}$ is given by

$$R^{\mathrm{sri}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and the related afterset fuzzy automaton $\mathscr{A}_2 = \mathscr{A}/R^{\mathrm{sri}} = (A_2, X, \delta^{A_2})$ has also three states and fuzzy transition relations $\delta_x^{A_2}$ and $\delta_y^{A_2}$ are given by

$$\delta_x^{A_2} = \delta_x^A \circ R^{\mathrm{sri}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta_y^{A_2} = \delta_y^A \circ R^{\mathrm{sri}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Further, the greatest strongly right invariant fuzzy quasi-order R_2^{sri} on $\mathscr{A}_2$ is given by

$$R_2^{\mathrm{sri}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and the afterset fuzzy automaton $\mathscr{A}_2/R_2^{\mathrm{sri}}$ is isomorphic to $\mathscr{A}_2$. Therefore, the number of states of $\mathscr{A}$ can not be reduced by means of strongly right invariant fuzzy quasi-orders.

6. Weakly right and left invariant fuzzy quasi-orders

In the previous sections we have considered right and left invariant fuzzy quasi-orders and some special types of these fuzzy quasi-orders. In this section we study some fuzzy quasi-orders which are more general than right and left invariant ones.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer. For any $u \in X^*$ we define fuzzy sets $\sigma_u^A, \tau_u^A \in L^A$ by

$$\sigma_u^A(a) = \bigvee_{b \in A} \sigma^A(b) \otimes \delta_*^A(b, u, a), \qquad \tau_u^A(a) = \bigvee_{b \in A} \delta_*^A(a, u, b) \otimes \tau^A(b),$$

for each $a \in A$, i.e., $\sigma_u^A = \sigma^A \circ \delta_u^A$ and $\tau_u^A = \delta_u^A \circ \tau^A$. Evidently, for the empty word $e \in X^*$ we have that $\sigma_e^A = \sigma^A$ and $\tau_e^A = \tau^A$. Fuzzy sets σ_u^A have been already used in [32], and they played a key role in determinization of fuzzy automata. By the same rule, for any $a \in A$ we define fuzzy languages $\sigma_a^A, \tau_a^A \in L^{X^*}$, i.e., $\sigma_a^A(u) = \sigma_u^A(a)$ and $\tau_a^A(u) = \tau_u^A(a)$, for every $u \in X^*$. Following terminology used in [18] for non-deterministic automata, we call σ_a^A the *left fuzzy language* of a, and τ_a^A the *right fuzzy language* of a. Left fuzzy languages have been already studied in [32, 34].

A fuzzy quasi-order R on A which is a solution to a system of fuzzy relation equations

$$R \circ \tau_u^A = \tau_u^A, \quad \text{for every } u \in X^*, \tag{68}$$

is called a *weakly right invariant fuzzy quasi-order* on the fuzzy recognizer $\mathscr{A}$, and if R is a solution to

$$\sigma_u^A \circ R = \sigma_u^A, \quad \text{for every } u \in X^*, \tag{69}$$

then it is called a *weakly left invariant fuzzy quasi-order* on $\mathscr{A}$. Fuzzy equivalences on $\mathscr{A}$ which are solutions to (68) will be called *weakly right invariant fuzzy equivalences*, and those which are solutions to (69) will be called *weakly left invariant fuzzy equivalences*.

We have the following

Theorem 6.1. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer. Then*

(a) *The set $\mathscr{Q}^{\mathrm{wri}}(A)$ of all weakly right invariant fuzzy quasi orders on $\mathscr{A}$ is a principal ideal of the lattice $\mathscr{Q}(A)$. The greatest element of this principal ideal is a fuzzy quasi-order R^{wri} on A defined by*

$$R^{\mathrm{wri}}(a, b) = \bigwedge_{u \in X^*} \tau_u^A(b) \to \tau_u^A(a), \quad \text{for all } a, b \in A. \tag{70}$$

Moreover, R^{wri} is the greatest solution to the system (68) in $\mathscr{R}(A)$.

(b) *Every weakly right invariant fuzzy quasi-order on $\mathscr{A}$ is a solution to the general system.*

(c) *Every right invariant fuzzy quasi-order on $\mathscr{A}$ is weakly right invariant.*

Proof. (a) Beeing an intersection of a family of fuzzy quasi-orders defined as in (24), R^{wri} is a fuzzy quasi-order. According to results from [74] (see also [63, 64, 65]), R^{wri} is the greatest solution to (68), and it is easy to check that solutions to (68) in $\mathscr{Q}(A)$ form an ideal of $\mathscr{Q}(A)$, and thus, they form a principal ideal of $\mathscr{Q}(A)$.

Let R be an arbitrary solution to (68) in $\mathscr{R}(A)$. The equality relation I on A is also a solution to (68), and by (12) we obtain that $(R \vee I)^\infty$ is a solution to (68). Since $(R \vee I)^\infty$ is a fuzzy quasi-order on A, we conclude that $R \leqslant (R \vee I)^\infty \leqslant R^{\mathrm{wri}}$, and therefore, R^{wri} is the greatest solution to (68) in $\mathscr{R}(A)$.

(b) Let R be an arbitrary weakly right invariant fuzzy quasi-order on $\mathscr{A}$. By induction on n we will prove that

$$R \circ \delta_{x_1}^A \circ R \circ \delta_{x_2}^A \circ R \circ \cdots \circ R \circ \delta_{x_n}^A \circ R \circ \tau^A = \delta_{x_1}^A \circ \delta_{x_2}^A \circ \cdots \circ \delta_{x_n}^A \circ \tau^A, \tag{71}$$

for every $n \in \mathbb{N}$ and all $x_1, x_2, \ldots, x_n \in X$. First we note that $\tau_e^A = \tau^A$, where $e \in X^*$ is the empty word, and by (68) we obtain that $R \circ \tau^A = \tau^A$. By this and by (68), for each $x \in X$ we have that

$$R \circ \delta_x^A \circ R \circ \tau^A = R \circ \delta_x^A \circ \tau^A = \delta_x^A \circ \tau^A,$$

25

and hence, (71) holds for $n = 1$. Suppose now that (71) holds for some $n \in \mathbb{N}$. Then by (71) and (68), for arbitrary $x_1, \ldots, x_n, x_{n+1} \in X$ we have that

$$
\begin{aligned}
R \circ \delta^A_{x_1} \circ R \circ \delta^A_{x_2} \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \delta^A_{x_{n+1}} \circ R \circ \tau^A &= \\
&= R \circ \delta^A_{x_1} \circ (R \circ \delta^A_{x_2} \circ \cdots \circ R \circ \delta^A_{x_n} \circ R \circ \delta^A_{x_{n+1}} \circ R \circ \tau^A) \\
&= R \circ \delta^A_{x_1} \circ (\delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \delta^A_{x_{n+1}} \circ \tau^A) \\
&= R \circ \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \delta^A_{x_{n+1}} \circ \tau^A \\
&= \delta^A_{x_1} \circ \delta^A_{x_2} \circ \cdots \circ \delta^A_{x_n} \circ \delta^A_{x_{n+1}} \circ \tau^A.
\end{aligned}
$$

Therefore, by induction we conclude that (71) holds for every $n \in \mathbb{N}$. Finally, it follows immediately by (71) that R is a solution to the general system.

(c) Let R be a right invariant fuzzy quasi-order on $\mathscr{A}$. For each $u \in X^*$ we have $R \circ \delta^A_u \circ R = \delta^A_u \circ R$, and also $R \circ \tau^A = \tau^A$, what implies $R \circ \tau^A_u = R \circ \delta^A_u \circ \tau^A = R \circ \delta^A_u \circ R \circ \tau^A = \delta^A_u \circ R \circ \tau^A = \delta^A_u \circ \tau^A = \tau^A_u$. Hence, R is the weakly right invariant. $\square$

Let us note that R^{wri} can be also represented by

$$
R^{\mathrm{wri}}(a, b) = \bigwedge_{u \in X^*} \tau^A_b(u) \to \tau^A_a(u), \quad \text{for all } a, b \in A, \tag{72}
$$

i.e., $R^{\mathrm{wri}}(a, b)$ can be interpreted as the degree of inclusion of a fuzzy language τ^A_b in the fuzzy language τ^A_a.

Analogously, we can define a fuzzy quasi-order R^{wli} on $\mathscr{A}$ by

$$
R^{\mathrm{wli}}(a, b) = \bigwedge_{u \in X^*} \sigma^A_u(a) \to \sigma^A_u(b) = \bigwedge_{u \in X^*} \sigma^A_a(u) \to \sigma^A_b(u), \quad \text{for all } a, b \in A, \tag{73}
$$

and we can prove that R^{wli} is the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}$, that every weakly left invariant fuzzy quasi-order on $\mathscr{A}$ is also a solution to the general system, and that every left invariant fuzzy quasi-order on $\mathscr{A}$ is weakly left invariant. We can also show that the greatest weakly right invariant fuzzy equivalence E^{wrie} on $\mathscr{A}$ is given by

$$
E^{\mathrm{wrie}}(a, b) = \bigwedge_{u \in X^*} \tau^A_u(a) \leftrightarrow \tau^A_u(b) = \bigwedge_{u \in X^*} \tau^A_a(u) \leftrightarrow \tau^A_b(u), \quad \text{for all } a, b \in A, \tag{74}
$$

and the greatest weakly left invariant fuzzy equivalence E^{wlie} on $\mathscr{A}$ is given by

$$
E^{\mathrm{wlie}}(a, b) = \bigwedge_{u \in X^*} \sigma^A_u(a) \leftrightarrow \sigma^A_u(b) = \bigwedge_{u \in X^*} \sigma^A_a(u) \leftrightarrow \sigma^A_b(u), \quad \text{for all } a, b \in A, \tag{75}
$$

Clearly, E^{wrie} is the natural fuzzy equivalence of R^{wri}, and E^{wlie} is the natural fuzzy equivalence of R^{wlie}. We will also call R^{wri} the *right Myhill-Nerode's fuzzy quasi-order* of $\mathscr{A}$, R^{wli} the *left Myhill-Nerode's fuzzy quasi-order* of $\mathscr{A}$, E^{wrie} the *right Myhill-Nerode's fuzzy eqivalence* of $\mathscr{A}$, and E^{wlie} the *left Myhill-Nerode's fuzzy eqivalence* of $\mathscr{A}$. Note that a fuzzy relation N_σ on the free monoid X^* defined in a similar way by

$$
N_\sigma(u, v) = \bigwedge_{a \in A} \sigma^A_u(a) \leftrightarrow \sigma^A_v(a) = \bigwedge_{a \in A} \sigma^A_a(u) \leftrightarrow \sigma^A_a(v), \quad \text{for all } u, v \in X^*,
$$

is called the *Nerode's fuzzy right congruence* on X^*. Nerode's fuzzy right congruences and Myhill's fuzzy congruences on free monoids associated with fuzzy automata have been studied in [32, 34].

The following example shows that there are weakly right invariant fuzzy quasi-orders which are not right invariant, and that weakly right invariant fuzzy quasi-orders generally give better reductions than right invariant ones, according to Remark 3.1.

