CHROMATIC NUMBER OF RANDOM KNESER HYPERGRAPHS

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Abstract. Recently, Kupavskii [*On random subgraphs of Kneser and Schrijver graphs. J. Combin. Theory Ser. A,* 2016.] investigated the chromatic number of random Kneser graphs $\mathrm{KG}_{n,k}(\rho)$ and proved that, in many cases, the chromatic numbers of the random Kneser graph $\mathrm{KG}_{n,k}(\rho)$ and the Kneser graph $\mathrm{KG}_{n,k}$ are almost surely closed. He also marked the studying of the chromatic number of random Kneser hypergraphs $\mathrm{KG}^r_{n,k}(\rho)$ as a very interesting problem. With the help of $\mathbb{Z}_p$-Tucker lemma, a combinatorial generalization of the Borsuk-Ulam theorem, we generalize Kupavskii's result to random general Kneser hypergraphs by introducing an almost surely lower bound for the chromatic number of them. Roughly speaking, as a special case of our result, we show that the chromatic numbers of the random Kneser hypergraph $\mathrm{KG}^r_{n,k}(\rho)$ and the Kneser hypergraph $\mathrm{KG}^r_{n,k}$ are almost surely closed in many cases. Moreover, restricting to the Kneser and Schrijver graphs, we present a purely combinatorial proof for an improvement of Kupavskii's results.

Also, for any hypergraph $\mathcal{H}$, we present a lower bound for the minimum number of colors required in a coloring of $\mathrm{KG}^r(\mathcal{H})$ with no monochromatic $K^r_{t,\ldots,t}$ subhypergraph, where $K^r_{t,\ldots,t}$ is the complete r-uniform r-partite hypergraph with tr vertices such that each of its parts has t vertices. This result generalizes the lower bound for the chromatic number of $\mathrm{KG}^r(\mathcal{H})$ found by the present authors [*On the chromatic number of general Kneser hypergraphs. J. Combin. Theory, Ser. B,* 2015.].

Keywords: random Kneser hypergraphs, chromatic number of hypergraphs, $\mathbb{Z}_p$-Tucker lemma

1. Introduction and Main Results

For positive integers n and k, by the symbols $[n]$ and $\binom{[n]}{k}$, we mean the set $\{1,\ldots,n\}$ and the set of all k-subsets of $[n]$, respectively. A hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H}),E(\mathcal{H}))$, where $V(\mathcal{H})$ is a finite nonempty set and $E(\mathcal{H})$ is a family of distinct nonempty subsets of $V(\mathcal{H})$. Respectively, the sets $V(\mathcal{H})$ and $E(\mathcal{H})$ are called the vertex set and the edge set of $\mathcal{H}$. If each edge of $\mathcal{H}$ has the cardinality r, then $\mathcal{H}$ is called r-uniform. A 2-uniform hypergraph is simply called a graph. Let $\mathcal{H}$ be an r-uniform hypergraph and $V_1,\ldots,V_r$ be pairwise disjoint subsets of $V(\mathcal{H})$. The hypergraph $\mathcal{H}[V_1,\ldots,V_r]$ is a subhypergraph of $\mathcal{H}$ whose vertex set and edge set are respectively $\bigcup\limits_{i=1}^{r} V_i$ and

$$E(\mathcal{H}[U_1,\ldots,U_r])=\left\{e\in E(\mathcal{H}):\ e\subseteq\bigcup_{i=1}^{r}U_i \text{ and } |e\cap U_i|=1 \text{ for each } i\in[r]\right\}.$$

For a positive integer $r\geq 2$, the Kneser hypergraph $\mathrm{KG}^r_{n,k}$ is a hypergraph which has the vertex set $\binom{[n]}{k}$, and whose edges are formed by the r-sets $\{e_1,\ldots,e_r\}$, where $e_1,\ldots,e_r$ are pairwise disjoint members of $\binom{[n]}{k}$. Kneser 1955 [14] conjectured that for $n\geq 2k$, the chromatic number of $\mathrm{KG}^2_{n,k}$ is $n-2k+2$. After more than 20 years, in a fascinating paper, Lovász [17] gave an affirmative answer to Kneser's conjecture using algebraic topology. Lovász's paper is known as the beginning of the study of combinatorial problems by using topological tools, which is called topological combinatorics. Later, in 1986, Alon, Frankl and Lovász [6] generalized Lovász's result to Kneser hypergraphs by proving that for $n \geq rk$,

$$\chi(\mathrm{KG}^r_{n,k}) = \left\lceil \frac{n - r(k-1)}{r-1} \right\rceil.$$

This result also gives a positive answer to a conjecture posed by Erdős [11]. Schrijver [20] improved Lovász's result by introducing a subgraph $\mathrm{SG}_{n,k}$ of $\mathrm{KG}^2_{n,k}$, called the Schrijver graph, which is a vertex critical graph having the same chromatic number as that of $\mathrm{KG}^2_{n,k}$. A *stable subset of* $[n]$ is a set $A \subseteq [n]$ such that for each $i \neq j \in A$, we have $2 \leq |i-j| \leq n-2$. Let $\binom{[n]}{k}_{stable}$ be the set of all stable k-subsets of $[n]$. The graph $\mathrm{SG}_{n,k} = \mathrm{KG}\left([n], \binom{[n]}{k}_{stable}\right)$ is called the Schrijver graph.

For a hypergraph $\mathcal{H}$ and a positive integer $r \geq 2$, the general Kneser hypergraph $\mathrm{KG}^r(\mathcal{H})$ is an r-uniform hypergraph with vertex set $E(\mathcal{H})$ and the edge set defining as follows;

$$E(\mathrm{KG}^r(\mathcal{H})) = \left\{\{e_1, \ldots, e_r\} \subseteq E(\mathcal{H}) :\ e_i \cap e_j = \varnothing \text{ for each } i \neq j \in [r]\right\}.$$

Throughout the paper, for $r = 2$, we speak about $\mathrm{KG}(\mathcal{H})$ and $\mathrm{KG}_{n,k}$ rather than $\mathrm{KG}^2(\mathcal{H})$ and $\mathrm{KG}^2_{n,k}$, respectively. The r-colorability defect of $\mathcal{H}$, denoted $\mathrm{cd}_r(\mathcal{H})$, is the minimum number of vertices should be excluded so that the induced subhypergraph on the remaining vertices is r-colorable. Note that if we set $K^k_n = ([n], \binom{[n]}{k})$, then $\mathrm{KG}^r(K^k_n) = \mathrm{KG}^r_{n,k}$ and $\mathrm{cd}_r(K^k_n) = n - r(k-1)$ for $n \geq rk$. Dol'nikov [10] (for $r = 2$) and Kříž [15] improved the results by Lovász [17] and Alon, Frankl and Lovász [6] by proving $\chi(\mathrm{KG}^r(\mathcal{H})) \geq \left\lceil \frac{\mathrm{cd}_r(\mathcal{H})}{r-1} \right\rceil$. A famous combinatorial counterpart of the Borsuk-Ulam theorem is Tucker lemma [21]. Matoušek [18] proved Lovász's theorem by use of Tucker lemma. He also presented a purely combinatorial proof for Tucker lemma, hence a purely combinatorial proof for Lovász's theorem. Ziegler [22] extended Tucker lemma to $\mathbb{Z}_p$-Tucker lemma with a proof which makes no use of topological tools. Using this lemma, Ziegler [22], inspired by Matoušek's proof, improved Dol'nikov-Kříž lower bound by a purely combinatorial approach. Next, Meunier [19] found a variant of $\mathbb{Z}_p$-Tucker lemma as an extension of Ziegler's result, which can be proved combinatorially as well. Using this lemma, he presented a combinatorial proof of Schrijver's result.

Remark. Note that since there is a purely combinatorial proof for $\mathbb{Z}_p$-Tucker lemma (Lemma A), see [19, 22], any combinatorial proof with the help of $\mathbb{Z}_p$-Tucker lemma can be seen as a purely combinatorial proof. In this point of view, all results in this paper are proved purely combinatorial.