Example 6.1. Let $\mathscr{L}$ be the Boolean structure and $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ a fuzzy recognizer over $\mathscr{L}$, where $A = \{1, 2, 3, 4\}$, $X = \{x\}$, σ^A is any fuzzy subset of A and δ_x^A, and τ^A are given by

$$
\delta_x^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
$$

For the sake of simplicity set $\tau^A = \tau$. As we have noted before, the greatest right invariant fuzzy equivalence on the fuzzy recognizer $\mathscr{A}$ is the greatest right invariant fuzzy equivalence on the fuzzy automaton (A, X, δ^A) contained in the fuzzy quasi-order R^τ. In this example we have

$$
R^\tau = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix},
$$

and hence, applying the procedure from Theorem 4.3 to R^τ we obtain that the greatest right invariant fuzzy equivalence R^{ri} on $\mathscr{A}$ is

$$
R^{\mathrm{ri}} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

On the other hand, we have that $\tau_e = \tau$ and

$$
\tau_x = \delta_x^A \circ \tau = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tau_{x^2} = \delta_x^A \circ \tau_x = \tau_x
$$

what means that $\tau_u = \tau_x$, for every $u \in X^*$, $u \neq e$, whence

$$
R^{\mathrm{wri}} = R^\tau \wedge R^{\tau_x} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.
$$

Hence, R^{ri} is strictly smaller than R^{wri}, and R^{ri} do not reduce the number of states of $\mathscr{A}$, whereas R^{wri} reduces $\mathscr{A}$ to a fuzzy recognizer $\mathscr{A}/R^{\mathrm{wri}} = (A_2, X, \delta^{A_2}, \sigma^{A_2}, \tau^{A_2})$ with two states, where $\delta_x^{A_2}$ and τ^{A_2} are given by

$$
\delta_x^{A_2} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tau^{A_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},
$$

and σ^{A_2} is defined as in (27).

However, although weakly right invariant and weakly left invariant fuzzy quasi-orders generally give better reductions than right invariant and left invariant ones, they have a serious shortcoming. For fuzzy automata and fuzzy recognizers over a locally finite complete residuated lattice, the greatest right and left invariant fuzzy equivalences can be computed in a polynomial time, using a procedure from Theorem 4.3, but computing the greatest weakly right and left invariant ones is computationally hard. Namely, any particular equation $R \circ \tau_u^A = \tau_u^A$ in (68) can be easily solved if the fuzzy set τ_u^A is given, but computing τ_u^A, for all $u \in X^*$, may be very hard. In fact, computing τ_u^A, for all $u \in X^*$, is nothing else than determinization

27

of the reverse fuzzy recognizer of $\mathscr{A}$, whereas computing σ_u^A, for all $u \in X^*$, is the determinization of $\mathscr{A}$ using a procedure developed in [32], called the *accessible fuzzy subset construction*. It is well-known that determinization of crisp non-deterministic recognizers may require an exponential time, because numbers of elements of the sets $\{\sigma_u^A \mid u \in X^*\}$ and $\{\tau_u^A \mid u \in X^*\}$ may be exponential in the number of states of $\mathscr{A}$, and in the case of fuzzy recognizers these sets may even be infinite. Conditions under which these sets must be finite have been determined in [32, 34]. Moreover, because of exponential growth in the number of states during determinization of non-deterministic recognizers, state reduction procedures are often used to decrease the number of states prior to determinization. But, here we have that determinization is needed prior to the state reduction by means of the greatest weakly right and left invariant fuzzy quasi-orders.

7. Alternate reductions

In this section we show that better reductions can be obtained alternating reductions by means of the greatest right and left invariant fuzzy quasi-orders, or the greatest weakly right and left invariant fuzzy quasi-orders. We show that even if any of these fuzzy quasi-orders separately do not reduce the number of states, alternating right and left invariant ones, or weakly right and left invariant ones, the number of states can be reduced.

Theorem 7.1. *Let $\mathscr{A}$ be a fuzzy automaton or a fuzzy recognizer, let R be a right invariant fuzzy quasi-order on $\mathscr{A}$ and let S be a fuzzy quasi-order on the set of states of $\mathscr{A}$ such that $R \leqslant S$. Then*

   (a) *S is a right invariant fuzzy quasi-order on $\mathscr{A}$ if and only if S/R is a right invariant fuzzy quasi-order on $\mathscr{A}/R$;*
   (b) *S is the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ if and only if S/R is the greatest right invariant fuzzy quasi-order on $\mathscr{A}/R$;*
   (c) *R is the greatest right invariant fuzzy quasi-order on $\mathscr{A}$ if and only if $\widetilde{R}$ is the greatest right invariant fuzzy quasi-order on $\mathscr{A}/R$.*

Proof. First we note that $R \leqslant S$ is equivalent to $R \circ S = S \circ R = S$.
   (a) Assume first that $\mathscr{A} = (A, X, \delta^A)$ is a fuzzy automaton. Consider any $a, b \in A$ and $x \in X$. Then

$$(\delta_x^{A/R} \circ S/R)(R_a, R_b) = \bigvee_{c \in A} \delta_x^{A/R}(R_a, R_c) \otimes S/R(R_c, R_b) = \bigvee_{c \in A} (R \circ \delta_x^A \circ R)(a, c) \otimes S(c, b)$$
$$= (R \circ \delta_x^A \circ R \circ S)(a, b) = (\delta_x^A \circ R \circ S)(a, b) = (\delta_x^A \circ S)(a, b),$$
(76)

and by the proof of Theorem 3.3 it follows that

$$(S/R \circ \delta_x^{A/R} \circ S/R)(R_a, R_b) = (S \circ \delta_x^A \circ S)(a, b).$$
(77)

Therefore, by (76) and (77) we obtain that (a) holds.
   Next, let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer. Then for any $a \in A$ we have that

$$(S/R \circ \tau^{A/R})(R_a) = \bigvee_{b \in A} S/R(R_a, R_b) \otimes \tau^{A/R}(R_b) = \bigvee_{b \in A} S(a, b) \otimes \tau^A(b) = (S \circ \tau^A)(a),$$

so $S/R \circ \tau^{A/R} = \tau^{A/R}$ if and only if $S \circ \tau^A = \tau^A$. Therefore, in this case we also have that (a) holds.
   (b) Let S be the greatest right invariant fuzzy quasi-order on $\mathscr{A}$. By (a), S/R is a right invariant fuzzy quasi-order on $\mathscr{A}/R$. Let Q be the greatest right invariant fuzzy quasi-order on $\mathscr{A}/R$. Define a fuzzy relation T on $\mathscr{A}$ by

$$T(a, b) = Q(R_a, R_b), \qquad \text{for all } a, b \in A.$$

It is easy to verify that T is a fuzzy quasi-order on $\mathscr{A}$. According to (a), $\widetilde{R}$ is a right invariant fuzzy quasi-order on $\mathscr{A}/R$, what implies $\widetilde{R} \leqslant Q$, and for arbitrary $a, b \in A$ we obtain that

$$R(a, b) = \widetilde{R}(R_a, R_b) \leqslant Q(R_a, R_b) = T(a, b),$$

28

what means that $R \leqslant T$. Therefore, we have that $Q = T/R$, and by (a) we obtain that T is a right invariant fuzzy quasi-order on $\mathscr{A}$, what implies $T \leqslant S$. Now, according to (33), we have that $Q = T/R \leqslant S/R$, and since S/R is a right invariant fuzzy quasi-order on $\mathscr{A}/R$, we conclude that $Q = S/R$, i.e., S/R is the greatest right invariant fuzzy quasi-order on $\mathscr{A}/R$.

Conversely, let S/R be the greatest right invariant fuzzy quasi-order on $\mathscr{A}/R$. According to (a), S is a right invariant fuzzy quasi-order on $\mathscr{A}$. Let T be the greatest right invariant fuzzy quasi-order on $\mathscr{A}$. Then we have that $R \leqslant S \leqslant T$, and by (a) it follows that T/R is a right invariant fuzzy quasi-order on $\mathscr{A}/R$, what yields $T/R \leqslant S/R$. Now, by (33) it follows that $T \leqslant S$, and hence, $T = S$, and we have proved that S is the greatest right invariant fuzzy quasi-order on $\mathscr{A}$.

(c) This assertion follows immediately by (b). $\quad\square$

Certainly, the previous theorem also holds for left invariant fuzzy quasi orders. Furthermore, we have that a similar theorem concerning weakly right invariant fuzzy quasi-orders is true:

Theorem 7.2. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer, let R be a weakly right invariant fuzzy quasi-order on $\mathscr{A}$ and let S be a fuzzy quasi-order on A such that $R \leqslant S$. Then*

(a) *S is a weakly right invariant fuzzy quasi-order on $\mathscr{A}$ if and only if S/R is a weakly right invariant fuzzy quasi-order on $\mathscr{A}/R$;*

(b) *S is the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}$ if and only if S/R is the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}/R$;*

(c) *R is the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}$ if and only if $\widetilde{R}$ is the greatest weakly right invariant fuzzy quasi-order on $\mathscr{A}/R$.*

Proof. (a) For arbitrary $a \in A$ and $u = x_1 \dots x_n \in X^*$, $x_1, \dots, x_n \in X$, by (71) we obtain that

$$\tau_u^{A/R}(R_a) = (\delta_u^{A/R} \circ \tau^{A/R})(R_a) = \bigvee_{b \in A} \delta_u^{A/R}(R_a, R_b) \otimes \tau^{A/R}(R_b)$$

$$= \bigvee_{b \in A} (R \circ \delta_{x_1}^A \circ R \circ \cdots \circ R \circ \delta_{x_n}^A \circ R)(a, b) \otimes (R \circ \tau^A)(b)$$

$$= (R \circ \delta_{x_1}^A \circ R \circ \cdots \circ R \circ \delta_{x_n}^A \circ R \circ \tau^A)(a)$$

$$= (\delta_{x_1}^A \circ \cdots \circ \delta_{x_n}^A \circ \tau^A)(a)$$

$$= \tau_u^A(a).$$

Next, for any $a \in A$ and $u \in X^*$ we have that

$$(S/R \circ \tau_u^{A/R})(R_a) = \bigvee_{b \in A} S/R(R_a, R_b) \otimes \tau_u^{A/R}(R_b) = \bigvee_{b \in A} S(a, b) \otimes \tau_u^A(b) = (S \circ \tau_u^A)(a).$$

Therefore $S/R \circ \tau_u^{A/R} = \tau_u^{A/R}$ if and only if $S \circ \tau_u^A = \tau_u^A$, and we have proved that (a) is true.

The assertion (b) can be proved similarly as (b) of Theorem 7.1, and (c) follows immediately by (b). $\quad\square$

Let $\mathscr{A}$ be a fuzzy automaton. A sequence $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of fuzzy automata we will call a $\mathscr{Q}^{\mathrm{ri}}$-*reduction* of $\mathscr{A}$ if $\mathscr{A}_1 = \mathscr{A}$ and for each $k \in \{1, 2, \dots, n-1\}$ we have that $\mathscr{A}_{k+1}$ is the afterset fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest right invariant fuzzy quasi-order on $\mathscr{A}_k$. Analogously, using left invariant fuzzy quasi-orders instead of right invariant ones we define a $\mathscr{Q}^{\mathrm{li}}$-*reduction* of $\mathscr{A}$, using strongly right and left invariant fuzzy quasi-orders we define a $\mathscr{Q}^{\mathrm{sri}}$-*reduction* and a $\mathscr{Q}^{\mathrm{sli}}$-*reduction* of $\mathscr{A}$, and using right and left invariant fuzzy equivalences we define a $\mathscr{E}^{\mathrm{ri}}$-*reduction* and a $\mathscr{E}^{\mathrm{li}}$-*reduction* of $\mathscr{A}$. If we consider fuzzy recognizers, in a similar way we define $\mathscr{Q}^{\mathrm{wri}}$- and $\mathscr{Q}^{\mathrm{wli}}$-*reductions*, as well as $\mathscr{Q}^{\mathrm{ri}}$- and $\mathscr{Q}^{\mathrm{li}}$-*reductions* of fuzzy recognizers.