Let $\mathbb{Z}_r = \{\omega^1, \ldots, \omega^r\}$ be a cyclic group with generator ω. For an $X = (x_1, \ldots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n$, an *alternating subsequence of* X is a sequence $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ ($i_1 < \cdots < i_k$) of nonzero terms of X such that $x_{i_j} \neq x_{i_{j+1}}$ for each $j \in [k-1]$. The maximum length of an alternating subsequence of X is called *the alternation number of* X, denoted $\mathrm{alt}(X)$. We define $\mathrm{alt}(0, \ldots, 0) = 0$. For each $i \in [r]$, let X^i be the set of all $j \in [n]$ such that $x_j = \omega^i$, that is, $X^i = \{j \in [n] \ :\ x_j = \omega^i\}$. Note that, by abuse of notation, we can write $X = (X^1, \ldots, X^r)$. For two signed vectors X and Y, by $X \subseteq Y$, we mean $X^i \subseteq Y^i$ for each $i \in [r]$. Let $\mathcal{H}$ be a hypergraph and let $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection. Define

$$\mathrm{alt}_r(\mathcal{H}, \sigma, q) = \max\left\{\mathrm{alt}(X) :\ X \in (\mathbb{Z}_r \cup \{0\})^n \ \text{ s.t. } |E(\mathcal{H}[\sigma(X^i)])| \leq q - 1 \text{ for all } i \in [r] \right\}.$$

Now, set

$$\mathrm{alt}_r(\mathcal{H}, q) = \min_{\sigma} \mathrm{alt}_r(\mathcal{H}, \sigma, q),$$

where the minimum is taken over all bijections $\sigma : [n] \longrightarrow V(\mathcal{H})$. Throughout the paper, for $q = 1$, we would use $\mathrm{alt}_r(\mathcal{H})$ rather than $\mathrm{alt}_r(\mathcal{H}, 1)$. The present authors [1], using the extension of

$\mathbb{Z}_p$-Tucker lemma by Meunier [19], improved Dol'nikov-Křížlower bound by proving that

$$\chi(\mathrm{KG}^r(\mathcal{H})) \geq \left\lceil \frac{|V(\mathcal{H})| - \mathrm{alt}_r(\mathcal{H})}{r-1} \right\rceil. \tag{1}$$

Using this lower bound, the chromatic numbers of several families of graphs and hypergraphs are computed, see [1, 2, 3, 4, 5, 13].

1.1. **Random Kneser Hypergraphs.** Let ρ be a real number, where $0 < \rho \leq 1$. The random general Kneser hypergraph $\mathrm{KG}^r(\mathcal{H})(\rho)$ is a random spanning subgraph of $\mathrm{KG}^r(\mathcal{H})$ containing each edge of $\mathrm{KG}^r(\mathcal{H})$ randomly and independently with probability ρ, i.e., each pairwise vertex-disjoint edges $e_1, \ldots, e_r \in E(\mathcal{H})$ form an edge of $\mathrm{KG}^r(\mathcal{H})(\rho)$ with probability ρ. The stability properties of random Kneser graphs $\mathrm{KG}_{n,k}(\rho)$ has been received a considerable attention in recent years, see for instances [7, 9]. In this regard, Bollobás, Narayanan, and Raigorodskii [9] proved a random analogue of the Erdős-Ko-Rado theorem. In detail, they proved that for a real number $\varepsilon > 0$ and an integer function $2 \leq k = k(n) = o(n^{\frac{1}{2}})$, there is a threshold $t(n) \in (0, 1]$ such that for $\rho \geq (1+\varepsilon)t(n)$ and $\rho \leq (1-\varepsilon)t(n)$, the quantity $\Pr\left(\alpha\left(\mathrm{KG}_{n,k}(\rho)\right) = \binom{n-1}{k-1}\right)$ respectively tends to 1 and 0 as n goes to infinity. They also asked what happens for larger k. Furthermore, they conjectured that if $\frac{k}{n}$ is bounded away from $\frac{1}{2}$, then such a random analogue of the Erdős-Ko-Rado theorem should continue to hold for some ρ bounded away from 1. This conjecture received an affirmative answer owing to the work by Balogh, Bollobás, and Narayanan [7]. They proved that the random analogue of the Erdős-Ko-Rado theorem is still true for each $k \leq (\frac{1}{2} - \varepsilon)n$.

In the rest of the paper, for simplicity of notation, for two functions $f(n)$ and $g(n)$, by $f(n) \gg g(n)$ or $g(n) \ll f(n)$, we mean $\lim\limits_{n \mapsto \infty} \dfrac{g(n)}{f(n)} = 0$. Also, the abbreviation a.s. stands for "almost surely", which means that the probability tends to 1 as n goes to infinity.

Recently, Kupavskii [16] studied the chromatic number of random Kneser graphs $\mathrm{KG}_{n,k}(\rho)$. He applied Gale's lemma [12], in a similar fashion as in Bárány's proof [8] of Lovász's theorem, to introduce an a.s. lower bound for the chromatic number of random Kneser graphs $\mathrm{KG}_{n,k}(\rho)$. The following theorem is the main result of Kupavskii's paper.

Theorem A. [16] *Let $k = k(n) \geq 2$ and $l = l(n) \geq 1$ be integer functions and $\rho = \rho(n) \in (0, 1]$ be a real function such that $d = n - 2k - 2l + 2 \geq 3$. Put $x = \left\lceil \frac{\binom{k+l}{k}}{d-1} \right\rceil$. If for some $\epsilon > 0$, we have $(1-\epsilon)\rho > x^{-2} n \ln 3 + 2x^{-1}(1 + \ln(d-1))$, then a.s. $\chi(\mathrm{SG}_{n,k}(\rho)) \geq d$.*

Kupavskii, at the end of his paper, marked the investigation of the chromatic number of random Kneser hypergraphs $\mathrm{KG}^r_{n,k}(\rho)$ as a very interesting problem. In this paper, we shall study the chromatic number of random general Kneser hypergraphs $\mathrm{KG}^r_{n,k}(\mathcal{H})(\rho)$. As the first main result of this paper, we extend Theorem A to the following theorem.

Theorem 1. *Let $\mathcal{A} = \{\mathcal{H}_m : m \in \mathbb{N}\}$ be a family of distinct hypergraphs and set $n = n(m) = |V(\mathcal{H}_m)|$. Let $r = r(n) \geq 2$, $t = t(n)$, $d = d(n)$, and $q = q(n)$ be integer functions, where $(d-1)(t-1) + 1 \leq q \leq (d-1)t$ and let $\rho = \rho(n) \in (0, 1]$ be a real function. Then we a.s. have*

$$\chi(\mathrm{KG}^r(\mathcal{H}_m)(\rho)) \geq \min\left\{\frac{|V(\mathcal{H}_m)| - \mathrm{alt}_r(\mathcal{H}_m, q)}{r-1}, d\right\}$$

provided that $n \ln(r+1) + rt(1 + \ln(d-1)) - \rho t^r \to -\infty$ as n tends to infinity.

Note that if we set $n = m$, $\mathcal{H}_n = K_n^k$, then $\mathrm{KG}^r(\mathcal{H}_n) = \mathrm{KG}^r_{n,k}$. Consequently, if we set $r = 2$, $d = n-2k-2l+2$, $q = \binom{k+l}{k}$, and $t = \left\lceil \frac{\binom{k+l}{k}}{d-1} \right\rceil$, then the previous theorem results in a slightly weaker version of Kupavskii's theorem (using Kneser graphs $\mathrm{KG}_{n,k}$ instead of Schrijver graphs $\mathrm{SG}_{n,k}$). Also, in general, for $n = m$, $\mathcal{A} = \left\{K_n^k : \ n \in \mathbb{N}\right\}$, $d = \left\lceil \frac{n-r(k+l-1)}{r-1} \right\rceil \geq 2$, and $q = \binom{k+l}{k}$, Theorem 1 implies that if $n\ln(r+1) + rt(1+\ln(d-1)) - \rho t^r \to -\infty$, then a.s. $\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq \min\left\{\frac{n-\mathrm{alt}_r(K_n^k,q)}{r-1}, d\right\}$. On the other hand, for the identity bijection $I : [n] \longrightarrow [n]$, since $q = \binom{k+l}{k}$, we have

$$\mathrm{alt}(K_n^k, q) \leq \mathrm{alt}_r(K_n^k, I, q) = r(k+l-1).$$

Consequently, we a.s. have $\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq \min\left\{\frac{n-r(k+1-1)}{r-1}, d\right\} = \frac{n-r(k+1-1)}{r-1}$ provided that

$$n\ln(r+1) + rt(1+\ln(d-1)) - \rho t^r \to -\infty.$$

This observation proves the next theorem provided that condition (I) holds. Therefore, to prove the next theorem, it suffices to consider just the second condition, which is discussed in Section 3.