Let us note that for each fuzzy finite automaton $\mathscr{A}$ there exists a $\mathscr{Q}^{\mathrm{ri}}$-reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n$ of $\mathscr{A}$ such that for every $\mathscr{Q}^{\mathrm{ri}}$-reduction $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_n, \mathscr{A}_{n+1}, \dots, \mathscr{A}_{n+m}$ of $\mathscr{A}$ which is a continuation of this reduction we have that

$$|\mathscr{A}_n| = |\mathscr{A}_{n+1}| = \cdots = |\mathscr{A}_{n+m}|,$$

i.e., all fuzzy automata $\mathscr{A}_{n+1}, \ldots, \mathscr{A}_{n+m}$ have the same number of states as $\mathscr{A}_n$. Also, there is a shortest $\mathscr{Q}^{\mathrm{ri}}$-reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of $\mathscr{A}$ having this property, which we will call the *shortest $\mathscr{Q}^{\mathrm{ri}}$-reduction* of $\mathscr{A}$, and then we will call $\mathscr{A}_n$ a $\mathscr{Q}^{\mathrm{ri}}$-*reduct* of $\mathscr{A}$, and we will cal n the *length* of this shortest $\mathscr{Q}^{\mathrm{ri}}$-reduction. If a fuzzy automaton $\mathscr{A}$ is its own $\mathscr{Q}^{\mathrm{ri}}$-reduct, then it is called $\mathscr{Q}^{\mathrm{ri}}$-*reduced*. Analogously we define a $\mathscr{Q}^{\mathrm{li}}$-*reduct* of $\mathscr{A}$ and a $\mathscr{Q}^{\mathrm{li}}$-*reduced* fuzzy automaton, as well as $\mathscr{Q}^{\mathrm{sri}}$- and $\mathscr{Q}^{\mathrm{sli}}$-*reducts*, $\mathscr{Q}^{\mathrm{sri}}$- and $\mathscr{Q}^{\mathrm{sli}}$-*reduced* fuzzy automata, $\mathscr{E}^{\mathrm{ri}}$- and $\mathscr{E}^{\mathrm{li}}$-*reducts*, $\mathscr{E}^{\mathrm{ri}}$- and $\mathscr{E}^{\mathrm{li}}$-*reduced* fuzzy automata, and other related notions. For fuzzy recognizers we similarly define $\mathscr{Q}^{\mathrm{wri}}$- and $\mathscr{Q}^{\mathrm{wli}}$-*reducts*, $\mathscr{Q}^{\mathrm{wri}}$- and $\mathscr{Q}^{\mathrm{wli}}$-*reduced* fuzzy recognizers, $\mathscr{Q}^{\mathrm{ri}}$- and $\mathscr{Q}^{\mathrm{li}}$-*reducts*, $\mathscr{Q}^{\mathrm{ri}}$- and $\mathscr{Q}^{\mathrm{li}}$-*reduced* fuzzy recognizers, and so forth.

The next theorem shows that length of the shortest $\mathscr{Q}^{\mathrm{ri}}$- and $\mathscr{Q}^{\mathrm{li}}$-reductions do not exceed 2.

Theorem 7.3. *A fuzzy recognizer (automaton) $\mathscr{A}$ is $\mathscr{Q}^{\mathrm{ri}}$-reduced if and only if the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$ is a fuzzy order.*

Consequently, for each fuzzy finite recognizer (automaton) $\mathscr{A}$, the afterset fuzzy recognizer (automaton) $\mathscr{A}/R^{\mathrm{ri}}$ is $\mathscr{Q}^{\mathrm{ri}}$-reduced.

Proof. Let $\mathscr{A}$ be $\mathscr{Q}^{\mathrm{ri}}$-reduced. If R^{ri} is not a fuzzy order, then $|\mathscr{A}/R^{\mathrm{ri}}| < |\mathscr{A}|$, what contradicts our starting hypothesis that $\mathscr{A}$ is $\mathscr{Q}^{\mathrm{ri}}$-reduced. Thus, we conclude that R^{ri} is a fuzzy order.

Conversely, let R^{ri} be a fuzzy order. Consider an arbitrary $\mathscr{Q}^{\mathrm{ri}}$-reduction $\mathscr{A}_1 = \mathscr{A}, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of $\mathscr{A}$. For each $k \in \{1, 2, \ldots, n\}$ let R_k^{ri} be the greatest right invariant fuzzy quasi-order on $\mathscr{A}_k$. By Theorem 7.1, for every $k \in \{2, \ldots, n\}$ we have that $R_k^{\mathrm{ri}} = \widetilde{R}_{k-1}^{\mathrm{ri}}$, so R_k^{ri} is a fuzzy order, and by the hypothesis, $R_1^{\mathrm{ri}} = R^{\mathrm{ri}}$ is a fuzzy order. Now, for every $k \in \{2, \ldots, n\}$ we have that $|\mathscr{A}_k| = |(\mathscr{A}_{k-1})/R_{k-1}^{\mathrm{ri}}| = |\mathscr{A}_{k-1}|$, and hence, $|\mathscr{A}| = |\mathscr{A}_1| = |\mathscr{A}_2| = \cdots = |\mathscr{A}_n|$. Therefore, the fuzzy recognizer (automaton) $\mathscr{A}$ is $\mathscr{Q}^{\mathrm{ri}}$-reduced.

Further, let $\mathscr{A}$ be an arbitrary fuzzy finite recognizer (automaton) and R^{ri} the greatest right invariant fuzzy quasi-order on $\mathscr{A}$. Then by Theorem 7.1 it follows that $\widetilde{R}^{\mathrm{ri}}$ is the greatest right invariant fuzzy quasi-order on the afterset fuzzy recognizer (automaton) $\mathscr{A}/R^{\mathrm{ri}}$, and since it is a fuzzy order, we conclude that $\mathscr{A}/R^{\mathrm{ri}}$ is $\mathscr{Q}^{\mathrm{ri}}$-reduced. $\square$

Similarly we prove the following:

Theorem 7.4. *A fuzzy recognizer $\mathscr{A}$ is $\mathscr{Q}^{\mathrm{wri}}$-reduced if and only if the greatest weakly right invariant fuzzy quasi-order R^{wri} on $\mathscr{A}$ is a fuzzy order.*

Consequently, for each fuzzy finite recognizer $\mathscr{A}$, the afterset fuzzy recognizer $\mathscr{A}/R^{\mathrm{wri}}$ is $\mathscr{Q}^{\mathrm{wri}}$-reduced.

If a fuzzy automaton $\mathscr{A} = (A, X, \delta^A)$ is $\mathscr{Q}^{\mathrm{ri}}$-reduced, that is, if the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$ is a fuzzy order, then the afterset fuzzy automaton $\mathscr{A}/R^{\mathrm{ri}}$ has the same cardinality as $\mathscr{A}$, but it is not necessary isomorphic to $\mathscr{A}$ (see Example 7.1). If the afterset fuzzy automaton $\mathscr{A}/R^{\mathrm{ri}}$ is isomorphic to $\mathscr{A}$, then $\mathscr{A}$ is called *completely $\mathscr{Q}^{\mathrm{ri}}$-reduced*. Analogously we define *completely $\mathscr{Q}^{\mathrm{li}}$-, $\mathscr{E}^{\mathrm{ri}}$- and $\mathscr{E}^{\mathrm{li}}$-reduced* fuzzy automata, as well as *completely $\mathscr{Q}^{\mathrm{ri}}$-, $\mathscr{Q}^{\mathrm{li}}$-, $\mathscr{Q}^{\mathrm{wri}}$-, and $\mathscr{Q}^{\mathrm{wli}}$-reduced* fuzzy recognizers.

Example 7.1 will show that even if a fuzzy recognizer or a fuzzy automaton $\mathscr{A}$ is $\mathscr{Q}^{\mathrm{wri}}$- and/or $\mathscr{Q}^{\mathrm{wli}}$-reduced, or it is $\mathscr{Q}^{\mathrm{ri}}$- and/or $\mathscr{Q}^{\mathrm{li}}$-reduced, it is still possible to continue reduction of the number of states of $\mathscr{A}$ alternating reductions by means of the greatest weakly right and left invariant fuzzy quasi-orders, or by means of the greatest right and left invariant fuzzy quasi-orders. For that reason we introduce the following concepts.

Let $\mathscr{A}$ be a fuzzy automaton. A sequence $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of fuzzy automata will be called an *alternate $\mathscr{Q}$-reduction* of $\mathscr{A}$ if $\mathscr{A}_1 = \mathscr{A}$ and for every $k \in \{1, 2, \ldots, n-2\}$ the following is true:

(1) $\mathscr{A}_{k+1}$ is the afterset fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest right invariant or the greatest left invariant fuzzy quasi-odred on $\mathscr{A}_k$;

(2) If $\mathscr{A}_{k+1}$ is the afterset fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest right invariant fuzzy quasi-order on $\mathscr{A}_k$, then $\mathscr{A}_{k+2}$ is the afterset fuzzy automaton of $\mathscr{A}_{k+1}$ w.r.t. the greatest left invariant fuzzy quasi-order on $\mathscr{A}_k$;

(3) If $\mathscr{A}_{k+1}$ is the afterset fuzzy automaton of $\mathscr{A}_k$ w.r.t. the greatest left invariant fuzzy quasi-order on $\mathscr{A}_k$, then $\mathscr{A}_{k+2}$ is the afterset fuzzy automaton of $\mathscr{A}_{k+1}$ w.r.t. the greatest right invariant fuzzy quasi-order on $\mathscr{A}_k$.

If $\mathscr{A}_2$ is the afterset fuzzy automaton of $\mathscr{A}_1$ w.r.t. the greatest right invariant fuzzy quasi-order on $\mathscr{A}_1$, then this alternate $\mathscr{Q}$-reduction is called an *alternate $\mathscr{Q}^{\mathrm{rl}}$-reduction*, and if $\mathscr{A}_2$ is the afterset fuzzy automaton of $\mathscr{A}_1$ w.r.t. the greatest left invariant fuzzy quasi-order on $\mathscr{A}_1$, then this alternate $\mathscr{Q}$-reduction is called an *alternate $\mathscr{Q}^{\mathrm{lr}}$-reduction*.

Note that for each fuzzy finite automaton $\mathscr{A}$ there exists an alternate $\mathscr{Q}^{\mathrm{rl}}$-reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of $\mathscr{A}$ such that for every alternate $\mathscr{Q}^{\mathrm{rl}}$-reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n, \mathscr{A}_{n+1}, \ldots, \mathscr{A}_{n+m}$ which is a continuation of this reduction we have that

$$|\mathscr{A}_n| = |\mathscr{A}_{n+1}| = \cdots = |\mathscr{A}_{n+m}|,$$

i.e., all fuzzy automata $\mathscr{A}_{n+1}, \ldots, \mathscr{A}_{n+m}$ have the same number of states as $\mathscr{A}_n$. Also, there is a shortest alternate $\mathscr{Q}^{\mathrm{rl}}$-reduction $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ of $\mathscr{A}$ having this property, which we will call the *shortest alternate $\mathscr{Q}^{\mathrm{rl}}$-reduction of $\mathscr{A}$*, and we will call $\mathscr{A}_n$ an *alternate $\mathscr{Q}^{\mathrm{rl}}$-reduct of $\mathscr{A}$*, whereas the number n will be called the *length* of the shortest alternate $\mathscr{Q}^{\mathrm{rl}}$-reduction of $\mathscr{A}$. Analogously we define the *shortest alternate $\mathscr{Q}^{\mathrm{lr}}$-reduction*, its length, and the *alternate $\mathscr{Q}^{\mathrm{lr}}$-reduct of $\mathscr{A}$*. Using the greatest right and left invariant fuzzy equivalences instead of the greatest right and left invariant fuzzy quasi-orders, we also define *alternate $\mathscr{E}$-reductions, alternate $\mathscr{E}^{\mathrm{rl}}$-and $\mathscr{E}^{\mathrm{lr}}$-reductions, alternate $\mathscr{E}^{\mathrm{rl}}$-and $\mathscr{E}^{\mathrm{lr}}$-reducts*, etc. For fuzzy recognizers, weakly right invariant and weakly left invariant fuzzy quasi-orders, similarly we define *alternate $\mathscr{Q}^{\mathrm{w}}$- and $\mathscr{E}^{\mathrm{w}}$-reductions, alternate $\mathscr{Q}^{\mathrm{wrl}}$- and $\mathscr{Q}^{\mathrm{wlr}}$-reductions, alternate $\mathscr{Q}^{\mathrm{wrl}}$- and $\mathscr{Q}^{\mathrm{wlr}}$-reducts*, as well as *alternate $\mathscr{Q}$-reductions, alternate $\mathscr{Q}^{\mathrm{rl}}$- and $\mathscr{Q}^{\mathrm{lr}}$-reductions, alternate $\mathscr{Q}^{\mathrm{rl}}$- and $\mathscr{Q}^{\mathrm{lr}}$-reducts* of fuzzy recognizers, and other related concepts.