Theorem 2. *Let $k = k(n)$, $r = r(n)$ and $l = l(n)$ be nonnegative integer functions and let $\rho = \rho(n)$ be a real function, where $2 \leq r \leq \frac{n}{k}$ and $\rho \in (0,1]$. For $d = \left\lceil \frac{n-r(k+l-1)}{r-1} \right\rceil \geq 2$ and $t = \left\lceil \frac{\binom{k+l}{k}}{d-1} \right\rceil$, we have a.s. $\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq d$ provided that at least one of the followings holds;*

(I) $n\ln(r+1) + rt(1+\ln(d-1)) - \rho t^r \to -\infty$
(II) $r(k+l)(\ln n + 1) + rt(1+\ln(d-1)) - \rho t^r \to -\infty$.

In Theorem A and Theorem 2, we deal with some quite complicated conditions which make this theorems difficult to use. To get rid of these difficulties, Kupavskii derived some corollaries from Theorem A having simpler conditions. In detail, he proved that a.s. $\chi(\mathrm{SG}_{n,k}(\rho)) \geq \chi(\mathrm{KG}_{n,k}) - 4$ provided that ρ is fixed and $k \gg n^{\frac{3}{4}}$. Also, for any fixed ρ and for $n - 2k \ll \sqrt{n}$, he improved this lower bound by proving that a.s. $\chi(\mathrm{SG}_{n,k}(\rho)) \geq \chi(\mathrm{KG}_{n,k}) - 2$. With a straightforward computation and by use of Theorem 2, one can extend Kupavskii's results to the Kneser hypergraphs $\mathrm{KG}^2_{n,k}$.

In the rest of this section, we consider some special cases of Theorem 2, which are easy to interpret. In this regard, we prove two corollaries (Corollary 2 and Corollary 3), which not only extend Kupavskii's results to random Kneser hypergraphs, but also improve it (when we deal with the case $r = 2$).

Corollary 1. *Let $\rho \in (0,1]$ be a real number. Also, let $k = k(n)$ and $r = r(n)$ be positive integer functions, where $2 \leq r \leq \frac{n}{k}$. If $k \gg n^{\frac{r}{2r-1}}(\ln n)^{\frac{1}{2r-1}}$, then a.s. $\chi(\mathrm{KG}^r_{n,k}(p)) \geq \left\lceil \frac{n-r(k+1)}{r-1} \right\rceil$. In particular, if $n^{\frac{r-1}{r}} \gg rn - r^2k$, then a.s. $\chi(\mathrm{KG}^r_{n,k}(p)) \geq \left\lceil \frac{n-rk}{r-1} \right\rceil$.*

Proof. To prove the assertion, it suffices to check that if at least one of two conditions in Theorem 2 holds for $l = 2$ and $l = 1$, respectively. Let us first deal with the case $l = 2$. We prove this case via Condition (II) of Theorem 2. To this end, we need to show that for $d = \left\lceil \frac{n-r(k+1)}{r-1} \right\rceil$ and $t = \left\lceil \frac{(k+2)(k+1)}{2(d-1)} \right\rceil$, we have $r(k+2)(\ln n + 1) + rt(1+\ln(d-1)) - \rho t^r \to -\infty$, which clearly holds, since $r(k+2)(\ln n + 1) = o(t^r)$ and $rt(1+\ln d) \leq rt(1+\ln n) = o(t^r)$.

For $l = 1$, note that $\frac{n^{\frac{r-1}{r}}}{r^2} \gg d = \left\lceil \frac{n-rk}{r-1} \right\rceil$ and for large enough n, we have $k \geq \frac{n}{2r}$; consequently,

$$t = \left\lceil \frac{k+1}{(d-1)} \right\rceil \gg \frac{\frac{n}{r}}{\frac{n^{\frac{r-1}{r}}}{r^2}} = r \cdot n^{\frac{1}{r}}.$$

Now, we clearly have $n \ln(r+1) = o(t^r)$ and $rt(1 + \ln d) \leq rt(1 + \ln n) = o(t^r)$. Using Condition (I) of Theorem 2, we have the proof completed. □

Next corollary is an immediate consequence of Corollary 1.

Corollary 2. *Let $\rho \in (0, 1]$ be a real number. Also, let $k = k(n)$ and $r = r(n)$ be positive integer functions, where $2 \leq r \leq \frac{n}{k}$. Then the following assertions hold.*

I) *If $k \gg n^{\frac{r}{2r-1}} (\ln n)^{\frac{1}{2r-1}}$, then a.s.*

$$\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq \begin{cases} \chi(\mathrm{KG}^r_{n,k}) - 4 & r = 2 \\ \chi(\mathrm{KG}^r_{n,k}) - 3 & r > 2. \end{cases}$$

In particular, if $n \not\equiv k, k+1 \pmod{r-1}$, then a.s $\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq \chi(\mathrm{KG}^r_{n,k}) - 2$.

II) *If $n^{\frac{r-1}{r}} \gg rn - r^2 k$, then a.s $\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq \chi(\mathrm{KG}^r_{n,k}) - 2$. In particular, if $n \not\equiv k \pmod{r-1}$, then a.s. $\chi(\mathrm{KG}^r_{n,k}(\rho)) \geq \chi(\mathrm{KG}^r_{n,k}) - 1$.*

Note that Kupavskii's result (Theorem A) provides an a.s. lower bound for the chromatic number of random Schrijver graphs $\mathrm{SG}_{n,k}(\rho)$, while Theorem 2 and Corollary 2 concern the chromatic number of random Kneser hypergraphs $\mathrm{KG}^r_{n,k}(\rho)$. The next theorem can be seen as a complementary statement for Theorem A.

Theorem 3. *Let $k = k(n) \geq 2$ and $l = l(n) \geq 0$ be integer functions and $\rho = \rho(n) \in (0, 1]$ be a real function such that $d = n - 2k - 2l + 2 \geq 2$. Put $t = \left\lceil \frac{\binom{k+l}{k}}{d-1} \right\rceil$. If $r(k+l)(\ln n + 1) + rt(1 + \ln(d-1)) - \rho t^r \to -\infty$, then a.s. $\chi(\mathrm{SG}_{n,k}(\rho)) \geq d$.*

Similar to the proof of Corollary 1 and by using Theorem 3 instead of Theorem 2, we can prove the next corollary, which is an improvement of Kupavskii's result.

Corollary 3. *Let $\rho \in (0, 1]$ be a real number and $k = k(n) \leq \frac{n}{2}$ be an integer function. If $k \gg n^{\frac{2}{3}} (\ln n)^{\frac{1}{3}}$, then a.s. $\chi(\mathrm{SG}_{n,k}(\rho)) \geq \chi(\mathrm{KG}_{n,k}) - 4$.*

1.2. **Coloring With No Monochromatic $K^r_{t,\ldots,t}$ Subhypergraph.** Let r and t be two integers, where $r \geq 2$ and $t \geq 1$ and let $\mathcal{H}$ be a hypergraph. Also, set $K^r_{t,\ldots,t}$ to be the complete r-uniform r-partite hypergraph with tr vertices such that each of its parts has t vertices. Next result concerns the minimum number of colors required in a coloring of $\mathrm{KG}^r(\mathcal{H})$ with no monochromatic $K^r_{t,\ldots,t}$ subhypergraph. For $t = 1$, any edge of $\mathrm{KG}^r(\mathcal{H})$ is a $K^r_{1,\ldots,1}$ subhypergraph of $\mathrm{KG}^r(\mathcal{H})$. Therefore, for $t = 1$, any coloring of $\mathrm{KG}^r(\mathcal{H})$ with no monochromatic $K^r_{1,\ldots,1}$ subhypergraph is just a proper coloring of $\mathrm{KG}^r(\mathcal{H})$. Note that for $t = q = 1$, $d = n$, the next theorem implies Inequality 1.

Theorem 4. *Let $\mathcal{H}$ be a hypergraph and $\sigma : [n] \longrightarrow V(\mathcal{H})$ be an arbitrary bijection. Also, let d, q, r and t be be positive integers, where $r \geq 2$ and $q \geq (d-1)(t-1) + 1$. Then any coloring of $\mathrm{KG}^r(\mathcal{H})$ with no monochromatic $K^r_{t,\ldots,t}$ uses at least $\min\left\{ \left\lceil \frac{n - \mathrm{alt}_r(\mathcal{H},\sigma,q)}{r-1} \right\rceil, d \right\}$ colors.*

For a given positive integer t, let l be the smallest nonnegative integer such that

$$q = \binom{k+l}{k} \geq \left(\left\lceil \frac{n - r(k+l-1)}{r-1} \right\rceil - 1 \right)(t-1) + 1.$$

Theorem 4 implies that any coloring of $\mathrm{KG}^r_{n,k}$ with no monochromatic $K^r_{t,\ldots,t}$ subhypergraph uses at least $\left\lceil \frac{n-r(k+l-1)}{r-1} \right\rceil$ colors. Note that the case $t=1$ concludes the chromatic number of Kneser hypergraphs $\mathrm{KG}^r_{n,k}$.

Plan. This paper is organized as follows. In Section 2, we introduce some tools which will be needed throughout the paper. Section 3 is devoted to the proof of main theorems. In the last section, we present a generalization of Theorem A with a purely combinatorial proof which implies this theorem immediately.

2. Tools

2.1. **Random General Kneser Hypergraphs.** Let $\mathcal{H}=(V(\mathcal{H}),E(\mathcal{H}))$ be a hypergraph and r,s,C be positive integers, where $r,s\geq 2$. Also, let $\sigma:[n]\longrightarrow V(\mathcal{H})$ be a bijection. Let $M\subseteq V(\mathcal{H})$ be an m-set, where $\sigma^{-1}(M)=\{i_1,\ldots,i_m\}$ and $i_1<\cdots<i_m$. By σ_M, we mean the following bijective map;

$$\begin{array}{rccc}\sigma_M: & [m] & \longrightarrow & M\\ & j & \longmapsto & \sigma(i_j).\end{array}$$

Define $\mathcal{T}=\mathcal{T}_{\mathcal{H},C,s,\sigma}$ to be a hypergraph with vertex set $V(\mathcal{H})$ and edge set

$$E(\mathcal{T})=\{M\subseteq V(\mathcal{H}):\ M\neq\varnothing \text{ and } |M|-\mathrm{alt}_s(\mathcal{H}[M],\sigma_M,q)>(s-1)C\}\,.$$

The next lemma, for $q=1$, is implicitly used in the proof of Theorem 1 in [1]. Also, a similar lemma is proved in [13]. However, for sake of completeness, we state it here with a proof.

Lemma 1. *Let $\mathcal{H}=(V(\mathcal{H}),E(\mathcal{H}))$ be a hypergraph and r,s,C and q be positive integers, where $r,s\geq 2$. Then for any bijection $\sigma:[n]\longrightarrow V(\mathcal{H})$, we have*

$$\mathrm{alt}_r(\mathcal{T},\sigma,1)\leq r(s-1)C+\mathrm{alt}_{rs}(\mathcal{H},\sigma,q).$$

Proof. If $\mathrm{alt}_r(\mathcal{T},\sigma,1)=0$, then there is nothing to prove. Therefore, we may assume that $\mathrm{alt}_r(\mathcal{T},\sigma,1)>0$.

For simplicity of notation, without loss of generality, suppose that $V(\mathcal{H})=[n]$ and $\sigma=I$ (the identity map). Therefore, for each $A=\{a_1,\ldots,a_m\}\subseteq[n]$ $(a_1<\cdots<a_m)$, we have

$$\begin{array}{rccc}I_A: & [m] & \longrightarrow & A\\ & i & \longmapsto & a_i.\end{array}$$

In view of the definition of $\mathrm{alt}_r(\mathcal{T},I,1)$, there is an $X=(X^1,\ldots,X^r)\in(\mathbb{Z}_r\cup\{0\})^n$ with $\mathrm{alt}(X)=|X|=\mathrm{alt}_r(\mathcal{T},I,1)$ and such that $E(\mathcal{H}[X^i])=\varnothing$ for each $i\in[r]$. It implies that $X^i\notin E(\mathcal{T})$ for each $i\in[r]$. Let I_0 be the set of all $i\in[r]$, such that $X^i\neq\varnothing$. Note that since $\mathrm{alt}_r(\mathcal{T},\sigma,1)>0$, we have $I_0\neq\varnothing$. Consequently, for each $i\in I_0$, we have

$$|X^i|-\mathrm{alt}_s(\mathcal{H}[X^i],I_{X^i},q)\leq(s-1)C.$$

It implies that for each $i\in I_0$, there is at least one $Y_i=(Y^{i,1},\ldots,Y^{i,s})\in(\mathbb{Z}_s\cup\{0\})^{|X^i|}$ such that

- $\mathrm{alt}(Y_i)=|Y_i|\geq|X^i|-(s-1)C$ and
- $|E(\mathcal{H}[I_{X^i}(Y^{i,j})])|<q$ for each $j\in[s]$.

Note that for each $i\in I_0$ and each $j\in[s]$, we have

$$Y^{ij}\subseteq\{1,\ldots,|X^i|\}\quad\text{ and }\quad I_{X^i}:\{1,\ldots,|X^i|\}\longrightarrow X^i.$$

For each $i\in[r]\setminus I_0$, set $I_{X^i}(Y^{i,1})=\cdots=I_{X^i}(Y^{i,s})=\varnothing$. Define

$$\begin{array}{rcl}Z&=&\left(I_{X^1}(Y^{1,1}),\ldots,I_{X^1}(Y^{1,s}),\ldots,I_{X^r}(Y^{r,1}),\ldots,I_{X^r}(Y^{r,s})\right)\\ &=&(Z^1,Z^2,\ldots,Z^{rs})\in(\mathbb{Z}_{rs}\cup\{\mathbf{0}\})^n.\end{array}$$

One can simply see that $\mathrm{alt}(Z) = |Z|$. This implies that

$$\begin{array}{rcl}\mathrm{alt}(Z) & = & \displaystyle\sum_{i=1}^{r}\sum_{j=1}^{s}|I_{X^i}(Y^{i,j})| \\ & = & \displaystyle\sum_{i\in I_0}|Y_i| \\ & \geq & \displaystyle\sum_{i\in I_0}\big(|X^i| - (s-1)C\big) \\ & \geq & |X| - r(s-1)C \\ & = & \mathrm{alt}_r(\mathcal{T},\sigma,1) - r(s-1)C.\end{array}$$

In view of the definition of $\mathrm{alt}_{rs}(\mathcal{H}, I, q)$ and since $|E(\mathcal{H}[Z^l])| < q$ for each $l \in [rs]$, we have

$$\mathrm{alt}_{rs}(\mathcal{H}, I, q) \geq \mathrm{alt}(Z) \geq \mathrm{alt}_r(\mathcal{T}, I, 1) - r(s-1)C,$$

as desired. □

Now, we are ready to state the main lemma, which has a key role in the paper. For the proof of this lemma, we need the following version of Z_p-Tucker lemma.

Lemma A. ($\mathbb{Z}_p$-Tucker lemma [19, 22]) *Let m, n, p, and α be nonnegative integers, where $m, n \geq 1$, $m \geq \alpha \geq 0$, and p is prime. Let*

$$\begin{array}{rrcl}\lambda: & (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\} & \longrightarrow & \mathbb{Z}_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X))\end{array}$$

be a map satisfying the following properties:

(i) *λ is a $\mathbb{Z}_p$-equivariant map, that is, for each $\varepsilon \in \mathbb{Z}_p$, we have $\lambda(\varepsilon X) = (\varepsilon\lambda_1(X), \lambda_2(X))$,*
(ii) *for $X_1 \subseteq X_2 \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, if $\lambda_2(X_1) = \lambda_2(X_2) \leq \alpha$, then $\lambda_1(X_1) = \lambda_1(X_2)$,*
(iii) *for $X_1 \subseteq \cdots \subseteq X_p \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$, if $\lambda_2(X_1) = \cdots = \lambda_2(X_p) \geq \alpha + 1$, then*

$$|\{\lambda_1(X_1), \ldots, \lambda_1(X_p)\}| < p.$$

Then $\alpha + (m - \alpha)(p - 1) \geq n$.