Consider now the following example.

Example 7.1. Let $\mathscr{L}$ be the Boolean structure and let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x, y\}$, and $\delta_x^A, \delta_y^A, \sigma^A$ and τ^A are given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta_y^A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Let us note that the fuzzy automaton (A, X, δ^A) has been already considered in Example 7.1.

The greatest weakly right invariant fuzzy quasi-order R^{wri} on $\mathscr{A}$ and related afterset fuzzy recognizer $\mathscr{A}_2 = \mathscr{A}/R^{\mathrm{wri}} = (A_2, X, \delta^{A_2}, \sigma^{A_2}, \tau^{A_2})$ are given by

$$R^{\mathrm{wri}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \delta_x^{A_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \delta_y^{A_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \sigma^{A_2} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \tau^{A_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

and the greatest weakly left invariant fuzzy quasi-order R_2^{wli} on $\mathscr{A}_2$ and related afterset fuzzy recognizer $\mathscr{A}_3 = \mathscr{A}_2/R^{\mathrm{wri}} = (A_3, X, \delta^{A_3}, \sigma^{A_3}, \tau^{A_3})$ are given by

$$R_2^{\mathrm{wli}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \delta_x^{A_3} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \delta_y^{A_3} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \sigma^{A_3} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \tau^{A_3} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It can be easily verified that both the greatest weakly right invariant fuzzy quasi-order and the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}_3$ coincide with the equality relation on A_3, and the afterset fuzzy recognizers of $\mathscr{A}_3$ w.r.t. these fuzzy quasi-orders are isomorphic to $\mathscr{A}_3$. By this it follows that none alternate $\mathscr{Q}^{\mathrm{w}}$-reduction decreases the number of states of $\mathscr{A}_3$, and we obtain that the sequence $\mathscr{A} = \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ is the shortest alternate $\mathscr{Q}^{\mathrm{wrl}}$-reduction of $\mathscr{A}$, and $\mathscr{A}_3$ is the alternate $\mathscr{Q}^{\mathrm{wrl}}$-reduct of $\mathscr{A}$.

On the other hand, the greatest weakly left invariant fuzzy quasi-order R^{wli} on $\mathscr{A}$ and the afterset fuzzy recognizer $\mathscr{A}'_2 = \mathscr{A}/R^{\mathrm{wli}} = (A'_2, X, \delta^{A'_2}, \sigma^{A'_2}, \tau^{A'_2})$ are given by

$$R^{\mathrm{wli}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \delta^{A'_2}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \delta^{A'_2}_y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \sigma^{A'_2} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \tau^{A'_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

and both the greatest weakly right invariant fuzzy quasi-order and the greatest weakly left invariant fuzzy quasi-order on $\mathscr{A}'_2$ coincide with R^{wli}, and the afterset fuzzy recognizers of $\mathscr{A}'_2$ w.r.t. these fuzzy quasi-orders are isomorphic to $\mathscr{A}'_2$. This means that none alternate $\mathscr{Q}^{\mathrm{w}}$-reduction of $\mathscr{A}'_2$ decreases the number of states of $\mathscr{A}'_2$, i.e., none alternate $\mathscr{Q}^{\mathrm{wlr}}$-reduction decreases the number of states of $\mathscr{A}$, and we obtain that $\mathscr{A}$ is its own alternate $\mathscr{Q}^{\mathrm{wlr}}$-reduct.

Let us note that R^{wri} and R^{wri}_2 are also the greatest right invariant fuzzy quasi-orders on fuzzy recognizers $\mathscr{A}$ and $\mathscr{A}_2$, as well as on fuzzy automata (A, X, δ^A) and (A_2, X, δ^{A_2}), and R^{wli} is also the greatest left invariant fuzzy quasi-order on the fuzzy recognizer $\mathscr{A}$ and the fuzzy automaton (A, X, δ^A). Therefore, everything we have shown for weakly right invariant and weakly left invariant fuzzy quasi-orders holds also for right invariant and left invariant ones.

Example 7.1 shows that even if a fuzzy recognizer $\mathscr{A}$ is $\mathscr{Q}^{\mathrm{wri}}$- and/or $\mathscr{Q}^{\mathrm{wli}}$-reduced, it is still possible to continue reduction of the number of states of $\mathscr{A}$ alternating reductions by means of the greatest weakly right and left invariant fuzzy quasi-orders. Namely, the fuzzy recognizer $\mathscr{A}$ from this example is both $\mathscr{Q}^{\mathrm{wri}}$- and $\mathscr{Q}^{\mathrm{wli}}$-reduced, but alternate $\mathscr{Q}^{\mathrm{wrl}}$-reduction decreases its number of states. The same example also shows that shortest alternate $\mathscr{Q}^{\mathrm{wrl}}$- and $\mathscr{Q}^{\mathrm{wlr}}$-reductions can have different lengths, and that alternate $\mathscr{Q}^{\mathrm{wrl}}$- and $\mathscr{Q}^{\mathrm{wlr}}$-reducts can have different number of states. Indeed, alternate $\mathscr{Q}^{\mathrm{wrl}}$-reduction reduces $\mathscr{A}$ from three to two states, whereas alternate $\mathscr{Q}^{\mathrm{wlr}}$-reduction do not decrease number of states of $\mathscr{A}$. The above remarks also hold for alternate $\mathscr{Q}^{\mathrm{rl}}$- and $\mathscr{Q}^{\mathrm{lr}}$-reductions.

The state reduction of non-deterministic automata and recognizers by means of right invariant and left invariant quasi-orders has been studied by Champarnaud and Coulon [17, 18], Ilie, Navarro and Yu [39], and Ilie, Solis-Oba and Yu [40] (see also [36, 37]). In these papers a non-deterministic recognizer $\mathscr{A}$ has been reduced using factor recognizers $\mathscr{A}/E_{R^{\mathrm{ri}}}$ and $\mathscr{A}/E_{R^{\mathrm{li}}}$ w.r.t. natural equivalences of R^{ri} and R^{li}, but none of the mentioned authors have considered afterset recognizers $\mathscr{A}/R^{\mathrm{ri}}$ and $\mathscr{A}/R^{\mathrm{li}}$. As we have noted earlier, recognizers $\mathscr{A}/E_{R^{\mathrm{ri}}}$ and $\mathscr{A}/R^{\mathrm{ri}}$, as well as $\mathscr{A}/E_{R^{\mathrm{li}}}$ and $\mathscr{A}/R^{\mathrm{li}}$, are not necessary isomorphic, but they have the same number of states and both of them are equivalent to $\mathscr{A}$. Therefore, it is all the same if we use $\mathscr{A}/E_{R^{\mathrm{ri}}}$ or $\mathscr{A}/R^{\mathrm{ri}}$, and $\mathscr{A}/E_{R^{\mathrm{li}}}$ or $\mathscr{A}/R^{\mathrm{li}}$. However, there are differences if we work with alternate reductions. For the recognizer $\mathscr{A}$ with three states given in Example 7.1, natural equivalences $E_{R^{\mathrm{ri}}}$ and $E_{R^{\mathrm{li}}}$ coincide with the equality relation, so alternate reductions by means of these equivalences do not decrease the number of states of $\mathscr{A}$, but the alternate $\mathscr{Q}^{\mathrm{wrl}}$-reduction of $\mathscr{A}$ gives a recognizer with two states. The same conclusion can be drawn for alternate $\mathscr{E}$-reductions. Equivalences E^{ri} and E^{li} on A also coincide with the equality relation, and none alternate $\mathscr{E}$-reduction decrease the number of states of $\mathscr{A}$.

In alternate $\mathscr{Q}^{\mathrm{w}}$-reductions considered in Example 7.1 we have obtained three consecutive members which are isomorphic, and by this fact we have concluded that none alternate $\mathscr{Q}^{\mathrm{w}}$-reduction can further decrease the number of states. A similar conclusion we can draw in cases when we obtain a fuzzy recognizer with only one state. However, we have no yet a general procedure to decide whether we have reached the smallest number of states in an alternate $\mathscr{Q}$- or $\mathscr{Q}^{\mathrm{w}}$-reduction. An exception are alternate $\mathscr{E}$- and $\mathscr{E}^{\mathrm{w}}$-reductions of non-deterministic automata and recognizers, for which there exists such general procedure. Indeed, if after two successive steps the number of states did not changed, then we can be sure that we have reached the smallest number of states and this alternate $\mathscr{E}$- or $\mathscr{E}^{\mathrm{w}}$-reduction is finished. In other words, an alternate $\mathscr{E}$-reduction finishes when we obtain a non-deterministic automaton which is both $\mathscr{E}^{\mathrm{ri}}$- and $\mathscr{E}^{\mathrm{li}}$-reduced, and this automaton is an alternate $\mathscr{E}$-reduct of the staring automaton. The same holds for alternate $\mathscr{E}^{\mathrm{w}}$-reductions of non-deterministic recognizers. Alternate $\mathscr{Q}$- and $\mathscr{Q}^{\mathrm{w}}$-reductions do not have this property even in the case of non-deterministic automata and recognizers, because making an afterset automaton or recognizer w.r.t. an order relation we change the transition relation and we obtain an automaton

or recognizer which is not necessary isomorphic to the original one, what makes possible to continue an alternate $\mathscr{Q}$- or $\mathscr{Q}^{\mathrm{w}}$-reduction and decrease the number of states (see again Example 7.1). The same conclusion can be drawn for alternate $\mathscr{Q}$-, $\mathscr{Q}^{\mathrm{w}}$-, $\mathscr{E}$- and $\mathscr{E}^{\mathrm{w}}$-reductions of fuzzy automata and recognizers.

Finally, let us give several remarks concerning strongly right and left invariant fuzzy quasi-orders. It can be easily verified that for every fuzzy quasi-order R on a fuzzy automaton $\mathscr{A}$, the fuzzy order $\widetilde{R}$ on the afterset fuzzy automaton $\mathscr{A}/R$ is strongly invariant, i.e., it is both strongly right and strongly left invariant. Consequently, for the greatest right invariant fuzzy quasi-order R^{ri} on $\mathscr{A}$, by Theorem 7.1 it follows that $\widetilde{R}^{\mathrm{ri}}$ is the greatest right invariant fuzzy quasi-order on $\mathscr{A}/R^{\mathrm{ri}}$, and hence, $\widetilde{R}^{\mathrm{ri}}$ is the greatest strongly right invariant fuzzy quasi-order on $\mathscr{A}/R^{\mathrm{ri}}$, and every right invariant fuzzy quasi-order on $\mathscr{A}/R^{\mathrm{ri}}$ is a strongly right invariant.

However, for the greatest strongly right invariant fuzzy quasi-order R^{sri} on $\mathscr{A}$ we have that $\widetilde{R}^{\mathrm{sri}}$ is a strongly right invariant fuzzy quasi-order on $\mathscr{A}/R^{\mathrm{sri}}$, but the next example shows that it is not necessary the greatest element of $\mathscr{Q}^{\mathrm{sri}}(\mathscr{A})$. For that reason, the analogue of Theorem 7.3 does not hold for strongly right invariant fuzzy quasi-orders, i.e., the afterset fuzzy automaton $\mathscr{A}/R^{\mathrm{sri}}$ is not necessary $\mathscr{Q}^{\mathrm{sri}}$-reduced, and contrary to $\mathscr{Q}^{\mathrm{ri}}$-reductions, a $\mathscr{Q}^{\mathrm{sri}}$-reduction does not necessary stop after its first step. This will be also shown by the next example.