Lemma 2. *Let d, q, r, and t be positive integers, where $d, r \geq 2$, and $q \geq (d-1)(t-1)+1$. Let $\mathcal{H}$ be a hypergraph and $\preceq$ be a total ordering on the power set of $V(\mathcal{H})$, which refines the partial ordering according to size. Moreover, let $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection. Then for any coloring $c : E(\mathcal{H}) \longrightarrow [C]$, where $1 \leq C < \min\{\frac{n-\mathrm{alt}_r(\mathcal{H},\sigma,q)}{r-1}, d\}$, there exists an r-tuple $(N_1, \ldots, N_r)$ with the following properties;*

- *$N_1, \ldots, N_r$ are pairwise disjoint subsets of $[n]$.*
- *For each $j \in [r]$, $|E(\mathcal{H}[\sigma(N_j)])| \geq q$.*
- *For each $j \in [r]$, there are t distinct edges $e_{1,j}, \ldots, e_{t,j} \subseteq \sigma(N_j)$ chosen from the last q largest edges in $E(\mathcal{H}[\sigma(N_j)])$ (according to the total ordering $\preceq$) such that all edges in $\{e_{i,j} : i \in [t]\ \&\ j \in [r]\}$ receive the same color $c(N_1, \ldots, N_r) \in [C]$.*

Proof. The proof is divided into two parts. First, we prove the theorem when r is prime. Then, we reduce the nonprime case to the prime case, which completes the proof.

First, assume that $r = p$ is a prime number. Consider an arbitrary coloring $c : E(\mathcal{H}) \longrightarrow [C]$ such that $1 \leq C < \min\{\frac{n-\mathrm{alt}_r(\mathcal{H},\sigma,q)}{r-1}, d\}$. Without loss of generality and for simplicity of notation, we may assume that $V(\mathcal{H}) = [n]$ and $\sigma = I$ is the identity map. Set $m = \mathrm{alt}_p(\mathcal{H}, I, q) + C$ and $\alpha = \mathrm{alt}_p(\mathcal{H}, I, q)$. Let

$$\begin{array}{rrcl}\lambda: & (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\} & \longrightarrow & \mathbb{Z}_p \times [m] \\ & X & \longmapsto & (\lambda_1(X), \lambda_2(X))\end{array}$$

be a map defining as follows.

- If $\text{alt}(X) \leq \alpha$, then define $\lambda_1(X)$ to be the first nonzero coordinate of X and $\lambda_2(X) = \text{alt}(X)$.
- If $\text{alt}(X) \geq \alpha + 1$, then, in view of the definition of $\text{alt}_p(\mathcal{H}, I, q)$, there is at least one $i \in [p]$ such that $|E(\mathcal{H}[X^i])| \geq q$. Choose $\omega^i \in \mathbb{Z}_p$ such that

$$X^i = \max\{X^j : \; |E(\mathcal{H}[X^j])| \geq q\},$$

where the maximum is taken according to the total ordering $\preceq$. Now, see all edges in $E(\mathcal{H}[X^i])$ as a chain (according to the total ordering $\preceq$) and consider the last q edges of this chain. In other words, if $E(\mathcal{H}[X^i]) = \{e_1, \ldots, e_m\}$, where $e_1 \prec \cdots \prec e_m$, then consider $e_{m-q+1}, \ldots, e_m$. Define $c(X)$ to be the most popular color amongst all colors assigned to these q edges. If there is more than one such a color, then choose the maximum one. Clearly the frequency of this color is at least $\lceil \frac{q}{C} \rceil \geq t$ (note that $\frac{q}{C} \geq \frac{q}{d-1} > t-1$). Define

$$\lambda(X) = (\omega^i, \alpha + c(X)).$$

It is straightforward to check that the map λ satisfies Property (i) and Property (ii) of Lemma A. Since

$$n - \text{alt}_p(\mathcal{H}, I, q) = n - \alpha > (m - \alpha)(p-1) = C(p-1),$$

the map λ does not satisfy Property (iii) of Lemma A. Thus, there is a chain $X_1 \subseteq \cdots \subseteq X_p \in (\mathbb{Z} \cup \{0\})^n \setminus \{\mathbf{0}\}$, such that $i = \lambda_2(X_1) = \cdots = \lambda_2(X_p) \geq \alpha + 1$ and $|\{\lambda_1(X_1), \ldots, \lambda_1(X_p)\}| = p$. Hence, we have $\{\lambda_1(X_1), \ldots, \lambda_1(X_p)\} = \mathbb{Z}_p$. Let $\pi : [p] \longrightarrow [p]$ be the bijection for which we have $\lambda_1(X_j) = \omega^{\pi(j)}$ for each $j \in [p]$. Define $N_j = X_j^{\pi(j)} \subseteq X_p^{\pi(j)}$ for each $j \in [p]$. Since the sets X_p^j's are pairwise disjoint, the sets N_j's are pairwise disjoint as well. In view of the definition of λ, for each $j \in [p]$, there are at least t edges $e_{1,j}, \ldots, e_{t,j} \subseteq N_j$ such that these edges are amongst the last q largest edges in $E(\mathcal{H}[N_j])$ and $c(e_{1,j}) = \cdots = c(e_{t,j}) = c(X_j) = i - \alpha$. It implies that all edges in $\{e_{i,j} : i \in [t] \;\&\; j \in [r]\}$ receive the same color $i - \alpha$. Clearly, for $c(N_1, \ldots, N_p) = i - \alpha$, the p-tuple $(N_1, \ldots, N_p)$ has the desired properties.

Lemma 3. *If Lemma 2 holds for $r = r_1$ and $r = r_2$, then it holds for $r = r_1 r_2$.*

Proof. Let $c : E(\mathcal{H}) \longrightarrow [C]$ be a coloring such that

$$1 \leq C < \min\left\{\frac{|V(\mathcal{H})| - \text{alt}_{r_1 r_2}(\mathcal{H}, \sigma, q)}{r_1 r_2 - 1}, d\right\}.$$

Note that this implies that $C < d$. Consider the hypergraph $\mathcal{T} = \mathcal{T}_{\mathcal{H}, C, r_2, \sigma}$. First, we define a coloring $f : E(\mathcal{T}) \longrightarrow [C]$. For each $M \in E(\mathcal{T})$, in view of the definition of $\mathcal{T}$, we have

$$|M| - \text{alt}_{r_2}(\mathcal{H}[M], \sigma_M, q) > (r_2 - 1)C.$$

Hence,

$$1 \leq C < \min\left\{\frac{|M| - \text{alt}_{r_2}(\mathcal{H}[M], \sigma_M, q)}{r_2 - 1}, d\right\}.$$

Consider the hypergraph $\mathcal{H}[M]$ and the coloring c restricted to the edges of $\mathcal{H}[M]$. Let $(N_1, \ldots, N_{r_2})$ be an r_2-tuple whose existence is ensured since we have assumed that Lemma 2 is true for $r = r_2$. Note that $N_1, \ldots, N_{r_2}$ are pairwise disjoint subsets of $\{1, \ldots, |M|\}$. Now, define $f(N) = c(N_1, \ldots, N_{r_2})$. In view of lemma 1, we have

$$\begin{array}{rcl} n - \text{alt}_{r_1}(\mathcal{T}, \sigma, 1) & \geq & n - r_1(r_2 - 1)C - \text{alt}_{r_1 r_2}(\mathcal{H}, \sigma, q) \\ & > & (r_1 r_2 - 1)C - r_1(r_2 - 1)C \\ & = & (r_1 - 1)C. \end{array}$$

It implies that

$$1 \leq C < \min\left\{\frac{|V(\mathcal{T})| - \mathrm{alt}_{r_1}(\mathcal{T},\sigma,1)}{r_1-1}, d\right\}.$$

Since Lemma 2 holds for $r = r_1$, if we set $t = q = 1$, then there are r_1 pairwise vertex-disjoint edges $M_1, \ldots, M_{r_1} \in E(\mathcal{T})$ such that $f(M_1) = \cdots = f(M_{r_1}) = i$. Now, for each $i \in [r_1]$, let $(N_{i,1}, \ldots, N_{i,r_2})$ be the r_2-tuple, which is used for the definition of $f(M_i)$. Now, one can see that the r_1r_2 tuple

$$P = (N_{1,1}, \ldots, N_{1,r_2}, \ldots, N_{r_1,1}, \ldots, N_{r_1,r_2})$$

with $c(P) = i$ has the desired properties. □

By induction, Lemma 3, and the fact that Lemma 2 is true for any prime number r, the proof is completed. □

2.2. **Random Kneser Hypergraphs and Schrijver Graphs.** In this subsection, we present two specializations of Lemma 2, which will be useful for computing the chromatic number of random Kneser hypergraphs $\mathrm{KG}^r_{n,k}(\rho)$ and random Schrijver Graphs $\mathrm{SG}_{n,k}(\rho)$.