Example 7.2. Let $\mathscr{L}$ be the Boolean structure and let $\mathscr{A} = (A, X, \delta^A)$ be a fuzzy automaton over $\mathscr{L}$, where $A = \{1, 2, 3\}$, $X = \{x\}$, and a fuzzy transition relation δ_x^A is given by

$$\delta_x^A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then the greatest strongly right invariant fuzzy quasi-order R^{sri} on $\mathscr{A}$ is given by

$$R^{\mathrm{sri}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

the afterset fuzzy automaton $\mathscr{A}_2 = \mathscr{A}/R^{\mathrm{sri}} = (A_2, X, \delta^{A_2})$ has two states, i.e., $A_2 = \{1, 2\}$, and a fuzzy transition relation $\delta_x^{A_2}$ is given by

$$\delta_x^{A_2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Consequently, the greatest strongly right invariant fuzzy quasi-order R_2^{sri} on $\mathscr{A}_2$ is given by

$$R_2^{\mathrm{sri}} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and it reduces $\mathscr{A}_2$ to a fuzzy automaton $\mathscr{A}_3 = \mathscr{A}_2/R_2^{\mathrm{sri}} = (A_3, X, \delta^{A_3})$ having only one state and a fuzzy transition relation $\delta_x^{A_3} = \begin{bmatrix} 1 \end{bmatrix}$. Therefore, the sequence $\mathscr{A} = \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ is the shortest $\mathscr{Q}^{\mathrm{sri}}$-reduction of $\mathscr{A}$.

This example also shows that the converse implication in (a) of Theorem 7.1 does not necessary hold for strongly right invariant fuzzy quasi-orders. Namely, if we assume that S is the universal relation on A, then we have that $S/R^{\mathrm{sri}} = R_2^{\mathrm{sri}}$ is a strongly right invariant fuzzy quasi-order on $\mathscr{A}/R^{\mathrm{sri}}$, but S is not a strongly right invariant fuzzy quasi-order on $\mathscr{A}$.

8. An example demonstrating some applications to fuzzy discrete event systems

In this section we give an example demonstrating some applications of weakly left invariant fuzzy quasi-orders to fuzzy discrete event systems. A more complete study of fuzzy discrete event systems will be a subject of our further work.

A *discrete event system* (*DES*) is a dynamical system whose state space is described by a discrete set, and states evolve as a result of asynchronously occurring discrete events over time [15, 31]. Such systems have significant applications in many fields of computer science and engineering, such as concurrent and distributed software systems, computer and communication networks, manufacturing, transportation and traffic control systems, etc. Usually, a discrete event system is modeled by a finite state automaton (deterministic or nondeterministic), with events modeled by input letters, and the behavior of a discrete event system is described by the language generated by the automaton. However, in many situations states and state transitions, as well as control strategies, are somewhat imprecise, uncertain and vague. To take this kind of uncertainty into account, Lin and Ying extended classical discrete event systems to *fuzzy discrete event systems* (*FDES*) by proposing a fuzzy finite automaton model [48, 49]. Fuzzy discrete event systems have been since studied in a number of papers [12, 13, 14, 44, 48, 49, 50, 51, 69, 71], and they have been successfully applied to biomedical control for HIV/AIDS treatment planning, robotic control, intelligent vehicle control, waste-water treatment, examination of chemical reactions, and in other fields.

In [48, 49], and later in [14, 44, 69, 71], fuzzy discrete event systems have been modeled by automata with fuzzy states and fuzzy inputs, whose transition function is defined over the sets of fuzzy states and fuzzy inputs in a deterministic way. In fact, such an automaton can be regarded as the determinization of a fuzzy automaton (defined as in this paper) by means of the accessible fuzzy subset construction (see [32, 34]). On the other hand, in [12, 13, 51] fuzzy discrete event systems have been modeled by fuzzy automata with single crisp initial states. In all mentioned papers membership values have been taken in the Gödel or product structure.

Here, a fuzzy discrete event system will be modeled by a fuzzy finite recognizer $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ over a complete residuated lattice $\mathscr{L}$, defined as in Section 2.2. Two kinds of fuzzy languages associated with this fuzzy recognizer play a key role in study of fuzzy discrete event systems. The first one is the *fuzzy language $L(\mathscr{A})$ recognized by $\mathscr{A}$*, which is defined as in (22) (or (23)), and the second one is the *fuzzy language $L_g(\mathscr{A})$ generated by $\mathscr{A}$*, which is defined by

$$L_g(\mathscr{A})(u) = \bigvee_{a,b \in A} \sigma^A(a) \otimes \delta_*^A(a, u, b) = \bigvee_{b \in A} (\sigma^A \circ \delta_u^A)(b) = \bigvee_{b \in A} \sigma_u^A(b), \tag{78}$$

for every $u \in X^*$. Intuitively, $L_g(\mathscr{A})(u)$ represents the degree to which the input word u causes a transition from some initial state to any other state. Two fuzzy recognizers $\mathscr{A}$ and $\mathscr{B}$ are called *language-equivalent* if $L(\mathscr{A}) = L(\mathscr{B})$ and $L_g(\mathscr{A}) = L_g(\mathscr{B})$.

Discrete event models of complex dynamic systems are built rarely in a monolithic manner. Instead, a modular approach is used where models of individual components are built first, followed by the composition of these models to obtain the model of the overall system. In the automaton modeling formalism the composition of individual automata (that model interacting system components) is usually formalized by the *parallel composition* of automata. Once a complete system model has been obtained by parallel composition of a set of automata, the resulting monolithic model can be used to analyze the properties of the system.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and $\mathscr{B} = (A, Y, \delta^B, \sigma^B, \tau^B)$ be fuzzy recognizers. The *product* of $\mathscr{A}$ and $\mathscr{B}$ is a fuzzy recognizer $\mathscr{A} \times \mathscr{B} = (A \times B, X \cap Y, \delta^{A \times B}, \sigma^{A \times B}, \tau^{A \times B})$, defined by

$$\delta^{A \times B}((a, b), x, (a', b')) = \delta^A(a, x, a') \otimes \delta^B(b, x, b'),$$
$$\sigma^{A \times B}(a, b) = \sigma^A(a) \otimes \sigma^B(b), \quad \tau^{A \times B}(a, b) = \tau^A(a) \otimes \tau^B(b), \tag{79}$$

for all $a, a' \in A$, $b, b' \in B$ and $x \in X \cap Y$, and the *parallel composition* of $\mathscr{A}$ and $\mathscr{B}$ is a fuzzy recognizer $\mathscr{A} \| \mathscr{B} = (A \times B, X \cup Y, \delta^{A\|B}, \sigma^{A\|B}, \tau^{A\|B})$, defined by

$$\delta^{A\|B}((a, b), x, (a', b')) = \begin{cases} \delta^A(a, x, a') \otimes \delta^B(b, x, b') & \text{if } x \in X \cap Y \\ \delta^A(a, x, a') & \text{if } x \in X \setminus Y \text{ and } b = b' \\ \delta^B(b, x, b') & \text{if } x \in Y \setminus X \text{ and } a = a' \text{ '} \\ 0 & \text{otherwise} \end{cases} \tag{80}$$

$$\sigma^{A\|B}(a, b) = \sigma^A(a) \otimes \sigma^B(b), \quad \tau^{A\|B}(a, b) = \tau^A(a) \otimes \tau^B(b),$$

for all $a, a' \in A$, $b, b' \in B$. Associativity is used to extend the definition of parallel composition to more than two automata.

In the parallel composition of fuzzy automata $\mathscr{A}$ and $\mathscr{B}$, a common input letter from $X \cap Y$ is executed in both automata simultaneously, what means that these two automata are synchronized on the common input letter. On the other hand, a private input letter from $X \setminus Y$ is executed in $\mathscr{A}$, while $\mathscr{B}$ is staying in the same state, and similarly for private letters from $Y \setminus X$. Clearly, if $X = Y$, then the parallel composition reduces to the product. However, even if $X \neq Y$, the parallel composition of fuzzy automata can be regarded as the product of suitable input extensions of these fuzzy automata, what will be shown in the sequel. If $X \cap Y = \emptyset$, then no synchronized transitions occur and $\mathscr{A} \| \mathscr{B}$ is the *concurrent behavior* of $\mathscr{A}$ and $\mathscr{B}$. This behavior is often termed the *shuffle* of $\mathscr{A}$ and $\mathscr{B}$.

Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer and let Y be an alphabet such that $X \subseteq Y$. Let us define a new transition function $\delta^{A_Y} : A \times Y \times A \to L$ by

$$\delta^{A_Y}(a, x, a') = \begin{cases} \delta^A(a, x, a') & \text{if } x \in X \\ 1 & \text{if } x \in Y \setminus X \text{ and } a = a' \ , \\ 0 & \text{otherwise} \end{cases} \tag{81}$$

for all $a, a' \in A$ and $x \in Y$. Then a fuzzy recognizer $\mathscr{A}_Y = (A, Y, \delta^{A_Y}, \sigma^A, \tau^A)$ is called a *Y-input extension* of $\mathscr{A}$. In other words, input letters from X cause in $\mathscr{A}_Y$ the same transitions as in $\mathscr{A}$, while those from $Y \setminus X$ cause $\mathscr{A}_Y$ to stay in the same state. Evidently, $\delta_u^{A_Y}$ is the equality relation on A, for each $u \in (Y \setminus X)^*$.

An operation frequently performed on words and languages is the so-called natural projection, which transforms words over an alphabet Y to words over a smaller alphabet $X \subseteq Y$. Formally, a *natural projection*, or briefly a *projection*, is a mapping $\pi_X : Y^* \to X^*$, where $X \subseteq Y$, defined inductively by

$$\pi_X(w) = \begin{cases} e & \text{if } w \in (Y \setminus X)^* \\ w & \text{if } w \in X^* \\ \pi_X(u)\pi_X(v) & \text{if } w = uv, \text{ for some } u, v \in Y^* \end{cases} \tag{82}$$

for each $w \in Y^*$ (cf. [15]). In other words, the word $\pi_X(w) \in X^*$ is obtained from w by deleting all appearances of letters from $Y \setminus X$.