Lemma 4. *Let n, k, r and l be nonnegative integers, where $r \geq 2$, $k \geq 1$, $n \geq rk$, and $d = \left\lceil \frac{n-r(k+l-1)}{r-1} \right\rceil \geq 2$. Set $t = \left\lceil \frac{\binom{k+l}{k}}{d-1} \right\rceil$. Then for any coloring $c : \binom{[n]}{k} \longrightarrow [C]$, where $1 \leq C < d$, there exists an r-tuple $(N_1, \ldots, N_r)$ with the following properties;*

- *$N_1, \ldots, N_r$ are pairwise disjoint $(k+l)$-subsets of $[n]$.*
- *For each $j \in [r]$, there are t distinct k-subsets $e_{1,j}, \ldots, e_{t,j} \subseteq N_j$ such that all members of $\{e_{i,j} : i \in [t] \,\&\, j \in [r]\}$ receive the same color $c(N_1, \ldots, N_r) \in [C]$.*

Proof. Consider an arbitrary coloring $c : \binom{[n]}{k} \longrightarrow [C]$, where $1 \leq C < d = \left\lceil \frac{n-r(k+l-1)}{r-1} \right\rceil$. Clearly, the assumption $d \geq 2$ implies that $n \geq r(k+l)$. Define the coloring $f : V(\mathrm{KG}^r_{n,k+l}) \longrightarrow [C]$ as follows. For each $(k+l)$-set $L \in V(\mathrm{KG}^r_{n,k+l})$, set $f(L)$ to be the most popular color (with respect to the coloring c) amongst the members of $\{A : |A| = k \text{ and } A \subseteq L\} \subseteq \binom{[n]}{k}$. If there is more than one such a color, then choose the maximum one. We already know that $\chi(\mathrm{KG}^r_{n,k+l}) = d$. Since $C < d = \chi(\mathrm{KG}^r_{n,k+l})$, the coloring f is not proper. Consequently, there are r pairwise disjoint $(k+l)$-sets $N_1, \ldots, N_r \subseteq [n]$ such that $f(N_1) = \cdots = f(N_r) = i \in [C]$. In view of the definition of f, one can simply see that $P = (N_1, \ldots, N_r)$ with $c(P) = i$ is the desired r-tuple. □

Also, we can have a similar statement for Schrijver graphs.

Lemma 5. *Let n, k and l be nonnegative integers, where $n \geq 2k \geq 2$, and $d = n - 2(k+l-1) \geq 2$. Set $t = \left\lceil \frac{\binom{k+l}{k}}{d-1} \right\rceil$. For any coloring $c : \binom{[n]}{k}_{stable} \longrightarrow [C]$ with $1 \leq C < d$, there is a pair (N_1, N_2) with the following properties.*

- *N_1 and N_2 are disjoint stable $(k+l)$-subsets of $[n]$.*
- *For $j = 1, 2$, there are t distinct stable k-sets $e_{1,j}, \ldots, e_{t,j} \subseteq N_j$ such that all members of $\{e_{i,j} : i \in [t] \,\&\, j \in [2]\}$ receive the same color $c(N_1, N_2) \in [C]$.*

Proof. Consider an arbitrary coloring $c : \binom{[n]}{k}_{stable} \longrightarrow [C]$, where $1 \leq C < d = n - 2(k+l-1)$. Define the coloring $f : V(\mathrm{SG}_{n,k+l}) \longrightarrow [C]$ as follows. For each stable $(k+l)$-set $L \in V(\mathrm{SG}_{n,k+l})$, set $f(L)$ to be the most popular color (with respect to the coloring c) amongst the members of $\{A : |A| = k \text{ and } A \subseteq L\} \subseteq \binom{[n]}{k}_{stable}$. If there is more than one such a color, then choose the maximum one. Since $C < \chi(\mathrm{SG}_{n,k+l}) = d$, there are two disjoint stable $(k+l)$-sets $N_1, N_2 \subseteq [n]$ such that $f(N_1) = f(N_2)$, which completes the proof. □

3. Proofs of Main Results

This section is completely devoted to the proof of main results.

Proof of Theorem 1. Assume that at least one of two mentioned conditions in the assertion of the theorem holds. For an arbitrary $m \in \mathbb{N}$, set $\mathcal{H}_m = \mathcal{H}$. Let $\preceq$ be a total ordering on the power set of $V(\mathcal{H})$, which refines the partial ordering according to size. Let $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection for which we have $\mathrm{alt}_r(\mathcal{H}, \sigma, q) = \mathrm{alt}_r(\mathcal{H}, q)$.

The Event E. Define $\mathbf{E}$ to be the event that $\mathrm{KG}^r(\mathcal{H})(\rho)$ has some proper C-coloring for some $1 \leq C < \min\left\{\frac{|V(\mathcal{H})|-\mathrm{alt}_r(\mathcal{H},\sigma,q)}{r-1}, d\right\}$. Clearly, to complete the proof, it suffices to show that $Pr(\mathbf{E}) \to 0$ as $m \to +\infty$.

Consider an arbitrary $P = (M_1, \ldots, M_r)$ such that $M_1, \ldots, M_r$ are pairwise disjoint subsets of $[n]$ and $|E(\sigma(M_i))| \geq q$ for each $i \in [r]$. Now, for each $i \in [r]$, see all edges in $E(\mathcal{H}[\sigma(M_i)])$ as a chain according to the total ordering $\preceq$ and consider the last q edges appearing in this chain. Let $U_i = U_i(P)$ be set of those edges.

The Event A. Define $\mathbf{A}(P)$ to be the event that for each $i \in [r]$, there is a t-subset $V_i \subseteq U_i$ such that the subhypergraph $\mathrm{KG}^r(\mathcal{H})(\rho)[V_1, \ldots, V_r]$ has no edge. Now, define the event $\mathbf{A}$ to be the union of all $\mathbf{A}(P)$'s, i.e.,

$$\mathbf{A} = \bigcup \mathbf{A}(P),$$

where the union is taken over all $P = (M_1, \ldots, M_r)$ such that $M_1, \ldots, M_r$ are pairwise disjoint subsets of $[n]$ and $|E(\sigma(M_i))| \geq q$ for each $i \in [r]$.

Let $c : E(\mathcal{H}) \longrightarrow [C]$ be a proper coloring for $\mathrm{KG}^r(\mathcal{H})(\rho)$, where $C < \min\left\{\frac{|V(\mathcal{H})|-\mathrm{alt}_r(\mathcal{H},\sigma,q)}{r-1}, d\right\}$. Consider the r-tuple $P = (N_1, \ldots, N_r)$ whose existence is ensured by Lemma 2. Without loss of generality, we may assume that $c(N_1, \ldots, N_r) = 1$. Now, for each $j \in [r]$, see all edges in $E(\mathcal{H}[\sigma(N_j)])$ as a chain according to the total ordering $\preceq$ and consider the last q edges appearing in this chain. Let $U_j = \{f_1^j, \ldots, f_q^j\}$ be set of those edges. For each $j \in [r]$, let $V_j \subseteq U_j$ be the set of edges receiving color 1. Clearly, the subhypergraph $\mathrm{KG}^r(\mathcal{H})[V_1, \ldots, V_r]$ is a monochromatic subhypergraph of $\mathrm{KG}^r(\mathcal{H})$. Therefore, since c is a proper coloring, $\mathrm{KG}^r(\mathcal{H})(\rho)[V_1, \ldots, V_r]$ has no edge, which implies that $\mathbf{A}(P) \subseteq \mathbf{A}$ is happened. Hence, we have $\mathbf{E} \subseteq \mathbf{A}$.