First we prove the following:

Lemma 8.1. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer, let Y be an alphabet such that $X \subseteq Y$, and let $\mathscr{A}_Y = (A, Y, \delta^{A_Y}, \sigma^A, \tau^A)$ be the Y-input extension of $\mathscr{A}$. Then for every $u \in Y^*$ we have that*

$$L_g(\mathscr{A}_Y)(u) = L_g(\mathscr{A})(\pi_X(u)) \ \text{ and } \ L(\mathscr{A}_Y)(u) = L(\mathscr{A})(\pi_X(u)).$$

Proof. An arbitrary word $u \in Y^*$ can be represented in the form $u = u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1}$, where $n \in \mathbb{N}$, $u_1, u_2, \ldots, u_{n+1} \in (Y \setminus X)^*$, and $v_1, v_2, \ldots, v_n \in X^*$, and clearly, $\pi_X(u) = v$, where $v = v_1 v_2 \cdots v_n$. Since $\delta_p^{A_Y}$ is the equality relation on A and $\delta_q^{A_Y} = \delta_q^A$, for all $p \in (Y \setminus X)^*$ and $q \in X^*$, then we have that

$$L_g(\mathscr{A}_Y)(u) = \bigvee_{a \in A}(\sigma^A \circ \delta_u^{A_Y})(a) = \bigvee_{a \in A}(\sigma^A \circ \delta_{u_1}^{A_Y} \circ \delta_{v_1}^{A_Y} \circ \delta_{u_2}^{A_Y} \circ \delta_{v_2}^{A_Y} \circ \cdots \circ \delta_{u_n}^{A_Y} \circ \delta_{v_n}^{A_Y} \circ \delta_{u_{n+1}}^{A_Y})(a)$$

$$= \bigvee_{a \in A}(\sigma^A \circ \delta_{v_1}^A \circ \delta_{v_2}^A \circ \cdots \circ \delta_{v_n}^A)(a) = \bigvee_{a \in A}(\sigma^A \circ \delta_v^A)(a) = L_g(\mathscr{A})(v) = L_g(\mathscr{A})(\pi_X(u)),$$

and similarly, $L(\mathscr{A}_Y)(u) = L(\mathscr{A})(\pi_X(u))$. $\square$

Now we prove the following:

Theorem 8.1. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ and $\mathscr{B} = (B, Y, \delta^B, \sigma^B, \tau^B)$ be fuzzy recognizers, let $Z = X \cup Y$, and let $\mathscr{A}_Z = (A, Z, \delta^{A_Z}, \sigma^A, \tau^A)$ and $\mathscr{B}_Z = (B, Z, \delta^{B_Z}, \sigma^B, \tau^B)$ be respectively their Z-input extensions.*

Then fuzzy recognizers $\mathscr{A} \| \mathscr{B}$ and $\mathscr{A}_Z \| \mathscr{B}_Z$ are isomorphic, and for each $u \in Z^*$ we have that

$$L_g(\mathscr{A} \| \mathscr{B})(u) = L_g(\mathscr{A}_Z)(u) \otimes L_g(\mathscr{B}_Z)(u) = L_g(\mathscr{A})(\pi_X(u)) \otimes L_g(\mathscr{B})(\pi_Y(u)), \tag{83}$$

$$L(\mathscr{A} \| \mathscr{B})(u) = L(\mathscr{A}_Z)(u) \otimes L(\mathscr{B}_Z)(u) = L(\mathscr{A})(\pi_X(u)) \otimes L(\mathscr{B})(\pi_Y(u)). \tag{84}$$

Proof. According to (81) and (80), for every $x \in Z = X \cup Y$ we have that

$$\delta^{A_Z \| B_Z}((a,b), x, (a',b')) = \delta^{A_Z}(a, x, a') \otimes \delta^{B_Z}(b, x, b')$$

$$= \begin{cases} \delta^A(a, x, a') \otimes \delta^B(b, x, b'), & \text{if } x \in X \cap Y \\ \delta^A(a, x, a') \otimes 1, & \text{if } x \in X \setminus Y \text{ and } b = b' \\ 1 \otimes \delta^B(b, x, b'), & \text{if } x \in Y \setminus X \text{ and } a = a' \end{cases}$$

$$= \delta^{A \| B}((a,b), x, (a',b')),$$

for every $a, a' \in A$, $b, b' \in B$. Since fuzzy recognizers $\mathscr{A}$ and $\mathscr{A}_Z$, as well as $\mathscr{B}$ and $\mathscr{B}_Z$, have the same fuzzy sets of initial and terminal states, we conclude that $\mathscr{A} \| \mathscr{B}$ and $\mathscr{A}_Z \| \mathscr{B}_Z$ are isomorphic. Moreover, according to Lemma 8.1, for each $u \in Z^* = (X \cup Y)^*$ we have that

$$L_g(\mathscr{A} \| \mathscr{B})(u) = L_g(\mathscr{A}_Z \| \mathscr{B}_Z)(u) = \bigvee_{(a,b),(a',b') \in A \times B} \sigma^{A \| B}(a, b) \otimes \delta^{A_Z \| B_Z}((a,b), u, (a',b'))$$

$$= \left(\bigvee_{a,a' \in A} \sigma^A(a) \otimes \delta^{A_Z}(a, u, a') \right) \otimes \left(\bigvee_{b,b' \in B} \sigma^B(b) \otimes \delta^{B_Z}(b, u, b') \right)$$

$$= L_g(\mathscr{A}_Z)(u) \otimes L_g(\mathscr{B}_Z)(u) = L_g(\mathscr{A})(\pi_X(u)) \otimes L_g(\mathscr{B})(\pi_Y(u)).$$

The rest of the assertion can be proved in a similar way. $\square$

In particular, if $X = Y$, i.e., if $\mathscr{A} \| \mathscr{B} = \mathscr{A} \times \mathscr{B}$, then by (83) and (84) it follows that

$$L_g(\mathscr{A} \times \mathscr{B})(u) = L_g(\mathscr{A})(u) \otimes L_g(\mathscr{B})(u), \tag{85}$$

$$L(\mathscr{A} \times \mathscr{B})(u) = L(\mathscr{A})(u) \otimes L(\mathscr{B})(u), \tag{86}$$

for every $u \in X^*$.

One of the key reasons for using automata to model discrete event systems is their amenability to analysis for answering various questions about the structure and behavior of the system, such as safety properties, blocking properties, diagnosability, etc. In the context of fuzzy automata we will consider blocking properties, which are originally concerned with the presence of deadlock and/or livelock in the automaton, i.e., with the problem of checking whether a terminal state can be reached from every reachable state.

A *prefix-closure* of a fuzzy language $f \in L^{X^*}$, denoted by $\overline{f}$, is a fuzzy language in L^{X^*} defined by

$$\overline{f}(u) = \bigvee_{v \in X^*} f(uv), \tag{87}$$

for any $u \in X^*$. It is easy to verify that the mapping $f \mapsto \overline{f}$ is a closure operator on L^{X^*}, i.e., for arbitrary $f, f_1, f_2 \in L^{X^*}$ we have that

$$f \leqslant \overline{f}, \quad \overline{\overline{f}} = \overline{f} \quad \text{and} \quad f_1 \leqslant f_2 \text{ implies } \overline{f_1} \leqslant \overline{f_2}. \tag{88}$$

A fuzzy language $f \in L^{X^*}$ is called *prefix-closed* if $f = \overline{f}$.

We have that the following is true:

Lemma 8.2. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer. Then*

$$L(\mathscr{A}) \leqslant \overline{L(\mathscr{A})} \leqslant L_g(\mathscr{A}) = \overline{L_g(\mathscr{A})}. \tag{89}$$

Proof. According to $L(\mathscr{A}) \leqslant L_g(\mathscr{A})$ and (88), it is enough to prove $\overline{L_g(\mathscr{A})} \leqslant L_g(\mathscr{A})$. Indeed, for arbitrary $a, b, c \in A$ and $u, v \in X^*$ we have that

$$\sigma^A(a) \otimes \delta_u^A(a, c) \otimes \delta_v^A(c, b) \leqslant \sigma^A(a) \otimes \delta_u^A(a, c) \leqslant L_g(\mathscr{A})(u),$$

what implies that

$$\overline{L_g(\mathscr{A})}(u) = \bigvee_{v \in X^*} L_g(\mathscr{A})(uv) = \bigvee_{v \in X^*} \bigvee_{a,b \in A} \sigma^A(a) \otimes \delta_{uv}^A(a, b)$$

$$= \bigvee_{v \in X^*} \bigvee_{a,b \in A} \bigvee_{c \in A} \sigma^A(a) \otimes \delta_u^A(a, c) \otimes \delta_v^A(c, b) = \bigvee_{a,c \in A} \left(\sigma^A(a) \otimes \delta_u^A(a, c) \right) \otimes \left(\bigvee_{v \in X^*} \bigvee_{b \in A} \delta_v^A(c, b) \right)$$

$$\leqslant \bigvee_{a,c \in A} \sigma^A(a) \otimes \delta_u^A(a, c) = L_g(\mathscr{A})(u),$$

for every $u \in X^*$. Therefore, $\overline{L_g(\mathscr{A})} \leqslant L_g(\mathscr{A})$. $\square$

It is worth noting that the fuzzy language $\overline{L(\mathscr{A})}$ can be represented by

$$\overline{L(\mathscr{A})}(u) = \bigvee_{v \in X^*} L(\mathscr{A})(uv) = \bigvee_{v \in X^*} \sigma^A \circ \delta_{uv}^A \circ \tau^A = \bigvee_{v \in X^*} \sigma^A \circ \delta_u^A \circ \delta_v^A \circ \tau^A = \bigvee_{v \in X^*} \sigma_u^A \circ \tau_v^A,$$

for every $u \in X^*$.

A fuzzy recognizer $\mathscr{A}$ is said to be *blocking* if $\overline{L(\mathscr{A})} < L_g(\mathscr{A})$, where the inequality is proper, and otherwise, if $\overline{L(\mathscr{A})} = L_g(\mathscr{A})$, then $\mathscr{A}$ is referred to as *nonblocking*. These concepts generalize related concepts of the crisp discrete event systems theory, where a crisp automaton is considered to be blocking if it can reach a state from which no terminal state can be reached anymore. This includes both the possibility of a deadlock, where an automaton is stuck and unable to continue at all, and a livelock, where an automaton continues to run forever without achieving any further progress.

When we work with parallel compositions, the term conflicting is used instead of blocking. Namely, fuzzy recognizers $\mathscr{A}$ and $\mathscr{B}$ are said to be *nonconflicting* if their parallel composition $\mathscr{A} \| \mathscr{B}$ is nonblocking, and otherwise they are said to be *conflicting*. The parallel composition of a set of automata may be blocking even if each of the individual components is nonblocking (cf. [15]), and hence, it is necessary to examine the transition structure of the parallel composition to answer blocking properties. But, the size of the state set of the parallel composition may in the worst case grow exponentially in the number of automata that are composed. This process is known as the curse of dimensionality in the study of complex systems composed of many interacting components.

The mentioned problems in analysis of large discrete event models may be mitigated if we adopt modular reasoning, which can make it possible to replace components in the parallel composition by smaller equivalent automata, and then to analyse a simpler system. Such an approach has been used in [54] in study of conflicting properties of crisp discrete event systems. Here we will show that every fuzzy recognizer $\mathscr{A}$ is conflict-equivalent with the afterset fuzzy recognizer $\mathscr{A}/R$ w.r.t. any weakly left invariant fuzzy quasi-order R on $\mathscr{A}$. This means that in the parallel composition of fuzzy recognizers every component can be replaced by such afterset fuzzy recognizer, what results in a smaller fuzzy recognizer to be analysed, and do not affect conflicting properties of the components.

Two fuzzy recognizers $\mathscr{A}$ and $\mathscr{B}$ are said to be *conflict-equivalent* if for every fuzzy recognizer $\mathscr{C}$ we have that $\mathscr{A} \| \mathscr{C}$ is nonblocking if and only if $\mathscr{B} \| \mathscr{C}$ is nonblocking, i.e., if $\mathscr{A}$ and $\mathscr{B}$ are nonconflicting (conflicting) with the same fuzzy recognizers (cf. [54]).

Now we are ready to state and prove the main results of this section.

Theorem 8.2. *Let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer and let R be a weakly left invariant fuzzy quasi-order on $\mathscr{A}$. Then fuzzy recognizers $\mathscr{A}$ and $\mathscr{A}/R$ are language-equivalent, and consequently, they are conflict-equivalent.*

Proof. As we already know, $L(\mathscr{A}) = L(\mathscr{A}/R)$. Moreover, according to the dual statement of (71), for an arbitrary $u = x_1 \cdots x_n \in X^*$, where $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have that

$$
\begin{aligned}
L_g(\mathscr{A}/R)(u) &= \bigvee_{b \in A} (\sigma^{A/R} \circ \delta_u^{A/R})(R_b) = \bigvee_{b \in A} (\sigma^{A/R} \circ \delta_{x_1}^{A/R} \circ \cdots \circ \delta_{x_n}^{A/R})(R_b) \\
&= \bigvee_{a_1, a_2, \ldots, a_n, b \in A} \sigma^{A/R}(R_{a_1}) \otimes \delta_{x_1}^{A/R}(R_{a_1}, R_{a_2}) \otimes \cdots \otimes \delta_{x_n}^{A/R}(R_{a_n}, R_b) \\
&= \bigvee_{a_1, a_2, \ldots, a_n, b \in A} (\sigma^A \circ R)(a_1) \otimes (R \circ \delta_{x_1}^A \circ R)(a_1, a_2) \otimes \cdots \otimes (R \circ \delta_{x_n}^A \circ R)(a_n, b) \\
&= \bigvee_{b \in A} (\sigma^A \circ R \circ \delta_{x_1}^A \circ R \circ \cdots \circ R \circ \delta_{x_n}^A \circ R)(b) \\
&= \bigvee_{b \in A} (\sigma^A \circ \delta_{x_1}^A \circ \cdots \circ \delta_{x_n}^A)(b) = \bigvee_{b \in A} (\sigma^A \circ \delta_u^A)(b) = L_g(\mathscr{A})(u),
\end{aligned}
$$

and therefore, $L_g(\mathscr{A}/R) = L_g(\mathscr{A})$. Hence, $\mathscr{A}$ and $\mathscr{A}/R$ are language-equivalent.

Next, let $\mathscr{B} = (B, Y, \delta^B, \sigma^B, \tau^B)$ be an arbitrary fuzzy recognizer, and let $Z = X \cup Y$. By the language-equivalence of $\mathscr{A}$ and $\mathscr{A}/R$ and Theorem 8.1, for every $u \in Z^* = (X \cup Y)^*$ we have that

$$
\begin{aligned}
L_g((\mathscr{A}/R)\|\mathscr{B})(u) &= L_g((\mathscr{A}/R)_Z)(u) \otimes L_g(\mathscr{B}_Z)(u) = L_g((\mathscr{A}/R))(\pi_X(u)) \otimes L_g(\mathscr{B})(\pi_Y(u)) \\
&= L_g(\mathscr{A})(\pi_X(u)) \otimes L_g(\mathscr{B})(\pi_Y(u) = L_g(\mathscr{A}_Z)(u) \otimes L_g(\mathscr{B}_Z)(u) = L_g(\mathscr{A}\|\mathscr{B})(u),
\end{aligned}
$$

and hence, $L_g((\mathscr{A}/R)\|\mathscr{B}) = L_g(\mathscr{A}\|\mathscr{B})$. Similarly we prove that $L((\mathscr{A}/R)\|\mathscr{B}) = L(\mathscr{A}\|\mathscr{B})$, and by this it follows that $\overline{L((\mathscr{A}/R)\|\mathscr{B})} = \overline{L(\mathscr{A}\|\mathscr{B})}$.

Hence, we have that $\overline{L(\mathscr{A}\|\mathscr{B})} = L_g(\mathscr{A}\|\mathscr{B})$ if and only if $\overline{L((\mathscr{A}/R)\|\mathscr{B})} = L_g((\mathscr{A}/R)\|\mathscr{B})$, what means that $\mathscr{A}$ and $\mathscr{A}/R$ are conflict-equivalent. $\square$

The following example shows that the previous theorem do not hold for weakly right invariant fuzzy quasi-orders, i.e., a fuzzy recognizer and its afterset fuzzy recognizer w.r.t. a weakly right invariant fuzzy quasi-order are not necessary language-equivalent nor conflict-equivalent.

Example 8.1. Let $\mathscr{L}$ be the Boolean structure and let $\mathscr{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ be a fuzzy recognizer over $\mathscr{L}$, where $A = \{1, 2, 3, 4\}$, $X = \{x\}$, and δ_x^A, σ^A and τ^A are given by

$$
\delta_x^A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \sigma^A = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}, \quad \tau^A = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.
$$

For every $u \in X^*$ we have that

$$
\overline{L(\mathscr{A})}(u) = L_g(\mathscr{A})(u) = \begin{cases} 1 & \text{if } u = e \text{ or } u = x \\ 0 & \text{if } u = x^n, \text{ for some } n \geq 2 \end{cases},
$$

and hence, the fuzzy recognizer $\mathscr{A}$ is nonblocking.

A fuzzy relation R on A given by

$$
R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix},
$$

is a weakly right invariant fuzzy quasi-order on $\mathscr{A}$ (it is just the greatest one), and the related afterset fuzzy recognizer is $\mathscr{A}/R = (A/R, X, \delta^{A/R}, \sigma^{A/R}, \tau^{A/R})$, where $\delta_x^{A/R}$, $\sigma^{A/R}$ and $\tau^{A/R}$ are given by

$$\delta_x^{A/R} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \sigma^{A/R} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad \tau^{A/R} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For every $u \in X^*$ we have that

$$\overline{L(\mathscr{A}/R)}(u) = \begin{cases} 1 & \text{if } u = e \text{ or } u = x \\ 0 & \text{if } u = x^n, \text{ for some } n \geqslant 2 \end{cases},$$

and $L_g(\mathscr{A}/R)(u) = 1$, for each $u \in X^*$. Hence, $\overline{L(\mathscr{A}/R)} < L_g(\mathscr{A}/R)$, and we have that the fuzzy recognizer $\mathscr{A}/R$ is blocking. We also have that $L_g(\mathscr{A}) \neq L_g(\mathscr{A}/R)$, what means that $\mathscr{A}$ and $\mathscr{A}/R$ are not language-equivalent.

Next, let $\mathscr{B} = (B, X, \delta^B, \sigma^B, \tau^B)$, where $B = \{b\}$, $\delta^B(b, x, b) = 1$, for each $x \in X$, and $\sigma^B(b) = \tau^B(b) = 1$. Then $\mathscr{A}\|\mathscr{B} = \mathscr{A} \times \mathscr{B}$, and by (85) and (86) it follows that

$$L_g(\mathscr{A}\|\mathscr{B}) = L_g(\mathscr{A}) \otimes L_g(\mathscr{B}) = L_g(\mathscr{A}), \quad L(\mathscr{A}\|\mathscr{B}) = L(\mathscr{A}) \otimes L(\mathscr{B}) = L(\mathscr{A}),$$
$$L_g((\mathscr{A}/R)\|\mathscr{B}) = L_g(\mathscr{A}/R) \otimes L_g(\mathscr{B}) = L_g(\mathscr{A}/R), \quad L((\mathscr{A}/R)\|\mathscr{B}) = L(\mathscr{A}/R) \otimes L(\mathscr{B}) = L(\mathscr{A}/R).$$

Therefore,

$$\overline{L(\mathscr{A}\|\mathscr{B})} = \overline{L(\mathscr{A})} = L_g(\mathscr{A}) = L_g(\mathscr{A}\|\mathscr{B}), \quad \overline{L((\mathscr{A}/R)\|\mathscr{B})} = \overline{L(\mathscr{A}/R)} < L_g(\mathscr{A}/R) = L_g((\mathscr{A}/R)\|\mathscr{B}),$$

what means that $\mathscr{A}\|\mathscr{B}$ is nonblocking and $(\mathscr{A}/R)\|\mathscr{B}$ is blocking, and hence, $\mathscr{A}$ and $\mathscr{A}/R$ are not conflict-equivalent.

9. Concluding remarks

In our recent paper we have established close relationships between the state reduction of a fuzzy recognizer and the resolution of a particular system of fuzzy relation equations. We have studied reductions by means of those solutions which are fuzzy equivalences. In this paper we demonstrated that in some cases better reductions can be obtained using the solutions of this system that are fuzzy quasi-orders. Although by Theorem 3.4 we have proved that in the general case fuzzy quasi-orders and fuzzy equivalences are equally good in the state reduction, we have shown that in some cases fuzzy quasi-orders give better reductions. The meaning of state reductions by means of fuzzy quasi-orders and fuzzy equivalences is in their possible effectiveness, as opposed to the minimization which is not effective. By Theorem 3.5 we have shown that minimization of some fuzzy recognizers can not be realized as its state reduction by means of fuzzy quasi-orders or fuzzy equivalences.

We gave a procedure for computing the greatest right invariant fuzzy quasi-order on a fuzzy automaton or fuzzy recognizer, which works if the underlying structure of truth values is a locally finite, but not only in this case. We also gave procedures for computing the greatest right invariant crisp quasi-order and the greatest strongly right invariant fuzzy quasi-order. They work for fuzzy automata over any complete residuated lattice. However, although these procedures are applicable to a larger class of fuzzy automata, we have proved that right invariant fuzzy quasi-orders give better reductions than right invariant crisp quasi-orders and strongly right invariant fuzzy quasi-orders. We also have studied a more general type of fuzzy quasi-orders, weakly right and left invariant ones. These fuzzy quasi-orders give better reductions than right and left invariant ones, but are harder to compute. In fact, weakly right and left invariant fuzzy quasi-orders on a fuzzy recognizer are defined by means of two systems of fuzzy relation equations whose resolution include determinization of this fuzzy recognizer and its reverse fuzzy recognizer.

Finally, we have shown that better results in the state reduction can be achieved if we alternate reductions by means of right and left invariant fuzzy quasi-orders, or weakly right and left invariant fuzzy quasi-orders. Furthermore, we show that alternate reductions by means of fuzzy quasi-orders give better results

than those by means of fuzzy equivalences. It is worth noting that the presented state reduction methods are based on the construction of the afterset fuzzy recognizer w.r.t. a fuzzy quasi-order, and we have proved that such approach gives better results in alternate reductions than approach by Champarnaud and Coulon, Ilie, Navarro and Yu, and Ilie, Solis-Oba and Yu, whose state reduction methods are based on the construction of the factor recognizer w.r.t. the natural equivalence of a quasi-order.

At the end of the paper we have demonstrated some applications of weakly left invariant fuzzy quasi-orders in conflict analysis of fuzzy discrete event systems. Another interesting problem is application of state reductions by means of fuzzy quasi-orders in fault diagnosis of discrete event systems. Since this problem is very complex and deserves special attention, it will be discussed in a separate paper.

Several questions remained unsolved, too. They include determining more precise conditions under which our iterative procedures for computing the greatest right and left invariant fuzzy quasi-orders terminate in a finite number of steps, finding alternative algorithms for computing the greatest right and left invariant fuzzy quasi-orders for use in cases where the mentioned iterative procedures do not terminate in a finite number of steps, as well as finding even faster algorithms for computing such fuzzy quasi-orders, and general procedures to decide whether we have reached the smallest number of states in alternate reductions, and so forth. All these issues will be topics of our future research.

References

[1] W. Bandler, L.J. Kohout, Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems, in: S.K. Wang, P.P. Chang (Eds.), Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems, Plenum Press, New York, 1980, pp. 341–367.

[2] N. C. Basak, A. Gupta, On quotient machines of a fuzzy automaton and the minimal machine, Fuzzy Sets and Systems 125 (2002) 223–229.

[3] R. Bělohlávek, Fuzzy Relational Systems: Foundations and Principles, Kluwer, New York, 2002.

[4] R. Bělohlávek, V. Vychodil, Fuzzy Equational Logic, Springer, Berlin-Heidelberg, 2005.

[5] U. Bodenhofer, A similarity-based generalization of fuzzy orderings preserving the classical axioms, Internat. J. Uncertain. Fuzziness Knowledge-based Systems 8 (2000) 593–610.

[6] U. Bodenhofer, Representations and constructions of similarity-based fuzzy orderings, Fuzzy Sets and Systems 137 (2003) 113–136.

[7] U. Bodenhofer, B. De Baets, J. Fodor, A compendium of fuzzy weak orders: Representations and constructions, Fuzzy Sets and Systems 158 (2007) 811–829.

[8] U. Bodenhofer, F. Klawonn, A formal study of linearity axioms for fuzzy orderings, Fuzzy Sets and Systems 145 (2004) 323–354.