Also, note that since $\mathcal{H}_m$'s are distinct, if $m \to +\infty$, then $n = n(m) \to +\infty$. Consequently, if we prove that $Pr(\mathbf{A}) \to 0$ as $n \to +\infty$, then we have $Pr(\mathbf{E}) \to 0$ as $m \to +\infty$, as desired. First, note that

$$\begin{array}{rcl} Pr(A) & \leq & \sum \binom{q}{t}^r (1-\rho)^{t^r} \\ & \leq & (r+1)^n \left(\frac{eq}{t}\right)^{rt} e^{-\rho t^r} \\ & \leq & e^{-\rho t^r + n\ln(r+1) + rt(1+\ln(d-1))}, \end{array}$$

where the summation is taken over all $P = (M_1, \ldots, M_r)$ such that $M_1, \ldots, M_r$ are pairwise disjoint subsets of $[n]$ and $|E(\mathcal{H}[\sigma(M_i)])| \geq q$ for each $i \in [r]$ (Note that the number of such P's is at most $(r+1)^n$). Thus, if

$$n\ln(r+1) + rt(1+\ln(d-1)) - \rho t^r \to -\infty,$$

then $Pr(\mathbf{A}) \to 0$. This completes the proof. $\square$

The next proof is almost the same as the prior proof. However, for the ease of reading, we state it here completely.

Proof of Theorem 2. In view of the discussion before the statement of Theorem 2, it is enough to consider that Condition (II) holds. For random Kneser hypergraph $\mathrm{KG}^r_{n,k}(\rho)$, similar to the proof of Lemma 1, we shall introduce two events $\mathbf{E}_n$ and $\mathbf{A}_n$.

The Event $\mathbf{E}_n$. Define $\mathbf{E}_n$ to be the event that $\mathrm{KG}^r_{n,k}(\rho)$ has some proper C-coloring for some $1 \leq C < d$. Clearly, to complete the proof, it suffices to show that $Pr(\mathbf{E}_n) \to 0$ as $n \to +\infty$. Consider an arbitrary $P = (M_1, \ldots, M_r) \in (\mathbb{Z}_r \cup \{0\})^n$ such that $M_1, \ldots, M_r$ are pairwise disjoint $(k+l)$-subsets of $[n]$. Let

$$U_i = U_i(P) = \{A \,:\, |A| = k,\ A \subseteq M_i\} \subseteq V(\mathrm{KG}^r_{n,k}).$$

The Event A. Define $\mathbf{A}_n(P)$ to be the event that for each $i \in [2]$, there is a t-subset $V_i \subseteq U_i$ such that the subhypergraph $\mathrm{KG}^r_{n,k}(\rho)[V_1, \ldots, V_r]$ has no edge. Now, define the event $\mathbf{A}_n$ to be the union of all $\mathbf{A}_n(P)$'s, i.e.,

$$\mathbf{A}_n = \bigcup \mathbf{A}_n(P),$$

where the union is taken over all $P = (M_1, \ldots, M_r) \in (\mathbb{Z}_r \cup \{0\})^n$ such that $M_1, \ldots, M_r$ are pairwise disjoint $(k+l)$-subsets of $[n]$.

Let $c : \binom{[n]}{k} \longrightarrow [C]$ be a proper coloring for $\mathrm{KG}^r_{n,k}(\rho)$, where $C < d$. Consider the r-tuple $P = (N_1, \ldots, N_r)$ whose existence is ensured by Lemma 4. Without loss of generality, we may assume that $c(N_1, \ldots, N_r) = 1$. Let

$$U_i = \{A \,:\, |A| = k,\ A \subseteq N_i\} = \{f^j_1, \ldots, f^j_q\},$$

where $q = \binom{k+l}{k}$. For each $j \in [r]$, let $V_j \subseteq U_j$ be the set of edges receiving color 1. Clearly, the subhypergraph $\mathrm{KG}^r_{n,k}[V_1, \ldots, V_r]$ is a monochromatic subhypergraph of $\mathrm{KG}^r_{n,k}$. Therefore, since c is a proper coloring, $\mathrm{KG}^r_{n,k}(\rho)[V_1, \ldots, V_r]$ has no edge, which implies that $\mathbf{A}_n(P) \subseteq \mathbf{A}_n$ is happened. Hence, we have $\mathbf{E}_n \subseteq \mathbf{A}_n$.

Consequently, if we prove that $Pr(\mathbf{A}_n) \to 0$ as $n \to +\infty$, then we have $Pr(\mathbf{E}_n) \to 0$ as $n \to +\infty$, as desired. First, note that

$$\begin{aligned}
Pr(A) &\leq \sum \binom{q}{t}^r (1-\rho)^{t^r} \\
&\leq \binom{n}{k+l}^r \left(\tfrac{eq}{t}\right)^{rt} e^{-\rho t^r} \\
&\leq \left(\tfrac{ne}{k+l}\right)^{r(k+l)} \left(\tfrac{eq}{t}\right)^{rt} e^{-\rho t^r} \\
&\leq e^{-\rho t^r + r(k+l)(\ln n + 1) + rt(1+\ln(d-1))},
\end{aligned}$$

where the summation is taken over all $P = (M_1, \ldots, M_r)$ such that $M_1, \ldots, M_r$ are pairwise disjoint $(k+l)$-subsets of $[n]$. (Note that the number of such P's is at most $\binom{n}{k+l}^r$). Thus, if

$$r(k+l)(\ln n + 1) + rt(1 + \ln(d-1)) - \rho t^r \to -\infty,$$

then $Pr(\mathbf{A}_n) \to 0$. This completes the proof. □

Proof of Theorem 3. If we set $r = 2$ and use Lemma 5 instead of Lemma 4, the proof follows by almost "copy-pasting" the proof of Theorem 2. □

Proof of Theorem 4. Let $c : V(\mathrm{KG}^r(\mathcal{H})) \longrightarrow [C]$ be a coloring such that $\mathrm{KG}^r(\mathcal{H})$ has no monochromatic $K^r_{t,\ldots,t}$ subhypergraph, where $1 \leq C < \min\left\{\frac{n-\mathrm{alt}_r(\mathcal{H},\sigma,q)}{r-1}, d\right\}$. Let $e_{i,j}$'s be the edges whose existence is ensured by Lemma 2. If we set $W_j = \{e_{1,j}, \ldots, e_{t,j}\}$, then the subhypergraph $\mathrm{KG}^r(\mathcal{H})[W_1, \ldots, W_r]$ is a monochromatic $K^r_{t,\ldots,t}$ subhypergraph of $\mathrm{KG}^r(\mathcal{H})$, a contradiction.□

4. Another Extension of Theorem A

Note that Kupavskii's result (Theorem A) concerns the chromatic number of Schrijver graphs, while if we use Theorem 1 (set $r = 2$ and $\mathcal{H}_m = ([n], \binom{[n]}{k}_{stable})$) to obtain a lower bound for the chromatic number of random Schrijver graphs, then it implies a lower bound which is not as well as the lower bound stated in Theorem A. Actually, one can see that this lower bound is the lower bound stated in Theorem A minus one. Motivated by the this discussion, in this section, we present another extension of Theorem 1, which immediately implies Theorem A. Actually, with some slightly modifications in the proof of Lemma 2, we can have a similar statement which is helpful for the case of Schrijver graphs.