[9] S. Burris, H. P. Sankappanavar, A Course in Universal Algebra, Springer-Verlag, New York, 1981.

[10] C. S. Calude, E. Calude, B. Khoussainov, Finite nondeterministic automata: Simulation and minimality, Theoretical Computer Science 242 (2000) 219–235.

[11] C. Câmpeanu, N. Sântean, S. Yu, Mergible states in large NFA, Theoretical Computer Science 330 (2005) 23–34.

[12] Y. Z. Cao, M. S. Ying, Supervisory control of fuzzy discrete event systems, IEEE Transactions on Systems, Man, and Cybernetics – Part B 35 (2005) 366–371.

[13] Y. Z. Cao, M. S. Ying, Observability and decentralized control of fuzzy discrete-event systems, IEEE Transactions Fuzzy Systems 14 (2006) 202–216.

[14] Y. Z. Cao, M. S. Ying, G. Q. Chen, State-based control of fuzzy discrete-event systems, IEEE Transactions on Systems, Man, and Cybernetics – Part B 37(2007) 410–424.

[15] C. G. Cassandras, S. Lafortune, Introduction to Discrete Event Systems, Springer, 2008.

[16] J.-M. Champarnaud, F. Coulon, Theoretical study and implementation of the canonical automaton, Technical Report AIA 2003.03, LIFAR, Université de Rouen, 2003.

[17] J.-M. Champarnaud, F. Coulon, NFA reduction algorithms by means of regular inequalities, in: Z. Ésik, and Z. Fülöp (eds.), DLT 2003, Lecture Notes in Computer Science 2710 (2003) 194–205.

[18] J.-M. Champarnaud, F. Coulon, NFA reduction algorithms by means of regular inequalities, Theoretical Computer Science 327 (2004) 241–253 (erratum: Theoretical Computer Science 347 (2005) 437–40).

[19] J.-M. Champarnaud, D. Ziadi, New finite automaton constructions based on canonical derivatives, in: S. Yu, A. Paun (eds.), CIAA 2000, Springer, Berlin, Lecture Notes in Computer Science 2088 (2001) 94–104.

[20] J.-M. Champarnaud, D. Ziadi, Computing the equation automaton of a regular expression in $\mathcal{O}(s^2)$ space and time, in: A. Amir, G. Landau (eds.), CPM 2001, Springer, Berlin, Lecture Notes in Computer Science 2089 (2001) 157–168.

[21] W. Cheng, Z. Mo, Minimization algorithm of fuzzy finite automata, Fuzzy Sets and Systems 141 (2004) 439–448.

[22] M. Ćirić, J. Ignjatović, S. Bogdanović, Fuzzy equivalence relations and their equivalence classes, Fuzzy Sets and Systems 158 (2007) 1295–1313.

[23] M. Ćirić, J. Ignjatović, S. Bogdanović, Uniform fuzzy relations and fuzzy mappings, Fuzzy Sets and Systems 160 (2009) 1054–1081.

[24] M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Factorization of fuzzy automata, in: E. Csuhaj-Varju, and Z. Ésik (eds.), FCT 2007, Lecture Notes in Computer Science 4639 (2007) 213–225.

[25] M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Fuzzy relation equations and reduction of fuzzy automata, Journal of Computer and System Sciences 76 (2010) 609–633.

[26] B. De Baets, E. Kerre, The cutting of compositions, Fuzzy Sets and Systems 62 (1994) 295–309.

[27] M. De Cock, E. E. Kerre, Fuzzy modifiers based on fuzzy relations, Information Sciences 160 (2004) 173–199.

[28] A. Dovier, C. Piazza, A. Policriti, An efficient algorithm for computing bisimulation equivalence, Theoretical Computer Science 311 (2004) 221–256.

[29] R. Gentilini, C. Piazza, A. Policriti, From bisimulation to simulation: coarsest partition problems, Journal of Automated Reasoning 31 (2003) 73–103.

[30] U. Höhle, Commutative, residuated ℓ-monoids, in: U. Höhle and E. P. Klement (Eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Boston, Dordrecht, 1995, pp. 53–106.

[31] B. Hrúz, M. C. Zhou, Modeling and control of discrete-event dynamical systems: with Petri nets and other tools, Springer, 2007.

[32] J. Ignjatović, M. Ćirić, S. Bogdanović, Determinization of fuzzy automata with membership values in complete residuated lattices, Information Sciences 178 (2008) 164–180.

[33] J. Ignjatović, M. Ćirić, S. Bogdanović, Fuzzy homomorphisms of algebras, Fuzzy Sets and Systems 160 (2009) 2345–2365.

[34] J. Ignjatović, M. Ćirić, S. Bogdanović, T. Petković, Myhill-Nerode type theory for fuzzy languages and automata, Fuzzy sets and Systems 161 (2010) 1288–1324.

[35] L. Ilie, S. Yu, Constructing NFAs by optimal use of positions in regular expressions, in: A. Apostolico, M. Takeda (eds.), CPM 2002, Springer, Berlin, Lecture Notes in Computer Science 2373 (2002) 279–288.

[36] L. Ilie, S. Yu, Algorithms for computing small NFAs, in: K. Diks et al. (eds): MFCS 2002, Lecture Notes in Computer Science 2420 (2002) 328–340.

[37] L. Ilie, S. Yu, Reducing NFAs by invariant equivalences, Theoretical Computer Science 306 (2003) 373–390.

[38] L. Ilie, S. Yu, Follow automata, Information and Computation 186 (2003) 140-162

[39] L. Ilie, G. Navarro, S. Yu, On NFA reductions, in: J. Karhumäki et al. (eds): Theory is Forever, Lecture Notes in Computer Science 3113 (2004) 112–124.

[40] L. Ilie, R. Solis-Oba, S. Yu, Reducing the size of NFAs by using equivalences and preorders, in: A. Apostolico, M. Crochemore, and K. Park (eds): CPM 2005, Lecture Notes in Computer Science 3537 (2005) 310–321.

[41] T. Jiang, B. Ravikumar, Minimal NFA problems are hard, SIAM J. Comput. 22 (6) (1993) 1117–1141.

[42] T. Kameda, P. Weiner, On the state minimization of nondeterministic finite automata, IEEE Trans. Computers C-19(7) (1970) 617–627.

[43] P. C. Kannellakis, S. A. Smolka, CCS expressions, finite state processes, and three problems of equivalence, Information and Computation 86 (1990), 43–68.

[44] E. Kilic, Diagnosability of fuzzy discrete event systems, Information Sciences 178 (2008) 858–870.

[45] G. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic, Prentice-Hall PTR, 1995.

[46] H. Lei, Y. M. Li, Minimization of states in automata theory based on finite lattice-ordered monoids, Information Sciences 177 (2007) 1413–1421.

[47] Y. M. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids, Fuzzy Sets and Systems 156 (2005) 68–92.

[48] F. Lin, H. Ying, Fuzzy discrete event systems and their observability, in: Proceedings of the 2001 IFSA/NAFIP Conference, 2001, pp. 1271–1276.

[49] F. Lin, H. Ying, Modeling and control of fuzzy discrete event systems, IEEE Transactions on Man, Systems and Cybernetics – Part B 32 (2002) 408–415.

[50] F. Lin, H. Ying, R. D. MacArthur, J. A. Cohn, D. Barth-Jones, L. R. Crane, Decision making in fuzzy discrete event systems, Information Sciences 177 (2007) 3749–3763.

[51] J. P. Liu, Y. M. Li, The relationship of controllability between classical and fuzzy discrete-event systems, Information Sciences 178 (2008) 4142–4151.

[52] N. Lynch, F. Vaandrager, Forward and backward simulations: Part I. Untimed systems, Information and Computation 121 (1995), 214–233.

[53] D. S. Malik, J. N. Mordeson, M. K. Sen, Minimization of fuzzy finite automata, Information Sciences 113 (1999) 323–330.

[54] R. Malik, D. Streader, S. Reeves, Fair testing revisited: A process-algebraic characterisation of conflicts, in: F. Wang (ed.), ATVA 2004, Lecture Notes in Computer Science 3299 (2004) 120–134.

[55] B. F. Melnikov, A new algorithm of the state-minimization for the nondeterministic finite automata, Korean J. Comput. Appl. Math. 6(2) (1999) 277–290.

[56] B. F. Melnikov, Once more about the state-minimization of the nondeterministic finite automata, Korean J. Comput. Appl. Math. 7(3) (2000) 655–662.

[57] R. Milner, A calculus of communicating systems, in G. Goos and J. Hartmanis (eds.), Lecture Notes in Computer Science 92 (1980), Springer.

[58] R. Milner, Communication and Concurrency, Prentice-Hall International, 1989.

[59] R. Milner, Communicating and Mobile Systems: the π-Calculus, Cambridge University Press, Cambridge, 1999.

[60] J. N. Mordeson, D. S. Malik, Fuzzy Automata and Languages: Theory and Applications, Chapman & Hall/CRC, Boca Raton, London, 2002.

[61] R. Paige, R. E. Tarjan, Three partition refinement algorithms, SIAM J. Comput. 16(6) (1987) 973–989.

[62] D. Park, Concurrency and automata on infinite sequences, in: P. Deussen (ed.), Proc. 5th GI Conf., Karlsruhe, Germany, Springer-Verlag, Lecture Notes in Computer Science 104 (1981), 167–183.

[63] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems 147 (2004) 363–383.

[64] I. Perfilieva, S. Gottwald, Solvability and approximate solvability of fuzzy relation equations, Internat. J. Gen. Systems 32 (2003) 361–372.
[65] I. Perfilieva, V. Novák, System of fuzzy relation equations as a continuous model of IF-THEN rules, Information Sciences 177 (2007) 3218–3227.
[66] T. Petković, Congruences and homomorphisms of fuzzy automata, Fuzzy Sets and Systems 157 (2006) 444–458.
[67] D. W. Qiu, Automata theory based on completed residuated lattice-valued logic (I), Science in China, Ser. F, 44 (6) (2001) 419–429.
[68] D. W. Qiu, Automata theory based on completed residuated lattice-valued logic (II), Science in China, Ser. F, 45 (6) (2002) 442–452.
[69] D. Qiu, Supervisory control of fuzzy discrete event systems: a formal approach, IEEE Transactions on Systems, Man and Cybernetics – Part B 35 (2005) 72–88.
[70] D. W. Qiu, Pumping lemma in automata theory based on complete residuated lattice-valued logic: A note, Fuzzy Sets and Systems 157 (2006) 2128–2138.
[71] D. W. Qiu, F. C. Liu, Fuzzy discrete-event systems under fuzzy observability and a test algorithm, IEEE Transactions on Fuzzy Systems 17 (3) (2009), 578–589.
[72] F. Ranzato, F. Tapparo, Generalizing the PaigeTarjan algorithm by abstract interpretation, Information and Computation 206 (2008) 620–651.
[73] M. Roggenbach, M. Majster-Cederbaum, Towards a unified view of bisimulation: a comparative study, Theoretical Computer Science 238 (2000) 81-130.
[74] E. Sanchez, Resolution of composite fuzzy relation equations, Information and Control 30 (1976) 38–48.
[75] D. Sangiorgi, On the origins of bisimulation, coinduction, and fixed points, Technical Report UBLCS-2007-24, Department of Computer Science, University of Bologna, 2007.
[76] H. Sengoku, Minimization of nondeterministic finite automata, Master thesis, Kyoto University, 1992.
[77] S. Yu, Regular languages, in: Handbook of Formal Languages (G. Rozenberg, A. Salomaa, eds.), Vol 1, Springer-Verlag, Berlin - Heidelberg, 1997, pp. 41–110.
[78] L. A. Zadeh, Toward a generalized theory of uncertainty (GTU) – an outline, Information Sciences 172 (2005) 1–40.
[79] L. A. Zadeh, Is there a need for fuzzy logic?, Information Sciences 178 (2008) 2751–2779.