Set $\mathbb{Z}_2 = \{+, -\}$. For each $X = (x_1, \ldots, x_n) \in \{+, -, 0\}^n$, define

$$X^+ = \{i \in [n] : \ x_i = +\} \quad \text{and} \quad X^- = \{i \in [n] : \ x_i = -\}.$$

Let $\mathcal{H}$ be a hypergraph and $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection. Define

$$\mathrm{salt}(\mathcal{H}, \sigma, q) = \max\left\{\mathrm{alt}(X) : \ X \in \{+, -, 0\}^n \ \text{ s.t. } \min_{\varepsilon\in\{+,-\}} (|E(\mathcal{H}[\sigma(X^\varepsilon)])|) \le q - 1\right\}.$$

Now, set

$$\mathrm{salt}(\mathcal{H}, q) = \min_{\sigma} \mathrm{salt}(\mathcal{H}, \sigma, q),$$

where the minimum is taken over all bijection $\sigma : [n] \longrightarrow V(\mathcal{H})$. For $q = 1$, we prefer to use $\mathrm{salt}(\mathcal{H})$ instead of $\mathrm{salt}(\mathcal{H}, 1)$. The present authors [1] proved that $n - \mathrm{salt}(\mathcal{H}) + 1$ is a lower bound for the chromatic number of $\mathrm{KG}(\mathcal{H})$. One can simply see that $\mathrm{salt}\left([n], \binom{[n]}{k}_{stable}\right) = 2k - 1$. Hence, in view of last mentioned lower bound, we have an exact lower bound for the chromatic number of Schrijver graphs $\mathrm{SG}_{n,k}$. Also, as it is expected, we can have the following lemma which is similar to Lemma 2.

Lemma 6. *Let d, q and t be a positive integers, where $d \ge 2$ and $q \ge (d - 1)(t - 1) + 1$. Let $\mathcal{H}$ be a hypergraph and $\preceq$ be a total ordering on the power set of $V(\mathcal{H})$, which refines the partial ordering according to size. Moreover, let $\sigma : [n] \longrightarrow V(\mathcal{H})$ be a bijection. Then for any coloring $c : E(\mathcal{H}) \longrightarrow [C]$, where $C < \min\{n - \mathrm{salt}(\mathcal{H}, \sigma, q) + 1, d\}$, there is an ordered pair (N_1, N_2) with the following properties.*

- *N_1, N_2 are pairwise disjoint subsets of $[n]$.*
- *For $j = 1, 2$, $|E(\mathcal{H}[\sigma(N_j)])| \ge q$.*
- *For $j = 1, 2$, there are t distinct edges $e_{1,j}, \ldots, e_{t,j} \subseteq \sigma(N_j)$ chosen from the last q largest edges in $\sigma(N_j)$ (according to the total ordering $\preceq$) such that all edges in $\{e_{i,j} : i \in [t] \ \& \ j \in [2]\}$ receive the same color $c(N_1, N_2)$.*

Sketch of Proof. Consider an arbitrary coloring $c : E(\mathcal{H}) \longrightarrow [C]$ such that $1 \le C < \min\{n - \mathrm{salt}(\mathcal{H}, \sigma, q) + 1, d\}$. Without loss of generality and for simplicity of notation, we may assume that $V(\mathcal{H}) = [n]$ and $\sigma = I$ is the identity map. In view of the definition of $\mathrm{salt}(\mathcal{H}, I, q)$, for any $X \in \{+, -, 0\}^n \setminus \{\mathbf{0}\}$ with $\mathrm{alt}(X) \ge \mathrm{salt}(\mathcal{H}, I, q) + 1$, we have $|E(\mathcal{H}[X^\varepsilon])| \ge q$ for each $\varepsilon \in \{+, -\}$. See all edges in $E(\mathcal{H}[X^\varepsilon])$ as a chain (according to the total ordering $\preceq$) and consider the last q edges of this chain and define $g(X^\varepsilon)$ to be the maximum most popular color amongst colors assigned to these q edges. Now, set $g(X) = \max\left(g(X^+), g(X^-)\right)$. Note that if there is an $X \in \{+, -, 0\}^n \setminus \{\mathbf{0}\}$ with $\mathrm{alt}(X) \ge \mathrm{salt}(\mathcal{H}, \sigma, q)+1$ and such that $g(X^+) = g(X^-) = i$, then for $N_1 = X^+$ and $N_2 = X^-$, one can simply see that the pair (N_1, N_2) with $c(N_1, N_2) = i$ has the desired properties. Therefore, we may assume that $g(X^+) \neq g(X^-)$ for each $X \in \{+,-,0\}^n \setminus \{\mathbf{0}\}$ with $\mathrm{alt}(X) \geq \mathrm{salt}(\mathcal{H}, I, q)+1$. Note that it implies that $g(X) \geq 2$ for each $X \in \{+,-,0\}^n \setminus \{\mathbf{0}\}$ with $\mathrm{alt}(X) \geq \mathrm{salt}(\mathcal{H}, I, q)+1$. Set $p=2$, $\alpha = \mathrm{salt}(\mathcal{H}, I, q)$ and $m = \mathrm{salt}(\mathcal{H}, I, q)+C-1$. Now, we are ready to define a map $\lambda : \{+,-,0\}^n \setminus \{\mathbf{0}\} \longrightarrow \{+,-\} \times [m]$. Consider an arbitrary $X \in \{+,-,0\}^n \setminus \{\mathbf{0}\}$. If $\mathrm{alt}(X) \leq \alpha$, then define $\lambda(X) = (\varepsilon, \mathrm{alt}(X))$, where ε is the first nonzero coordinate of X. If $\mathrm{alt}(X) \geq \alpha+1$, then set $\lambda(X) = (\varepsilon, \alpha+g(X)-1)$, where ε is $+$ if $g(X^+) > g(X^-)$ and is $-$ otherwise. By the same approach as in the proof of Lemma 2, the proof follows with no difficulty. □

With the same approach as we used to derive Theorem 1 from Lemma 2, we can prove the following theorem from Lemma 6.

Theorem 5. *Let $\mathcal{A} = \{\mathcal{H}_m : m \in \mathbb{N}\}$ be a family of distinct hypergraphs and set $n = n(m) = |V(\mathcal{H}_m)|$. Also, let $t = t(n)$, $d = d(n)$, and $q = q(n)$ be integer functions, where $d \geq 2$ and $(d-1)(t-1)+1 \leq q \leq (d-1)t$, and let $\rho = \rho(n) \in (0,1]$ be a real function. Then we a.s. have*

$$\chi(\mathrm{KG}(\mathcal{H}_m)(\rho)) \geq \min\{|V(\mathcal{H}_m)| - \mathrm{salt}(\mathcal{H}_m, q)+1, d\}$$

provided that $n \ln 3 + 2t(1+\ln(d-1)) - \rho t^2 \to -\infty$ as n tends to infinity.

Since $\mathrm{KG}\left([n], \binom{[n]}{k}_{stable}\right) = \mathrm{SG}_{n,k}$ and $\mathrm{salt}\,\mathrm{KG}\left([n], \binom{[n]}{k}_{stable}\right) = 2k-1$, Theorem 5 implies Theorem A. Hence, it is a generalization of Theorem A with a combinatorial proof. Also, similar to the proof of Theorem 2, we have a purely combinatorial proof for the Kupavskii's heorem (Theorem A).

It might be intriguing that we state Theorem 5 just in the case of graphs while it seems that these results remain true even for hypergraphs. Actually, For a hypergraph $\mathcal{H}$, we can naturally generalize $\mathrm{salt}(\mathcal{H})$ to $\mathrm{salt}_r(\mathcal{H})$. However, for any hypergraph $\mathcal{H}$, the value of $\mathrm{salt}_r(\mathcal{H})$ is equal to $|V(\mathcal{H})|$, which clearly makes this generalization useless.

In the proof of Theorem 4, if we set $r=2$ and use Lemma 6 instead of Lemma 2, then we have the following theorem. It should be mentioned that this result is already proved in [1] for $t=1$.

Theorem 6. *Let $\mathcal{H}$ be a hypergraph and $\sigma : [n] \longrightarrow V(\mathcal{H})$ be an arbitrary bijection. Also, let d, q and t be be positive integers, where $q \geq (d-1)(t-1)+1$. Then any coloring of $\mathrm{KG}(\mathcal{H})$ with no monochromatic $K_{t,t}$ uses at least $\min\{n - \mathrm{salt}(\mathcal{H}, \sigma, q)+1, d\}$ colors.*

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